



GEOMETRIC QUANTIZATION OF FINITE TODA SYSTEMS AND COHERENT STATES

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Abstract. Adler had showed that the Toda system can be given a coadjoint orbit description. We quantize the Toda system by viewing it as a single orbit of a multiplicative group of lower triangular matrices of determinant one with positive diagonal entries. We get a unitary representation of the group with square integrable polarized sections of the quantization as the module. We find the Rawnsley coherent states after completion of the above space of sections. We also find non-unitary finite dimensional quantum Hilbert spaces for the system. Finally we give an expression for the quantum Hamiltonian for the system.

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1. Introduction

The connection between finite Toda system and coadjoint orbits was first explored by Adler [1]. We summarize the introduction to the Toda system as in [1]. The Hamiltonian considered is $H = \frac{1}{2} \sum_{i=1}^n y_i^2 + \sum_{i=1}^n e^{x_i - x_{i+1}}$, $x_0 = x_{n+1}$. The Hamiltonian equations of motion are $\dot{x}_i = y_i$, $\dot{y}_i = e^{x_{i-1} - x_i} - e^{x_i - x_{i+1}}$, $i = 1, \dots, n$.

Define

$$a_i = \frac{1}{2} e^{\frac{1}{2}(x_i - x_{i+1})}, \quad i = 1, \dots, n - 1, \quad b_i = \frac{1}{2} y_i, \quad i = 1, \dots, n.$$

Note that $a_i > 0$ for $i = 1, \dots, n - 1$.

Adler showed that the Hamiltonian equation of motion corresponds to a Lax equation and gave explicit expression for the integrals of motion which Poisson commute w.r.t. the following Poisson bracket

$$\{f, g\} = \sum_{i=1}^n (a_{i-1} g_{a_{i-1}} - a_i g_{a_i}) f_{b_i} + \sum_{i=1}^n a_i (g_{b_i} - g_{b_{i+1}}) f_{a_i}$$

where \cdot means that terms with undefined elements, i.e., terms involving a_0, a_n, b_{n+1} .

(**Note:** In page 222 of [1], there is a misprint in the formula for the Poisson bracket. The correct formula for the Poisson bracket, as written above, is given on page 225 in the same paper).

Adler goes on to show that the same system has a coadjoint orbit description of the group of lower triangular matrices of non-zero diagonal. In fact, one can restrict the action to that of lower triangular matrices of determinant 1 and positive diagonal elements. The system is homeomorphic to $\mathbb{R}_+^{n-1} \times \mathbb{R}^{n-1}$, just described by $a_i > 0, i = 1, \dots, n-1$ and $b_i, i = 1, \dots, n$ such that $b_1 + b_2 + \dots + b_n = c, c$ a constant.

We describe this precisely and show that the Toda system corresponds to a single orbit. We explore it further to geometrically quantize the Toda system. This is possible since the orbit has a symplectic structure whose induced Poisson bracket corresponds exactly to the Poisson bracket mentioned above.

In two famous papers [3], [4], Kostant describes the Kostant-Souriau quantization in general and for coadjoint orbits in particular. Using his construction, we were able to construct an infinite dimensional Hilbert-space of polarized sections of the quantum bundle (which is trivial in our case). The polarized sections are square integrable functions of $a_i, i = 1, \dots, (n-1)$ only. In this construction we modified the usual volume (given by the symplectic form) by an exponential decay. The group of lower triangular matrices with determinant 1 and positive diagonal entries acts on this Hilbert space giving in fact a unitary representation. By Kostant's result on general coadjoint orbits, [3], [4], this exactly corresponds to the Hilbert space of geometric quantization of the orbit and hence the Toda system.

Next, we construct Rawnsley's coherent states [5], of the Toda system corresponding to this quantization. There is another definition of coherent states for orbits. The coherent states are obtained by moving any "vacuum" vector by the group action. We show for orbit quantization, these two notions of coherent states coincide. We also construct (in the last section) finite dimensional representations which are not unitary.

The general reference we have followed about geometric quantization is the book by Woodhouse [9]. It would be interesting to relate this quantization with other quantizations of the Toda system [8] which are relevant to physics.

The quantization of Toda systems using geometric methods has already been worked out by Reyman and Semenov-Tian-Shansky [6,7] and Kharchev and Lebedev [2]. The coherent states have not been considered in this context as far as the authors know.

It would be interesting to see if the same idea of geometric quantization goes through for KdV type systems. The symplectic structure in this case has been derived by Adler [1].

2. The Orbit Corresponding to Toda System

A Toda system [1] is characterized by a upper triangular matrix of the form

$$A = \begin{bmatrix} b_1 & a_1 & & & \\ & b_2 & a_2 & & \\ & & \ddots & \ddots & \\ & & & b_{n-1} & a_{n-1} \\ & & & & b_n \end{bmatrix} \quad (1)$$

where the trace is constant and each a_i is positive and all other matrix elements are zero.

The space of upper triangular matrix is the dual lie algebra of multiplicative of group of lower triangular matrix. There is an coadjoint action. Since Toda matrices are upper triangular we attempt to find its orbit structure. The orbit structure is described in the following proposition.

Proposition 1. *The Toda space described above is a coadjoint orbit of the multiplicative group of lower triangular matrix with positive diagonals.*

Proof: In general the coadjoint action of l invertible and lower triangular on an upper triangular matrix u described in [1] is given by

$$l(u) = [lul^{-1}]_+ \quad (2)$$

where $[\]_+$ mean projection to the upper triangular part.

We find the orbits of the following matrices C_{ii+1} where only $(C_{ii+1})_{ii+1} > 0$ and rest of the terms are zero. Let L be any invertible lower triangular matrix given by $L = (L_{ij})$ then the matrix $LC_{12}L^{-1}_+$ is given by

$$\begin{aligned} ([LC_{12}L^{-1}]_+)_{11} &= -\frac{L_{21}}{L_{22}}c \\ ([LC_{12}L^{-1}]_+)_{12} &= \frac{L_{11}}{L_{22}}c \\ ([LC_{12}L^{-1}]_+)_{22} &= \frac{L_{21}}{L_{22}}c \end{aligned} \quad (3)$$

where c is the only nonzero entry of C_{12} that is $(C_{12})_{12} = c$. All other terms of the above matrix is zero.

Now if $\frac{L_{21}}{L_{22}}$ and $\frac{L_{11}}{L_{22}}$ are allowed to vary freely we will get a two dimensional manifold. Proceeding further by induction we can prove that the orbit of C with diagonal entries zero, all $(i, i + 1)$ entry greater than zero and rest entries zero is $2(n - 1)$ dimensional. The inductive argument is presented in the lemma bellow. If the Toda matrix is T then

$$T = D + C \quad (4)$$

where D is diagonal and C is a matrix of the type described above. The action of the invertible lower triangular matrices with positive diagonals will fix D and take the C to a $2(n - 1)$ dimensional manifold. Since the orbit map from G to the Toda space is of constant rank then by constant rank theorem the image (the orbit) is a manifold of dimension equal to the constant rank of the differential of the orbit map. Since each image is a $2(n - 1)$ dimensional manifold, the rank of the differential is $2(n - 1)$. Since the Toda space is $2(n - 1)$ dimensional the differentials are surjections. So by submersion theorem local open sets of G map surjectively by the orbit map to open sets of Toda space. Thus the orbits are open.

Since by above orbits are open each orbit is open and closed. This is because if we take an orbit O , it is open by the above argument and its complement is open as it is the union of other orbits. We have the set O is open and so is its complement, making it an open and closed set.

Since the Toda system is a connected set there will be only one orbit corresponding to it. ■

Lemma 2. *The coadjoint orbit of the matrix of type C is of dimension $2(n - 1)$.*

Proof: We proceed by induction. We first try to understand the case $C_{12} + C_{23}$

$$\begin{aligned} ([L(C_{12} + C_{23})L^{-1}]_+)_{11} &= -\frac{L_{21}}{L_{22}}c_1, & ([L(C_{12} + C_{23})L^{-1}]_+)_{12} &= \frac{L_{11}}{L_{22}}c_1 \\ ([L(C_{12} + C_{23})L^{-1}]_+)_{22} &= \frac{L_{21}}{L_{22}}c_1 - \frac{L_{32}}{L_{33}}c_2 & & (5) \\ ([L(C_{12} + C_{23})L^{-1}]_+)_{23} &= \frac{L_{22}}{L_{33}}c_2, & ([L(C_{12} + C_{23})L^{-1}]_+)_{33} &= \frac{L_{32}}{L_{33}}c_2. \end{aligned}$$

Here c_1 and c_2 are the values of non zero entries of C_{12} and C_{23} that is $(C_{12})_{12} = c_1$ and $(C_{23})_{23} = c_2$. All other terms of the above matrix is zero. Now for fixing a value of $\frac{L_{32}}{L_{33}}c_2$ the first row entries are unaffected so restriction of the orbit for

fixed $\frac{L_{32}}{L_{33}}c_2$ has at least two dimensions. Now as we change $\frac{L_{32}}{L_{33}}c_2$ we get one more dimension and thus the orbit will be at least three dimensional and since orbits are symplectic and hence even dimensional, this orbit will be at least four dimensional.

Now assuming the induction hypothesis that orbit $C_{12} + \dots + C_{kk+1}$ is at least $2k$ dimensional. Since the orbit of C_{k+1k+2} (which is similar to C_{12} case discussed above) is two dimensional, we can show $C_{12} + \dots + C_{kk+1} + C_{k+1k+2}$ will have orbit at least $2(k+1)$ dimensional (following the same line of argument as in the 4-d case.)

So the orbit of the matrix C in the Toda equation (4) above is at least $2(n-1)$ dimensional and since it lies in the Toda system which is $2(n-1)$ dimensional and the orbit of C is exactly $2(n-1)$ dimensional. ■

3. The Symplectic Form

We summarize the description of the symplectic form if the coadjoint orbit as in [1]. We also provide a formula for it. It is important to note that it corresponds to the Poisson bracket of the Toda system (given in page 225, [1]). This has been shown in the same paper.

Let G be the group of lower triangular matrices with non-zero diagonal entries. Its Lie algebra L is the lower triangular matrices.

As in [1], we may identify the dual of L , namely L^* , with the upper triangular matrices, using the pairing $\langle A, B \rangle = \text{Tr}(AB)$. $g \in G$ acts on L via conjugation. By duality, it acts on $l^* \in L^*$ $g : l^* \rightarrow [g^{-1}l^*g]_+$, where $[\cdot, \cdot]_+$ denotes the projection operator of setting all terms below the diagonal equal to zero, [1]. This is the coadjoint representation of G . The orbit of this action is through $l^* \in L^*$ is $\theta_{l^*} = \{[g^{-1}l^*g]_+; g \in G\}$. The tangent space of θ_{l^*} at l^* is described by $T_{l^*}\theta_{l^*} = \{[l^*, l]_+; l \in L\}$. The Kostant-Kirillov two-form ω associated with the orbit space θ_{l^*} is

$$\omega([l^*, l_1]_+, [l^*, l_2]_+)(l^*) = \langle l^*, [l_1, l_2] \rangle = \langle [l^*, l_1]_+, l_2 \rangle.$$

When one writes $l_1^* = [l^*, l_1]_+$ and $l_2^* = [l^*, l_2]_+$ and solves for the symplectic form, one gets that the symplectic form is

$$\omega = \sum_{i=1}^{n-1} \frac{1}{a_i} d(a_i) \wedge d(\sum_{j=1}^i b_j).$$

This form is of integral cohomology class, because it is exact.

Also, the symplectic form can be written in a standard form by a change of coordinates. Recall $a_i > 0$ for $i = 1, \dots, n - 1$. Let $q_i = \log(a_i)$ and $p_i = \sum_{j=1}^i b_j$. Then the symplectic form is the standard symplectic form on $\mathbb{R}^{2(n-1)}$, namely,

$$\sum_{i=1}^{n-1} d(q_i) \wedge d(p_i).$$

4. Polarization of the Complexified Lie Algebra

Here we describe the notion of polarization in a Lie algebra as in Kostant [4]. A polarization at g of an element of the dual lie algebra of a group G is a complex subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ (the complexification of \mathfrak{g}) which satisfies the following conditions

- 1) the subalgebra is stable under $\text{Ad } G_g$
- 2) $\dim_{\mathbb{C}}(\frac{\mathfrak{g}_{\mathbb{C}}}{\mathfrak{h}}) = n$
- 3) $\langle g, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$ and
- 4) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

If we restrict our self to the group G_1 of lower triangular matrices with non-negative diagonal entries and determinant one then its Lie-algebra \mathfrak{g} is the lower triangular matrices with trace zero.

Proposition 3. *The lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$, \mathfrak{h} of complex lower triangular matrices with diagonal zero is a polarization at any g in Toda space.*

This involves trivial checking.

4.1. Quantization

Here we recall the notion of geometric quantization as in Woodhouse [9]. A classical system is given by a pair (M, ω) where M is a symplectic manifold and ω a symplectic form. Suppose that the cohomology class of ω is integral then geometric pre-quantization is the construction of a line bundle L over M whose curvature $\rho(\nabla)$ (of a certain connection of L with covariant derivative ∇), is proportional to ω . The connection is given by a symplectic potential i.e., a one form θ such that $\omega = d(\theta)$ locally. The construction of the line bundle is always possible as long as the cohomology class of ω is proportional to an integral class. The Hilbert space of prequantization is the completion of the space of square integrable sections of

L . The Hilbert space of prequantization described above is too large for most purposes. Geometric quantization involves construction of a polarization (a certain type of distribution) of the symplectic manifold such that we now take polarized sections (depending on the above distribution) of the line bundle, yielding a finite dimensional Hilbert space in most cases.

Polarizations on the Manifold are complex involutory distributions F of $\dim n$ such that

- 1) $\omega(F_p F_p) = 0$ at all $p \in X$ and
- 2) $F + \bar{F}$ is an involutory distribution of constant dimension on M . Polarized sections are sections satisfying $\nabla_{\bar{X}} s = 0$ where X lies in F .

The following method of quantization developed by Kostant and Souriau, assigns to function $f \in C^\infty(M)$, an operator, $\hat{f} = \nabla_{X_f} + 2\pi i f$ acting on the Hilbert space of prequantization described above. Here ∇ is the covariant derivative of the connection θ of the bundle and X_f is the Hamiltonian vector field with respect to the symplectic form ω .

Functions mapping polarized sections to polarized sections are the ones which survive quantization. A sufficient condition for a function f to do this is that its Hamiltonian vector field X_f involutes with the polarization. This can be seen from the following formula

$$\nabla_{\bar{X}} \hat{f} s = \hat{f}(\nabla_{\bar{X}} s) + \nabla_{[X, X_f]} s \quad (6)$$

in which X lies in the polarization and s is a polarized section [9].

4.2. Polarization of the Complexified Lie Algebra and that on the Manifold

We have discussed two polarizations. One on the complexified lie algebra and the the other on the manifold through involutory distributions. When there is a Hamiltonian group action on the manifold M of a group G the fundamental vector fields are Hamiltonian and thus yielding functions. When M is a Hamiltonian- G space the assignment is a lie agebra homomorphism. In case of codajoint orbits lie algebra polarisation (described above) induces a polarisation in the manifold through fundamental vector fields. In such a case whole lie algebra \mathfrak{g}_G maps to polarized functions thus giving representation of the lie algebra to the space of polarized sections, [4].

Since the Toda space T is a coadjoint orbit from what proved above there is a Hamiltonian group action of G_1 the group of invertible lower triangular matrices

with non-negative diagonal entries of determinant one. It can be checked that restricting to G_1 does not reduce the orbit since $G = DG_1$ where D are scalar matrices which lies in stabilizer of the coadjoint action on the Toda space. Since our polarization comes from a lie algebra polarization \mathfrak{h} (lower triangular matrix with diagonal zero) which satisfies certain criterions (Section 4), the polarized sections will be a module of a representation of $\mathfrak{g}_{\mathbb{C}}$. Exponentiating this action we get a representation of the group [4]. (More details in Section 6).

5. Polarized Sections

When the manifold M is a coadjoint orbit the Hamiltonian functions due to fundamental vector fields can be evaluated easily. The Hamiltonian function for a polarized fundamental vector fields is given by

$$H(l^*) = \text{Trace}(ll^*) \quad (7)$$

where $l \in \mathfrak{h}$, as discussed above. Since in our case diagonal elements of $l \in \mathfrak{h}$ are zero and restricting to real matrices we have

$$H(l^*) = \sum_{i=1}^{n-1} l_{1+i} l_{ii+1}^*. \quad (8)$$

Going to a_i and b_i coordinates we get

$$H(l^*) = \sum_{i=1}^{n-1} l_{1+i} a_i. \quad (9)$$

Recall that the symplectic form is

$$\omega = \sum_{i=1}^{n-1} \frac{1}{a_i} d(a_i) \wedge d\left(\sum_{j=1}^i b_j\right).$$

Substituting

$$p_i = \sum_{j=1}^i b_j \quad (10)$$

we have $\omega = \sum_{i=1}^{n-1} \frac{1}{a_i} d(a_i) \wedge d(p_i)$. So from above

$$X_H = \sum_{i=1}^{n-1} a_i l_{i+1} \partial p_i. \quad (11)$$

Thus the polarisation is given by the fields ∂_{p_i} . Choosing the symplectic potential (or the connection form) as

$$\theta = \sum_{i=1}^{n-1} \frac{d(a_i)}{a_i} p_i$$

the polarized sections will coincide with the solutions to the following PDE

$$(d(s)(X)) + i\theta(X)s = 0$$

in which the section s is just a smooth function due to triviality of the bundle and

$$X = \sum_i y_i(a, p) \partial_{p_i}$$

while y_i are arbitrary and

$$(a, p) = (a_1, a_2, \dots, a_{n-1}, p_1, \dots, p_{n-1}).$$

Since

$$\theta(X) = 0, \quad s = s(a, p)$$

solves the equation

$$\sum_{i=1}^{n-1} y_i(a, p) \partial_{p_i} s(a, p) = 0. \quad (12)$$

This is because $X = \overline{X}$ in this case. Since ω is exact and the Toda space is contractible, we have a trivial bundle and the sections are functions and from the above, the polarised functions depend only on a_i .

6. Unitary Representation

6.1. Induced Representations

Let us recall the definition of induced representation. Suppose G is a topological group and H is a closed subgroup of G . Suppose π is a representation of H over the vector space V . Then G acts on the product $G \times V$ as follows

$$g' \cdot (g, v) = (g'g, v)$$

where g and g' are elements of G and $v \in V$.

Define on $G \times V$ the equivalence relation

$$(g, v) \sim (gh, \pi(h^{-1})(v)) \quad \text{for all } h \in H.$$

Note that this equivalence relation is invariant under the action of G . Consequently, G acts on $(G \times V)/\sim$. The latter is a vector bundle over the quotient space G/H with H as the structure group and V as the fiber. Let W be the space of sections of this vector bundle. This is the vector space underlying the induced representation $\text{Ind}_H^G \pi$. The group G acts on W as follows

$$(g \cdot \phi)(\bar{k}) = g \cdot (\phi(g^{-1}\bar{k})) \quad \text{for } g \in G, \phi \in W, \bar{k} \in G/H.$$

6.2. Unitary Representation of G on the Space Of Sections

We refer to the theory developed in [3] and [4]. When M is a coadjoint orbit of a group G and $g \in M$ from theorem 5 in [4] we have a bijection between $\mathfrak{L}_c(M, \omega)$ and \mathfrak{L}_g , (notation as in [4]). Here \mathfrak{L}_g is the set of characters from stabilizer group G_g to the unit circle at the fiber at the point g of the corresponding element $\mathfrak{l} \in \mathfrak{L}_c(M, \omega)$. Calling the character $\eta^{\mathfrak{l}}$ we get a induced representation of the whole group on the space of sections [4].

6.2.1. Prequantization

For prequantization, in the definition of induced representation we take $G = G_1$, $H = G_T$, the stabilizer subgroup of T in the dual of the Lie algebra. T is a point on the coadjoint orbit G/G_T . The representation of H is given by the character $\eta^{\mathfrak{l}}$. As mentioned in Theorem 5 in [4], the unitary representations of G defined by exponentiation of $\gamma^{\mathfrak{l}} \circ \lambda$ is $\text{ind}_G \eta^{\mathfrak{l}}$. Recall that here $\lambda(x)(T) = \langle T, x \rangle$, which, in our case, is $\text{Tr}(xT)$ where x is in the Lie algebra and T in its dual, both are given by matrices. Also $\gamma^{\mathfrak{l}}(\phi)s = (\nabla_{\zeta_\phi} + 2\pi i\phi)s$, s is a section of the prequantum line bundle and ϕ is a smooth function on the coadjoint orbit.

6.2.2. Quantization

We refer to the last paragraph of [4]. We chose our polarization \mathfrak{h} which satisfied the criterion of Kostant's polarization (see section 4). Then, the representation $\text{ind}(\eta^{\mathfrak{l}}, \mathfrak{h})$ of G corresponds to the unitary representation obtained from exponentiation of $\gamma_F^{\mathfrak{l}}(\mathfrak{h}) \circ \lambda$. (Notation as in [4]).

6.2.3. Unitary Representation

The space of polarized sections of the Toda system are functions depending on variables a_i . The action of the induced representation on the inner product is given

by

$$\int_T |g^* s(a_1 \dots a_{n-1})|^2 |h(x)|^2 d(V) = \int_T |s(a_1 \dots a_{n-1})|^2 |h(gx)|^2 g^*(d(V)) \quad (13)$$

where $g \in G_1$ and T and $d(V)$ is the orbit and its volume form respectively. In our case T is the Toda space and $d(V)$ is its volume form induced by the symplectic form and h the Hermitian structure compatible with the symplectic potential or the connection form. From [3], and [4], a Hermitian structure h is α invariant if

$$2\pi i(\alpha(s_0) - \alpha(\bar{s}_0)) = d(\log |h|^2) \quad (14)$$

where s_0 is the section defining the trivialization. In our trivialization, $\alpha(s_0) = \theta$ where θ is real and hence $|h|$ is constant.

If we take $d(V)$ to be ω^{n-1} we have a g invariant volume since ω is g invariant. But then square integrals of functions depending on a_i s will blow up since b_i s vary over a unbounded domain. This will make the inner product to blow up. So we multiply by a normalizing factor

$$d(V) = e^{-\sum_{i=1}^{n-1} p_i^2} (\omega)^{n-1} \quad (15)$$

where $p_i = \sum_{j=1}^i b_j$. The action of the group G_1 on the Toda space can be inferred from the Section 2. The variables a_i and b_i transform in following way when an element g acts on it

$$g(a_i) = \frac{L_{ii}}{L_{i+1i+1}} a_i. \quad (16)$$

For $i > 1$

$$g(b_i) = b_i + \frac{L_{ii-1}}{L_{ii}} a_{i-1} - \frac{L_{i+1i}}{L_{i+1i+1}} a_i. \quad (17)$$

For $i = 1$

$$g(b_1) = b_1 - \frac{L_{21}}{L_{22}} a_1 \quad (18)$$

where $g_{ij} = L_{ij}$. Substituting $p_i = \sum_{j=1}^i b_j$ we have

$$g(p_i) = p_i - \frac{L_{i+1i}}{L_{i+1i+1}} a_i, \quad g^*(d(V)) = e^{-\sum_{i=1}^{n-1} (p_i - \frac{L_{i+1i}}{L_{i+1i+1}} a_i)^2} \omega^{n-1}. \quad (19)$$

So the inner product (gs, gs) is given by the following expression

$$\int_0^\infty \dots \int_0^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty |s(a_1 \dots a_{n-1})|^2 e^{-\sum_{i=1}^{n-1} (p_i - \frac{L_{i+1i}}{L_{i+1i+1}} a_i)^2} \omega^{n-1}. \quad (20)$$

We first perform the p_i -integrals. Since the p_i coordinates range from $(-\infty, \infty)$, making a substitution $y_i = p_i - \frac{L_{i+1i}}{L_{i+1i+1}}a_i$ we get

$$(gs, gs) = (s, s). \quad (21)$$

This means g is a unitary representation. From last page of [4] we have this representation is the exponential of the lie algebra representation defined through connections and Hamiltonian functions described above. Since the p_i parts of the integral evaluates to constant the inner product is

$$(s, s) = c \int_0^\infty \dots \int_0^\infty |s(a_1 \dots a_{n-1})|^2 \frac{d(a_1) \dots d(a_{n-1})}{a_1 \dots a_{n-1}}. \quad (22)$$

Thus the Hilbert space of quantization, \mathfrak{H} , is the completion of smooth functions of a_i 's are such that the above integral is finite.

Remark 4. *Since lower triangular matrices L with positive diagonal entries is given by DG_1 where G_1 is the group of our representation and D scalar and since D has a trivial action on coadjoint orbit our representations can be extended to the whole group L .*

7. Coherent States

7.1. Rawnsley Coherent States

In [5] Rawnsley defined coherent states corresponding to geometric quantization of compact Kahler manifolds. We repeat the construction for our case (namely, \mathfrak{L}^2 -completion of smooth functions).

In our case the pre-Hilbert space consists of sections which are smooth functions of a_i such that the integral in the above section is finite. After completing the space w.r.t. the \mathfrak{L}^2 norm we get a Hilbert space \mathfrak{H} . By substituting $x_i = \log(a_i)$ we find that the Hilbert space lies in $\mathfrak{L}^2(\mathbb{R}^{n-1})$.

Consider the associated C^* principal bundle $L_0 \rightarrow M$ of L . Take an element $q \in L_0$ define a function Δ_q on the pre-Hilbert of space sections described above such that $\Delta_q(s).q = s(\pi q)$.

Since we have a trivial bundle the smooth sections are smooth functions and $\Delta_q(s)$ is a constant times the evaluation map of s at $\pi(q)$.

Lemma 5. *The evaluation map at a point on square integrable smooth functions in \mathfrak{L}^2 norm is continuous.*

Proof: Suppose s_n converge to s and s_n and s smooth then s_n converge point wise. Suppose not implies there exist x such that $s_n(x)$ does not converge to $s(x)$. This implies there exist a sub sequence s_{n_k} and an ϵ such that $|s(x) - s_{n_k}(x)| > \epsilon$. Now $s^{-1}(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2})$ is an open set U of positive measure. So from the above $|s - s_{n_k}|_2$ is greater than $\epsilon m(U)^{\frac{1}{2}}$ where $m(U)$ is the Lebesgue measure of U and $|\dots|_2$ is the \mathcal{L}^2 norm. This contradicts \mathcal{L}^2 convergence of s_n to s . ■

Lemma 6. *Any continuous linear functional f on a pre-Hilbert V space extends to a continuous linear functional \tilde{f} on a Hilbert space \bar{V} .*

Proof: This can be found in any standard functional analysis text book but we give a proof for the reader's convenience. Suppose we have a sequence s_n in V which converges to s in \bar{V} then $f(s_n)$ converges to $\tilde{f}(s)$ by definition. Now if we have a sequence \tilde{s}_n which converges to s where \tilde{s}_n in \bar{V} , then we can find $s_n \in V$ such that $|s_n - \tilde{s}_n|_{\bar{V}} < \frac{1}{n}$ and $|f(s_n) - \tilde{f}(\tilde{s}_n)| < \frac{1}{n}$. Since \tilde{s}_n converge to s then s_n converge to s so by definition $f(s_n)$ converges to $\tilde{f}(s)$. Since we have chosen s_n such that $\tilde{f}(\tilde{s}_n)$ and $f(s_n)$ converge to the same point we have $\tilde{f}(\tilde{s}_n)$ converges to $\tilde{f}(s)$. ■

Corollary 7. *The functional Δ_q extend to continuous linear functional $\tilde{\Delta}_q$ on the completion.*

So by Riesz representation theorem there is a section e_q such that $\tilde{\Delta}_q(s)_q = (e_q, s)_q$, where $(,)$ is the inner product on the Hilbert space. For q and q_1 in the same fiber e_q and e_{q_1} differ by a multiplication by a constant. Thus we get a map from the base to the projectivization of the Hilbert space. For every x in M the projectivized state it maps to is what is called a *Rawnsley* coherent state.

Remark 8. *Recall that the Δ_q 's are evaluation maps on the space of smooth square integrable functions (where q is the element one on the fiber above $\pi(q)$ corresponding to the global trivialization). Suppose s_n is a sequence of smooth sections (in our case functions) converging in \mathcal{L}^2 norm to s , then s_n will converge pointwise to a function s' . Because of the continuous extension of Δ_q (which is just the evaluation for the q we have chosen) we have $s'(\pi(q)) = \tilde{\Delta}_q(s)$. Now since \mathcal{L}^2 convergence implies pointwise convergence of a subsequence outside a set of zero measure, s_n converge to s' in \mathcal{L}^2 .*

Though \mathcal{L}^2 space are not evaluative we can choose the function s' as a representative, i.e., the function $s'(\pi(q)) = \tilde{\Delta}_q(s)$ where q is again the element one on the fiber above $\pi(q)$ with respect to the global trivialization, $\tilde{\Delta}_q$ is the extension of Δ_q ,

s is an element of \mathcal{L}^2 lying in completion and s' a representative of s described as above. Hence taking these special s' we get an evaluation of elements of the completion. For defining coherent states, the particular choice of q does not matter, since any two choices of q will differ by multiplication by a constant.

Since we have a trivial bundle the smooth sections are functions and from above the Hilbert space \mathfrak{H} also consists of evaluative functions so that the inner product with Rawnsley coherent vectors for the special q at x is just the evaluation of the function at x . Now take an orthonormal basis of \mathfrak{H} , ϕ_i with $i \in I$ for some index set I . Then the dot product of ϕ_i with a function f_x (denoting the coherent state which evaluates at x) is $\phi_i(x)$. So the coherent state at x is represented by the vector be $f_x = \sum_{i \in I} \phi_i(x) \phi_i$.

7.2. Coherent States from Group Actions and the Relation to Rawnsley Coherent States

Given the unitary G action on the Hilbert space of sections, states of the form $G\psi_0$, where ψ_0 is the “vacuum” state, are also called coherent states.

Consider a generic section s of the bundle L on the manifold (coadjoint orbit in our case) which is nonzero outside a set of zero measure. Let the set on which it is nonzero be called U . Let $x \in U$, then $s(x) \in L_0$ and so we can have $e_{s(x)}$. So from above we have

$$(e_{s(x)}, s_1) = \frac{s_1(x)}{s(x)}. \quad (23)$$

For any smooth section s_1 in the Hilbert space. Now take $e_{s(x)}$ and let $g \in G$ act on it. For $ge_{s(x)}$ to be a Rawnsley’s coherent vector a good guess for its value will be $ce_{gs(x)}$. So we have

$$(ge_{s(x)}, s_1) = c \frac{s_1(gx)}{s(gx)} \quad (24)$$

but by unitarity of the action

$$(ge_{s(x)}, s_1) = (e_{s(x)}, g^{-1}s_1) = \frac{g^{-1}s_1(x)}{s(x)}. \quad (25)$$

So by the above equations, for the two notions to agree, we should have

$$c \frac{s_1(gx)}{s(gx)} = \frac{g^{-1}s_1(x)}{s(x)}. \quad (26)$$

In general the coherent states obtained from group action may not be the same as Rawnsley coherent states. But there are cases when they are the same.

Let G be the group of translations acting on \mathbb{R}^{2n} with standard symplectic form. Further let $g \in G$ act by translation by w , let $x = 0$ and let $\frac{s_1}{s}$ be represented by the function $\Phi(z)$. Then it can be seen the left hand term of the above equation with $c = 1$ is $\Phi(w)$ and right hand side is $\Phi((z - w) = 0)$ which is $\Phi(w)$. So the two kinds of coherent states are same in case of the above group action.

We have the following proposition for geometric quantization of coadjoint orbits.

Proposition 9. *For Kostant-Souriau quantization of coadjoint orbits, the two notions of coherent states are the same.*

Proof: In this case the representation module is the space of sections and the representation is the induced representation of a character from stabilizer of an element of the orbit to the fiber of the orbit. In this case case the two notions of coherent states are the same as can be seen below.

By definition of induced representation, (see Section 6.1), $g^{-1} \cdot \phi(\bar{k}) = g^{-1} \cdot (g\bar{k}, v')$ $= (\bar{k}, v')$ where v' is the representative of $\phi(g\bar{k})$ at $g\bar{k}$ when the section is pulled back to $G \times V$. Now let the element x be as described above and let L_x is the fiber on which the character acts and let $s_1 = \phi$ and $\bar{k} = x$. So $v' = \frac{s_1(gx)}{s(gx)} c'$ where c' is the representative of $s(gx)$ at gx and $g^{-1} s_1(x) = v' \frac{s(x)}{c_1} = \frac{s_1(gx)}{s(gx)} \frac{c'}{c_1} s(x)$ where c_1 is the representative of $s(x)$ at x . Thus, $\frac{g^{-1} s_1(x)}{s(x)} = \frac{s_1(gx)}{s(gx)} \frac{c'}{c_1}$. In the local trivialization defined by $s(x)$, one can always take $c_1 = c' = 1$. Thus in equation (7.4) $c = 1$. ■

8. Finite Dimensional Representations

We can have finite representations of the group G_1

$$s(a_1 \dots a_{n-1}) = (a_1 \dots a_{n-1}) P_m(a_1 \dots a_{n-1}) \quad (27)$$

where P_m are polynomial in $a_i s$ of degree m . It can be checked $g(s(x)) = s(g^{-1}x)$ will keep the above space of sections invariant. But they are not square integrable. When we normalize by $e^{-\sum_{i=1}^{n-1} a_i^2}$ we may loose unitarity. So we get finite representations which are not necessarily unitary.

Remark 10. *Normalization of volume is equivalent to changing the Hermitian structure.*

Suppose a non-zero section s locally trivializes a bundle (L, α) where α is a connection in the associated line bundle L_0 . A Hermitian structure compatible with

the connection is an Hermitian metric $\langle \rangle$ such that

$$\zeta(\langle s, s \rangle) = \langle \nabla_\zeta s, s \rangle + \langle s, \nabla_\zeta s \rangle$$

and

$$\langle \nabla_\zeta s, s \rangle + \langle s, \nabla_\zeta s \rangle = 2\pi i \langle \alpha(\zeta)s, s \rangle - 2\pi i \langle s, \alpha(\zeta)s \rangle$$

where s is a section which locally trivializes the bundle.

From this we get (see Proposition 1.9.1 Kostant [3])

$$d\langle s, s \rangle = 2\pi i (\alpha(s) - \overline{\alpha(s)}).$$

In the case of a trivial bundle with s_0 the section which gives the trivialization, if the connection $\alpha(s_0)$ is real, then we have $d\langle s_0, s_0 \rangle = 0$.

Thus $\langle s_0, s_0 \rangle = c$, where c is a constant. Let us take $c = 1$.

Taking another section $s = \frac{s}{s_0} s_0$ we have

$$\langle s, s \rangle = \left| \frac{s}{s_0} \right|^2.$$

So in this setting we get a unitary character η^1 and taking the volume form to be ω^{n-1} (which is invariant under the group action), we get a unitary representation on the space of square integrable sections.

In our case the polarized sections are not square integrable with ω^{n-1} as the volume form, but we do have a representation by virtue of the fact they are sections and because of the polarization chosen. When we normalize the volume, we change the Hermitian structure. However η^1 may not be a compatible unitary character in this Hermitian structure so we may not get unitary representation in general.

But in the case where we normalized by $e^{-\sum_{i=1}^{n-1} p_i^2}$, we do get unitary representation.

9. The Quantum Hamiltonian of the Toda System

By substituting $q_i = \log(a_i)$, $i = 1, \dots, n-1$ and $p_i = \sum_{j=1}^i b_j$, $i = 1, \dots, n-1$ the Hamiltonian described in introduction is given by

$$H = 2 \sum_{i=1}^{n-1} (p_i - p_{i-1})^2 + 4 \sum_{i=1}^n e^{2q_i} \quad (28)$$

with $p_0 = 0$. Now quantizing p_i and q_i (using the formula $\hat{f} = \nabla_{X_f} + 2\pi i f$ for $f = p_i$ and q_i respectively), we get $\hat{p}_i = -i\partial q_i$ and $\hat{q}_i = q_i$. Substituting in the expression for the Hamiltonian H , we get

$$\hat{H} = -2\partial q_1^2 - 2\sum_{i=2}^{n-1} (\partial q_i - \partial q_{i-1})^2 + 4\sum_{i=1}^{n-1} e^{2q_i}. \quad (29)$$

Let $q_i = z_i - z_{i+1}$ for $i < n - 1$ and $q_{n-1} = z_{n-1}$.

Then

$$\partial_{z_i} = \partial q_i - \partial q_{i-1}$$

for $n - 1 \geq i > 1$ and

$$\partial_{z_1} = \partial q_1, \quad z_{i+1} - z_1 = -\sum_{j=1}^i q_j \text{ for } n - 1 > i > 0.$$

Further

$$\hat{H} = -2\sum_{i=1}^{n-1} (\partial_{z_i})^2 + 4\left(\sum_{i=1}^{n-2} e^{2(z_i - z_{i+1})} + e^{2z_{n-1}}\right). \quad (30)$$

As in [2], [7], the eigenfunctions can possibly be related to Whittaker functions. Also the spectrum needs to be found. This is work in progress.

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