



## MATHEMATICS IN CAGING OF ROBOTICS

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Communicated by Jan J. Slawianowski

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**Abstract.** It is a crucial problem in robotics field to cage an object using robots like multifingered hand. However the problem what is the caging for general geometrical objects and robots has not been well-described in mathematics though there were many rigorous studies on the methods how to cage an object by certain robots. In this article, we investigate the caging problem more mathematically and describe the problem in terms of recursion of the simple euclidean moves. Using this description, we show that the caging has the degree of difficulty which is closely related to a combinatorial problem and a wire puzzle. It implies that in order to capture an object by caging, from a practical viewpoint the difficulty plays an important role.

*MSC:* 51M04, 57N15, 57N35, 70B15, 70E60

*Keywords:* Caging, euclidean move, robotics, wire puzzle

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### 1. Introduction

In robotics fields, caging is a type of grasping where robots capture an object by surrounding or hooking it. Thus the caging problem based on the shape of the robots and the object is addressed via geometrical representation. Its mathematical description has been partially studied with a focus on methodology how to cage an object by certain robots. Though it is rigorous, it is not suitable for arbitrary target objects and robots. In this article, we propose an essential of caging to describe arbitrary target objects and robots from a mathematical viewpoint, and then it naturally leads us to a degree of difficulty of escaping and caging. It is a novel concept of the caging which is connected with practical approaches.

Caging or holding an object has been discussed in mathematics field in [3, 21], and has been applied to robotic manipulation in parallel. Rimon and Blake [16] raised a caging by two circular robots driven by one parameter in two dimensional planar space, and formulated its conditions. Wang and Kumar [19] proposed caging by multi-robot cooperation with mathematical abstract formulas. More than three dimensional caging problem is formulated by [14], although only circular and spherical robots are referred. There studies discuss existence of object's free movable

space closed by the robots. Hence path connectivity of the free space for the captured object [17] is an important matter to investigate whether the object is caged or able to escape. As mentioned above, confinement of caging formation is studied previously in [14, 16, 17], particularly for problems of two dimensional caging. Additionally although path connectivity can be examined by using probabilistic search algorithm [5, 11], the difficulty of caging constraint is not quantitatively qualified.

Therefore this article aims to describe caging problem in robotics mathematically in order to apply the formulation to arbitrary robots and objects. We reveal difficulty of caging constraint mathematically based on the euclidean moves. For examples, using the euclidean moves, control of multiple chains of links has been studied rigorously [12, 20]. In this article, we focus only on the caging problem weakly connected with such works.

From a mathematical viewpoint, we recall the fundamental fact that a compact connected  $n$ -dimensional topological manifold  $M$  in  $n$ -dimensional euclidean space  $\mathbb{E}^n$  could be wrapped by the  $(n-1)$ -dimensional sphere  $S^{n-1}$ , which corresponds to holding in the real world. This fact is described well by the homotopy group  $\pi_{n-1}$ . It is based on the mathematical fact that  $S^{n-1}$  in  $\mathbb{E}^n$  divides  $\mathbb{E}^n$  into its inner side and outer side. The subspace  $M$  has the configuration space  $[M]$  by the euclidean moves  $\psi$ , i.e.,  $[M] = \{\psi(M)\}$ . Let us fix the sphere  $S^{n-1}$  and then it divides  $[M]$  to the part  $[M]_{\text{in}}$  of the inner side of  $S^{n-1}$ , that of the outer side,  $[M]_{\text{out}}$ , and the intersection part  $[M]_{\text{int}}$ , i.e.,  $[M] = [M]_{\text{in}} \amalg [M]_{\text{out}} \amalg [M]_{\text{int}}$  and  $[M]_{\text{int}} := \{M' \in [M] ; M' \cap S^{n-1} \neq \emptyset\}$ . If  $[M]_{\text{in}}$  is not empty, it means that  $S^{n-1}$  holds  $M \in [M]_{\text{in}}$ .

On the other hand, caging is to divide the configurations space  $[M]$  using an  $(n-2)$  dimensional topological manifold  $\mathcal{K}$ . Then there arises a problem how  $\mathcal{K}$  with two-codimension can divide the configuration space  $[M]$  mathematically. This is a fundamental problem on caging.

There are many mathematical studies on this problem to cage some proper geometrical objects using special  $\mathcal{K}$  as mentioned above. Further Fruchard and Zamfirescu [6, 21] have studied the geometrical conditions whether a circle holds a convex object  $[M]$ . Rodriguez, Mason and Ferry [17] investigated geometrical and topological properties the multifinger cages rigorously.

However there was no investigation on the above fundamental problem for general  $M$  and  $\mathcal{K}$ . Though the path connectivity of an object  $M$  in  $\mathbb{E}^n \setminus \mathcal{K}$  studied in [5, 11, 17] is an essential property of caging, it is not sufficient. For example if we regard a wire puzzle constituting two pieces as a pair of a robot  $\mathcal{K}$  and a target object  $M$ ,  $M$  is not caged by  $\mathcal{K}$  because there is a path between  $M$  “in” of  $\mathcal{K}$

and that of the outer side of  $\mathcal{K}$ . However a wire puzzle should be considered as an example of caging and holding from a practical viewpoint. It is related to the probabilistic treatment of the path connectivity [5, 11]. However the path space is very complicate in general [2] and it is very difficult to assign a probabilistic measure in the path space.

In order to introduce a degree of escaping mathematically, in this article, we describe the fundamental problem more mathematically, and then we show that caging is represented in terms of recursion of the simple euclidean moves, i.e., the piecewise euclidean move defined in Definitions 8 and 14. The move means that caging is classified countably and naturally leads us to the degree of difficulty of escaping of  $[M]$  from  $\mathcal{K}$ . It is closely related to a combinatorial explosion and the wire puzzle. It means that there might be a difference between practical caging and complete caging. When we capture a complicate object by caging, we propose that the difficulty should be proactively considered from a practical viewpoint. Further if we treat the euclidean moves probabilistically, we could assign a natural measure on the moves.

## 2. Mathematical Preliminaries

Let us consider the  $n$ -dimensional real euclidean space  $\mathbb{E}^n$  and the  $n$ -dimensional euclidean group  $\text{SE}(n) := \mathbb{R}^n \rtimes \text{SO}(n)$ . The element  $g$  of  $\text{SE}(n)$  acts on each point  $x$  of  $\mathbb{E}^n$  by  $\Phi_g(x) \in \mathbb{E}^n$  such that for  $g, g' \in \text{SE}(n)$ ,  $\Phi_{g'g}(x) = \Phi_{g'}\Phi_g(x)$  and for the identity element  $e \in \text{SE}(n)$ ,  $\Phi_e(x) = \text{Id}(x) = x$  [1, 7]. The action  $\Phi_g$  of  $g \in \text{SE}(n)$  on  $\mathbb{E}^n$  has the matrix representation: for  $g$ , there are element  $u \in \mathbb{R}^n$  and  $A \in \text{SO}(n)$  such that for  $x \in \mathbb{E}^n$

$$\Phi_g(x) = \begin{pmatrix} 1 & 0 \\ u & A \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ Ax + u \end{pmatrix}.$$

In this article, let us refer a  $\ell$ -dimensional topological manifold in  $\mathbb{E}^n$ ,  $\ell$ -subspace. We say that the  $\ell$ -subspaces  $M$  and  $M'$  of  $\mathbb{E}^n$  are *congruent* if there is an element  $g$  of  $\text{SE}(n)$  such that  $\Phi_g(M) = M'$ . We denote it by  $M \simeq M'$ . Let  $[M]$  be the configuration space of  $M$ , i.e.,  $[M] := \{\Phi_g(M) ; g \in \text{SE}(n)\}$ . The quotient space of the set of the  $n$ -subspaces divided by  $\text{SE}(n)$  is denoted by  $\mathfrak{M}$ , which classifies the shape of subspace in  $\mathbb{E}^n$ . We call  $\mathfrak{M}$  *moduli of the shapes*,  $[M] \in \mathfrak{M}$ .

We consider a continuous map  $\psi : [0, 1] \rightarrow \text{SE}(n)$ , i.e.,  $\psi \in \mathcal{C}^0([0, 1], \text{SE}(n))$  with a fixing point  $\psi(0) = \text{Id}$ , where  $\mathcal{C}^0(N, F)$  means the set of  $F$ -valued continuous functions over  $N$ .

For given  $\psi \in \mathcal{C}^0([0, 1], \text{SE}(n))$ , the action  $\Phi_\psi$  on  $\mathbb{E}^n$  parameterized by  $t \in [0, 1]$  is called *orbit by  $\psi$* .

**Lemma 1.** For an  $n$ -subspace  $M \subset \mathbb{E}^n$ , the action  $\Phi_{\psi(t)}(M)$  induces a congruent family  $\{\Phi_{\psi(t)}(M)\}_{t \in [0,1]}$

For the Lie group  $\text{SE}(n)$ , there is its Lie algebra  $\mathfrak{se}(n)$  whose element  $\mathfrak{g}$  satisfies  $\exp(\mathfrak{g}) \in \text{SE}(n)$ ; for the economy of notations, we use the same notation  $\mathfrak{g}$  for its matrix representation.

**Lemma 2.** For  $\mathfrak{g} = \begin{pmatrix} 0 & 0 \\ \xi & \omega \end{pmatrix} \in \mathfrak{se}(n)$ , where  $\omega \in \mathfrak{so}(n)$  and  $\xi \in \mathbb{R}^n$

$$\exp(\mathfrak{g}) = e^{\mathfrak{g}} = \begin{pmatrix} 1 & 0 \\ v(\omega)\xi & e^\omega \end{pmatrix}$$

$$\text{where } v(\omega) := \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \omega^k = \int_0^1 e^{s\omega} ds$$

$$\omega v(\omega) = e^\omega - I_n$$

and where  $I_n$  is the  $n \times n$  unit matrix. If  $\omega$  is regular,  $v(\omega) = (e^\omega - I_n)\omega^{-1}$ .

**Proof:** The straightforward computation leads  $\mathfrak{g}^k = \begin{pmatrix} 0 & 0 \\ \omega^{k-1}\xi & \omega^k \end{pmatrix}$  and thus we have the result. ■

Here we recall some of the properties of  $\mathfrak{so}(n)$  [9]

**Lemma 3.** 1.  $\dim_{\mathbb{R}} \mathfrak{so}(n) = \frac{n(n-1)}{2}$

2. the matrix representation of an element  $\omega$  of  $\mathfrak{so}(n)$  is given by  $(\omega_{ij})$  such that  ${}^t\omega = -\omega$ , i.e.,  $\omega_{ij} = -\omega_{ji}$  ( $i, j = 1, 2, \dots, n$ )

3. the maximal rank of the matrix representation of  $\mathfrak{so}(n)$  is  $(n-1)$  if  $n = \text{odd}$  and  $n$  otherwise, and

4. for the matrix representation  $\omega \in \mathfrak{so}(n)$ , we have the natural decomposition

$$\mathbb{R}^n = \omega(\mathbb{R}^n) \oplus \ker \omega$$

by considering  $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , i.e., the cokernel  $\text{coker } \omega$  agrees with the kernel of  $\omega$ .

**Proof:** Items 1, 2, and 3 are obvious as in [9, p.63]. Using the euclidean inner product, let us show that  $\omega(\mathbb{R}^n)^\perp = \ker \omega$  which is equivalent with item 4. Let

us consider an element  $x$  of  $\omega(\mathbb{R}^n)^\perp = (\text{img}\omega)^\perp$ , which means  $(x, \omega y) = 0$  for every  $y \in \mathbb{R}^n$ , and equivalently  $({}^t\omega x, y)$  vanishes. The element  $x$  must belongs to  $\ker {}^t\omega$  or  $\ker \omega$  because of  ${}^t\omega = -\omega$ . ■

Since for  $\mathfrak{g} \in \mathfrak{se}(n)$  and  $t \in [0, 1]$ ,  $t\mathfrak{g}$  belongs to  $\mathfrak{se}(n)$ , by using this relation between  $\text{SE}(n)$  and  $\mathfrak{se}(n)$ , we consider a euclidean move  $\psi(t) = \exp(t\mathfrak{g})$ , which we call *the simple euclidean move*.

**Lemma 4.** For  $t \in [0, 1]$  and  $\mathfrak{g} = \begin{pmatrix} 0 & 0 \\ \xi & \omega \end{pmatrix} \in \mathfrak{se}(n)$

$$\exp(t\mathfrak{g}) = \begin{pmatrix} 1 & 0 \\ v_t(\omega)\xi & e^{t\omega} \end{pmatrix}$$

where  $v_t(\omega) = tv(t\omega) = \int_0^t e^{s\omega} ds$  satisfying

$$\omega v_t(\omega) = e^{t\omega} - I_n.$$

If  $\omega$  is regular,  $v_t(\omega) = (e^{t\omega} - I_n)\omega^{-1}$ .

**Proof:**  $v_t(\omega) = tv(t\omega) = \sum_{k=1}^{\infty} \frac{1}{k!} t^k \omega^{k-1} = t \int_0^1 e^{st\omega} ds$ . By replacing  $s$  with  $st$ ,

$$v_t(\omega) = \int_0^t e^{s\omega} ds. \quad \blacksquare$$

Though the action of the euclidean move should be regarded as a “rotation” in the projective space  $\mathbb{P}\mathbb{R}^n$  via  $\text{SL}(n, \mathbb{R})$  in  $\mathbb{R}^{n+1}$ , the simple euclidean move is given by the following lemma

**Lemma 5.** For  $t \in [0, 1]$  and  $\mathfrak{g} = \begin{pmatrix} 0 & 0 \\ \xi & \omega \end{pmatrix} \in \mathfrak{se}(n)$ , the orbit of  $x \in \mathbb{E}^n$  by the simple euclidean move  $\psi(t) := e^{t\mathfrak{g}}$ , i.e.,  $\psi(t)x = e^{t\omega}x + v_t(\omega)\xi$ , is reduced to

$$\psi(t)x = e^{t\omega}(x + \xi_0) - \xi_0 + \xi_1 t \quad (1)$$

where  $\xi = \omega\xi_0 + \xi_1$  for  $\xi_1 \in \ker\omega$  and  $\xi_0 \in \mathbb{R}^n$ . It means the following

1. If  $\xi_1$  vanishes,  $\psi(t)$  means a rotation by  $e^{t\omega}$ , at a center  $-\xi_0 \in \mathbb{E}^n$  by identifying  $\mathbb{R}^n$  with  $\mathbb{E}^n$  as a set.
2. If  $\omega$  vanishes,  $\psi(t)$  is the translation

$$\psi(t)x = x + \xi t.$$

3. If both  $\omega$  and  $\xi_1$  do not vanish, it is a mixed move of the rotation and the translation.

**Proof:** Since this action satisfies

$$\omega\psi(t)x = e^{t\omega}(\omega x + \xi) - \xi$$

and we have the decomposition  $\xi = \omega\xi_0 + \xi_1$  where  $\xi_1$  is the kernel of  $\omega$  from Lemma 3, we obtain the identity (1) by substituting  $\xi$  into  $\psi(t)x$ . ■

**Lemma 6.** *The map  $\exp$  from  $\mathfrak{se}(n)$  to  $\text{SE}(n)$  is surjective.*

**Proof:** Though it can be directly proved from the Lemma 3, it is also obtained from (1) by setting  $t = 1$ . Then its surjectivity is can be directly proved by the triangulation of the matrix  $\omega$  though it is a little bit complicate due to the maximal tori in  $\text{SO}(n)$ . ■

**Corollary 7.** *For every pair of congruent  $n$ -subspaces  $M$  and  $M'$ , there is a simple euclidean move  $\psi(t) = \exp(t\mathfrak{g})$  of  $\mathfrak{g} \in \mathfrak{se}(n)$  such that  $\Phi_{\psi(1)}(M) = M'$ .*

**Definition 8.** *The euclidean move  $\psi(t) \in \text{SE}(n)$  which is given by a collection of the simple euclidean moves  $\{\exp(t\mathfrak{g}_i) ; \mathfrak{g}_i \in \mathfrak{se}(n), t \in [0, 1]\}_{i=1, \dots, \ell}$ , i.e.,*

$$\psi(t) = \exp\left(\frac{t - t_{j-1}}{t_j - t_{j-1}}\mathfrak{g}_j\right) e^{\mathfrak{g}_{j-1}} \dots e^{\mathfrak{g}_1}$$

for  $t \in [t_{j-1}, t_j]$ , where  $t_j := \frac{j}{\ell}$ , we call  $\psi(t)$  the piecewise euclidean move.

### 3. Mathematics of Caging

Let us consider the subspace  $\mathcal{K} \subset \mathbb{E}^n$  whose codimension is two, i.e.,  $\mathcal{K}$  is  $(n-2)$ -subspace in  $\mathbb{E}^n$ .  $\mathcal{K}$  might be decomposed to  $p$  connected parts

$$\mathcal{K} = \prod_{i=1}^p \mathcal{K}_i.$$

Hereafter  $\mathcal{K}$  is fixed.

**Remark 9.** The caging is to restrict some  $n$ -subspace in  $\mathbb{E}^n$  using the other subspace  $\mathcal{K}$  whose codimension is two, which is modeled after figures, wires in three dimensional case. The subspace  $\mathcal{K}$  is modeled by them. Thus from a practical

viewpoint, we do not consider wild geometries such as the Hilbert curve but we neither exclude them in this article. For such a wild object, some of the following results might be trivial.

For given  $\mathcal{K}$ , we let  $[M]_{\mathcal{K}^c}$  be the subset of the configuration space  $[M]$  whose element is disjoint to  $\mathcal{K}$ , i.e.,  $[M]_{\mathcal{K}^c} := \{M' \in [M] ; M' \subset \mathcal{K}^c\}$ , and  $\mathfrak{M}_{\mathcal{K}^c}$  be the family of  $[M]_{\mathcal{K}^c}$ .

**Definition 10.** *Let  $M$  and  $M'$  be congruent  $n$ -subspaces in  $\mathcal{K}^c$  i.e.,  $M$  and  $M'$  belongs to  $[M]_{\mathcal{K}^c} \in \mathfrak{M}_{\mathcal{K}^c}$ , and there exists  $g \in \text{SE}(n)$  satisfying  $\Phi_g(M) = M'$ . If there is  $\psi \in \mathcal{C}^0([0, 1], \text{SE}(n))$  such that its congruent family  $\{\Phi_{\psi(t)}(M)\}_{t \in [0, 1]}$  satisfies the conditions*

1.  $\Phi_{\psi(0)} = \text{Id}$
2.  $\Phi_{\psi(1)} = \Phi_g$  and
3.  $\Phi_{\psi(t)}(M) \cap \mathcal{K} = \emptyset$  for every  $t \in (0, 1)$

*we say that  $M$  and  $M'$  are  $\mathcal{K}^c$ -congruent, denoted by  $M \simeq_{\mathcal{K}^c} M'$ , and  $\Phi_{\psi}$  is a  $\mathcal{K}^c$ -congruent homotopy for  $M$  and  $M'$ . If we cannot find  $\psi \in \mathcal{C}^0([0, 1], \text{SE}(n))$  satisfying the above conditions, we say that  $M$  and  $M'$  are not  $\mathcal{K}^c$ -congruent, denoted by  $M \not\simeq_{\mathcal{K}^c} M'$ .*

We note that the element  $[M]_{\mathcal{K}^c}$  of  $\mathfrak{M}_{\mathcal{K}^c}$  is the set of the congruent subspaces.

**Definition 11.** *If every element  $M$  of  $[M]_{\mathcal{K}^c}$  is  $\mathcal{K}^c$ -congruent each other, we say that  $[M]_{\mathcal{K}^c}$  is a  $\mathcal{K}^c$ -congruent set and we cannot cage  $[M]_{\mathcal{K}^c} \in \mathfrak{M}_{\mathcal{K}^c}$  for  $\mathcal{K}$ .*

*If we find a pair  $M$  and  $M'$  of  $[M]_{\mathcal{K}^c}$  which are not  $\mathcal{K}^c$ -congruent, we call  $[M]_{\mathcal{K}^c}$  a complete  $\mathcal{K}^c$ -caging set, and we say that we can completely cage  $[M]_{\mathcal{K}^c} \in \mathfrak{M}_{\mathcal{K}^c}$ .*

**Proposition 12.** *The moduli of shapes  $\mathfrak{M}_{\mathcal{K}^c}$  is decomposed to*

$$\mathfrak{M}_{\mathcal{K}^c} = \mathfrak{M}_{\mathcal{K}^c}^{(0)} \coprod \mathfrak{M}_{\mathcal{K}^c}^{(1)}$$

*where  $\mathfrak{M}_{\mathcal{K}^c}^{(0)}$  is the family of the  $\mathcal{K}^c$  congruent sets of  $\mathfrak{M}_{\mathcal{K}^c}$  and  $\mathfrak{M}_{\mathcal{K}^c}^{(1)}$  is the family of complete  $\mathcal{K}^c$ -caging sets.*

**Proof:** The decomposition is obvious from the definition. ■

**Remark 13.** If  $\mathcal{K}$  of  $n = 3$  case is a space-filling curve like the Hilbert curve such that it is dense in  $\mathbb{E}^n$ ,  $\mathfrak{M}_{\mathcal{K}^c}$  itself is the empty set though we are not concerned with such a case. However it should be noted that if  $\mathfrak{M}_{\mathcal{K}^c}$  is not empty,  $[M]_{\mathcal{K}^c} \in \mathfrak{M}_{\mathcal{K}^c}$  has a non-trivial geometrical structure generally. In fact, Fruchard and Zamfirescu [6,21] considered the similar problem in which the  $\mathcal{K}$  is a circle and  $[M]_{\mathcal{K}^c} (\subset [M])$  is of a convex object, though in general  $[M]$  is not convex from a practical point of view.

Now in order to find a path from  $M$  to  $M'$ , we express the euclidean move  $\psi$  in terms of the piecewise euclidean move in Definition 8.

**Definition 14.** Let  $M$  and  $M'$  be  $\mathcal{K}^c$ -congruent such that  $\Phi_g(M) = M'$  of  $g \in \text{SE}(n)$ . If  $g$  is decomposed to the piecewise euclidean move, i.e.,

$$g = g_\ell \circ g_{\ell-1} \circ \cdots \circ g_2 \circ g_1$$

satisfying the conditions

1. For each  $g_i$ , we have  $\mathfrak{g}_i \in \mathfrak{se}(n)$  satisfying  $g_i = \exp(\mathfrak{g}_i)$  and
2. each  $\psi_i \in \mathcal{C}^0([0, 1], \text{SE}(n))$  given by  $\psi_i(t) = \exp(t\mathfrak{g}_i)$  for  $t \in [0, 1]$  is  $\mathcal{K}^c$ -congruent homotopy for  $\Phi_{g_{i-1} \circ \cdots \circ g_2 \circ g_1}(M)$  and  $\Phi_{g_i \circ \cdots \circ g_2 \circ g_1}(M)$

we say that  $\Phi_g$  is reduced to  $\ell$ -SE( $n$ ) action.

If  $\Phi_g$  is reduced to  $\ell$ -SE( $n$ ) action but cannot be reduced to  $(\ell - 1)$ -SE( $n$ ) action, we say that  $\Phi_g$  is  $\ell$ -th type and  $M$  and  $M'$  are  $\ell$ -th  $\mathcal{K}^c$ -congruent if  $\Phi_g$  is  $\ell$ -th type.

Further if  $M$  and  $M'$  are congruent but not  $\mathcal{K}^c$ -congruent, or cannot be  $\ell$ -th  $\mathcal{K}^c$ -congruent of finite  $\ell$ , we say that  $M$  and  $M'$  are  $\infty$ -th  $\mathcal{K}^c$ -congruent.

We should note that these components  $\psi_i$ 's are given as the rotations at certain points or the translations from Lemma 5.

**Proposition 15.** For given an  $n$ -subspace  $M \subset \mathcal{K}^c$ , the configuration space  $[M]_{\mathcal{K}^c}$  of  $\mathfrak{M}_{\mathcal{K}^c}$  is decomposed to

$$[M]_{\mathcal{K}^c} = \prod_{\ell=1}^{\infty} [M]_{\mathcal{K}^c}^{(\ell)}$$

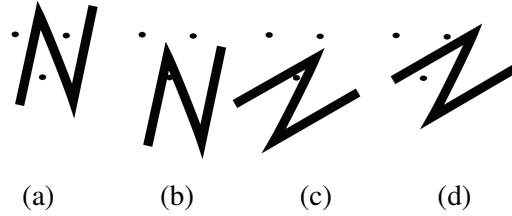
where  $[M]_{\mathcal{K}^c}^{(\ell)}$  is the set of subspaces of the  $\ell$ -th  $\mathcal{K}^c$ -congruence to  $M$ .

Now, we state the main theorem, which should be compared with Corollary 7.



**Theorem 16.** *In general  $\mathcal{K}^c$ -congruent  $n$ -subspaces  $M$  and  $M'$  are not the first  $\mathcal{K}^c$ -congruent.*

**Proof:** An example is illustrated in Fig. 1 of  $n = 2$  case, in which dots mean  $\mathcal{K}$  and a  $N$ -shaped object corresponds to  $M$ . They are  $\mathcal{K}^c$ -congruent to the outer one but it is obvious that the  $N$ -shaped object (a) in Fig. 1 is not the first  $\mathcal{K}^c$ -congruent to (d).



**Figure 1.** An example: The three dots mean  $\mathcal{K}$  and a  $N$ -shaped object corresponds to  $M$  and (a), (b), (c), and (d) show the piecewise euclidean moves of the  $N$ -shaped object.

This example can be extended to  $M \times \mathbb{R}^{n-2}$  and  $\mathcal{K} \times \mathbb{R}^{n-2}$ . ■

**Remark 17.** The decomposition to  $\mathfrak{M}_{\mathcal{K}^c}^{(0)}$  and  $\mathfrak{M}_{\mathcal{K}^c}^{(1)}$  is related to the link problem and the knot theory [10]. Thus  $\mathcal{K}^c$ -congruence is a profound problem. Some of kinds of caging are surely based on the braid group in the knot theory when the fundamental group  $\pi_1(M)$  of  $[M]$  is not trivial.

As mentioned in the Introduction from the practical viewpoint, we are concerned with the  $\ell$ -th  $\mathcal{K}^c$ -congruence rather than the  $\mathcal{K}^c$ -congruence itself. Theorem 16 means that for a  $\mathcal{K}^c$ -congruent configuration  $[M]_{\mathcal{K}^c}$ , there might be  $M \in [M]_{\mathcal{K}^c}$  so that we cannot move  $M$  to  $M' \in [M]_{\mathcal{K}^c}$  by the simple euclidean move. It reminds us of parking in a small garage and the wire puzzle. It is also closely related to the probabilistic treatment of connected path in  $[M]_{\mathcal{K}^c}$  [5, 11].

Let us consider  $M$  and  $M'$  are  $\ell$ -th  $\mathcal{K}^c$ -congruent,  $M \simeq_{\mathcal{K}^c} M'$ . If  $\ell$  is not small, it is not easy to find the  $\mathcal{K}^c$ -congruent homotopy  $\Phi_\psi$  such that  $\Phi_{\psi(1)}(M) = M'$ .

In the some situations, the  $i$ -th simple euclidean move in a piecewise euclidean move is restricted, and thus, countable in a certain sense, but  $\ell$  is not small. To find the piecewise euclidean moves connecting  $M$  and  $M'$  is basically difficult because of the combinatorial explosion. It is the origin of the difficulty of the wire puzzle. In other words, caging problem is connected with the difficulty of combinatorial problem.

This fact implies that if we want to restrict some geometrical objects  $[M] \in \mathfrak{M}$  by using the figures or something in a daily life, we do not need a complete caging in Definition 11 but we have to find whether it is not small  $\ell$  of  $\ell$ -th  $\mathcal{K}^c$ -congruence.

In other words, roughly speaking, there is the degree of the difficulty to take  $M$  in the inside of  $\mathcal{K}$  to the outside of  $\mathcal{K}$ , though the inside and the outside are not rigorous mathematically. We need not discriminate between the higher type of  $\mathcal{K}^c$ -congruence and a complete  $\mathcal{K}^c$ -caging set in a daily life.

Though it is very difficult to introduce the measure of the path space in general, we could measure the difficulty of caging and the probabilistic treatment of connected path in  $[M]_{\mathcal{K}^c}$  [5, 11].

Further this problem is related to a geometrical problem known as the Kakeya problem [4].

In order to express the practical caging, we introduce another concept.

**Definition 18.** *For given positive integer  $\ell_0$ , if we find  $n$ -subspaces  $M$  and  $M'$  in  $\mathcal{K}^c$  such that  $M$  and  $M'$  are congruent but not  $\ell$ -th  $\mathcal{K}^c$ -congruent for  $\ell < \ell_0$ , we say that  $\mathcal{K}$  dissociates  $M$  and  $M'$  by  $\ell_0$ -th caging.*

**Remark 19.** Practically, for a given geometrical object  $[M] \in \mathfrak{M}$ , it is very important to find a configuration  $\mathcal{K}$  which dissociates  $M$  and  $M'$  by  $\ell_0$ -th caging.

More practically, the first caging is much more important than higher  $\ell$ -th caging case because the concerned shape  $[M]$  is not so complicate. Even for the first caging, it is not easy to find whether it is the first caging or not because the dimension of  $SE(n)$  is six if  $n = 3$ . Determination of the first caging means the determination of topological property of  $SE(n)$ . For example, if we reduce the determination of continuous space to finite problem by expressing concerned area and  $\mathcal{K}$  in terms of the voxels for 100 points per one-dimension, we have to deal with  $100^6$  data, which is huge and one can not deal with it practically in this stage. Thus in order to avoid the problems, there are several attempts and proposals including the C-closure method. The second named author has investigated the problem using the C-closure concept [19]. In another way, intuitive geometrical features such as loop shape [15] and *double fork and neck* [18] help us derive sufficient conditions for caging constraint. In the computations, since the representation of the euclidean move and especially rotation is crucial, it is important to use the recent results [8, 13] for the study.

## Acknowledgments

We thank Professor S. Kaji for critical comments including the Takeya problem. The third named author was supported by the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant # 16K05187).

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