COVERING MAPS AND IDEAL EMBEDDINGS OF COMPACT HOMOGENEOUS SPACES

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Abstract. The notion of ideal embeddings was introduced by the author at the Third Pacific Rim Geometry Conference held at Seoul in 1996. Roughly speaking, an ideal embedding is an isometrical embedding which receives the least possible amount of tension from the surrounding space at each point.

In this article, we study ideal embeddings of irreducible compact homogenous spaces in Euclidean spaces. Our main result states that if $\pi : M \to N$ is a covering map between two irreducible compact homogeneous spaces with $\lambda_1(M) \neq \lambda_1(N)$, then N does not admit an ideal embedding in a Euclidean space, although M could.

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1. Introduction

According to Nash's embedding theorem [15], every Riemannian manifold can be isometrically embedded in a Euclidean space with sufficiently large codimension. In other words, every Riemannian manifold can live in the Euclidean world if the codimension was sufficiently large.

Related to Nash's theorem, my main question raised in [4, 5] is the following.

Main Question. *Can a given Riemannian manifold live in a Euclidean world ideally?*

More precisely, can a given Riemannian manifold be isometrically embedded in a Euclidean space in such way that it receives the least possible tension from the surrounding space at each point?

It is well-known that the mean curvature vector field of a submanifold is exactly the tension field for an isometric immersion of a Riemannian manifold in another Riemannian manifold.

A major concern of my main question is whether we can determine when a Riemannian manifold can be embedded in a Euclidean space ideally. Several answers to this question had been obtained (cf. [4–6, 8, 9, 11] among others).

For a compact Riemannian manifold M, we denote the first positive eigenvalue of the Laplacian Δ_M of M by $\lambda_1(M)$.

By a *covering map* $\pi : M \to N$ between two Riemannian manifolds we mean *a covering map which is isometric*.

The following result on λ_1 was proved by Yoshiji [16].

Theorem 1. *Every non-orientable compact manifold M, except the real projective plane $\mathbb{RP}^2$, admits a Riemannian metric g for which the first eigenvalue coincides with that of its Riemannian double cover $\hat{M}$, i.e., one has $\lambda_1(M) = \lambda_1(\hat{M})$ with respect to the metric g and its covering metric $\hat{g}$ on $\hat{M}$.*

A Riemannian manifold is called a *homogeneous space* if the group of isometries of M acts transitively on M. For instance, symmetric spaces are homogenous spaces (cf. [13]).

It is well-known that every homogeneous Riemannian manifold is complete (cf. [14, Theorem 4.6]). In this article, a compact homogeneous Riemannian manifold with irreducible isotropy action is simply called an *irreducible compact homogeneous space*.

Homogeneous spaces are important. From the point of view of the *Erlangen program*, in a homogeneous space one may understand that "all points are the same". For example, Euclidean space and projective space are in natural ways homogeneous spaces for their respective symmetry groups. The same is true of the models found of non-Euclidean geometry of constant curvature, such as hyperbolic space.

In this article, we will investigate the Main Question further. In particular, by studying covering maps between irreducible homogeneous spaces, we are able to obtain a solution to the Main Question for irreducible compact homogeneous spaces.

The main result of this article is the following.

Main Theorem. *Let $\pi : M \to N$ be a covering map between two irreducible compact homogeneous spaces. If $\lambda_1(M) \neq \lambda_1(N)$, then N does not admit an ideal embedding in any Euclidean space, regardless of codimension.*

In the last section, we give a simple example to illustrate that, under the hypothesis of the Main Theorem, M may admit an ideal embedding in some Euclidean space, although N cannot.

2. δ-Invariants and a Fundamental Inequality

In order to define ideal embedding, we need to recall the notion of δ-invariants (also known as the *Chen invariants*) and the fundamental inequality of Euclidean submanifolds.

Let M be a Riemannian n-manifold. Let $K(\pi)$ denote the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For a given orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let L be a subspace of T_pM of dimension $r \geq 2$ and let $\{e_1, \ldots, e_r\}$ be an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of L by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \qquad 1 \leq \alpha, \beta \leq r.$$

Given an integer $k \geq 1$, we denote by $\mathcal{S}(n, k)$ the finite set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. We put $\mathcal{S}(n) = \cup_{k \geq 1} \mathcal{S}(n, k)$.

For each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, Chen defined the δ-*invariant* $\delta(n_1, \ldots, n_k)$ in [5, 7, 9] as

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\} \qquad (1)$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \ldots, k$. In particular, we have

$\delta(\emptyset) = \tau$ ($k = 0$, the trivial δ-invariant)

$\delta(2) = \tau - \inf K$, where K is the sectional curvature

$\delta(n-1)(p) = \max Ric(p)$, where Ric is the Ricci curvature of M.

We shall point out that the invariant $\delta(2)$ was introduced earlier in [2, 3].

The δ-curvatures are very different in nature from the "classical" scalar and Ricci curvatures simply due to the fact that both scalar and Ricci curvatures are the "total sum" of sectional curvatures on a Riemannian manifold. In contrast, the δ-curvature invariants are obtained from the scalar curvature by throwing away a certain amount of sectional curvature. For history and motivation on δ-invariants, see author's surveys [8, 9, 11].

For an isometric immersion of a Riemannian n-manifold M into the Euclidean m-space $\mathbb{E}^m$. Let h and $\vec{H}$ denote the second fundamental form and the mean curvature vector of M in $\mathbb{E}^m$, respectively. Then h and $\vec{H}$ are defined respectively by

$$h(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y \tag{2}$$

$$\vec{H} = \frac{1}{n} \, \mathrm{trace}(h) \tag{3}$$

for vector fields X and Y tangent to M, where $\tilde{\nabla}$ and ∇ denote the Levi-Civita connection on $\mathbb{E}^m$ and on M^n, respectively. We put $H = |\vec{H}|$.

The fundamental inequality for Euclidean submanifolds obtained in [5, 6, 9] is the following sharp inequality.

Theorem 2. *For any isometric immersion of a Riemannian n-manifold M into a Euclidean m-space $\mathbb{E}^m$ and for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k) H^2 \tag{4}$$

where $c(n_1, \ldots, n_k)$ is the positive real number defined by

$$c(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum n_j)}{2(n + k - \sum n_j)}. \tag{5}$$

3. Ideal Embeddings - Best Ways of Living

The fundamental inequality (4) in Theorem 2 provides us the prime control on the most important extrinsic curvature, the squared mean curvature H^2, by the δ-invariant $\delta(n_1, \ldots, n_k)$ of the Riemannian manifold M.

If we put

$$\hat{\Delta}_0(M) = \max \left\{ \Delta(n_1, \ldots, n_k); \, (n_1, \ldots, n_k) \in \mathcal{S}(n) \right\} \tag{6}$$

with

$$\Delta(n_1, \ldots, n_k) = \frac{\delta(n_1, \ldots, n_k)}{c(n_1, \ldots, n_k)} \tag{7}$$

then, for any isometric immersion $x : M \to \mathbb{E}^m$, Theorem 2 yields

$$H^2 \geq \hat{\Delta}_0(M) \tag{8}$$

at each point $p \in M$.

Inequality (8) allowed us to introduce the following notion of ideal immersions in [5, 6, 9].

Definition 3. An isometric immersion of a Riemannian n-manifold M in $\mathbb{E}^m$ is called an *ideal immersion* if it satisfies the equality case of (8) identically.

A *Maximum Principle* on ideal immersions from [5] states that if an isometric immersion $\phi : N \to \mathbb{E}^m$ of a Riemannian n-manifold M into $\mathbb{E}^m$ satisfies the equality sign of (6) for a given k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, then it is an ideal immersion automatically.

Physical Interpretation of Ideal Immersions. An isometric immersion $x : M \to \mathbb{E}^m$ is an *ideal immersion* means that the submanifold M receives the least possible amount of tension (given by $\hat{\Delta}_0(M)$) from the surrounding space at each point on M. This is due to (8) and the well-known fact that the mean curvature vector field is exactly the tension field for an isometric immersion of a Riemannian manifold in another Riemannian manifold. Thus the squared mean curvature at each point on the submanifold simply measures the amount of tension the submanifold is receiving from the surrounding space at that point.

For this reason, an ideal embedding of a Riemannian manifold M in a Euclidean space is also known as a *best way of living* of M (see [4, 5, 9]).

It was shown in [10] that some Riemannian manifolds may admit more than one ideal embedding in a Euclidean space.

A major problem on ideal embeddings is to determine whether a given Riemannian manifold admits or doesn't admit an ideal embedding in a Euclidean space.

4. Proof of the Main Theorem

Let $\pi : M \to N$ be a covering map from a compact Riemannian n-manifold onto another. Suppose that $f \in C^\infty(N)$ is an eigenvalue function of the Laplacian Δ_N associated with the first positive eigenvalue $\lambda_1(N)$ of N. Then we have

$$\Delta_M(f \circ \pi) = \lambda_1(N)f \circ \pi. \tag{9}$$

It follows from (9) that $\lambda_1(N)$ is an eigenvalue of the Laplacian Δ_M of M. Thus we have

$$\lambda_1(N) \geq \lambda_1(M). \tag{10}$$

Now, suppose that $\pi : M \to N$ is a covering map from a compact irreducible homogeneous space M onto another irreducible compact homogeneous space N and suppose that $\lambda_1(M) \neq \lambda_1(N)$. Then, by combining $\lambda_1(M) \neq \lambda_1(N)$ with

(10) we obtain

$$\lambda_1(N) > \lambda_1(M). \tag{11}$$

On the other hand, from Theorem 6.3 of [6] or Theorem 14.6 of [9] we know that the first positive eigenvalue $\lambda_1(M)$ of the irreducible homogeneous space M satisfies

$$\lambda_1(M) \geq n\hat{\Delta}_0(M). \tag{12}$$

Since M is assumed to be a compact homogeneous Riemannian manifold, $\hat{\Delta}_0(M)$ defined in (6) is a constant on M. Because the covering map $\pi : M \to N$ preserves the metric tensors g_M and g_N of M and N, i.e., $\pi^* g_N = g_M$, we know that the invariant $\hat{\Delta}_0(N)$ of N must equal $\hat{\Delta}_0(M)$ on M. Therefore, after combining (11) and (12) we obtain

$$\lambda_1(N) > n\Delta_0(N). \tag{13}$$

From Theorem 6.6 of [6] or Theorem 14.7 of my book [9] we also know that an irreducible compact homogeneous space M admits an ideal immersion into some Euclidean space if and only if it satisfies

$$\lambda_1(M) = n\hat{\Delta}_0(M). \tag{14}$$

Consequently, inequality (13) together with [9, Theorem 14.7] imply that N does not admit an ideal embedding into Euclidean spaces, regardless of codimension. This completes the proof of the Main Theorem.

5. An Example

Now, we provide a simple example to illustrate that under the hypothesis of the Main Theorem, M may admit an ideal embedding in some Euclidean space, although N cannot.

Let $\mathbb{S}^n(1)$ and $\mathbb{RP}^n(1)$ denote the n-sphere and the real projective n-space of constant sectional curvature one. It is well-known that there is a two-fold covering map $\pi : \mathbb{S}^n(1) \to \mathbb{RP}^n(1)$ which carries each pair of antipodal points on $\mathbb{S}^n(1)$ to a single point in $\mathbb{RP}^n(1)$.

Since both

$$\mathbb{S}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n) \qquad \text{and} \qquad \mathbb{RP}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n) \times \{\pm 1\}$$

are irreducible compact homogeneous spaces and moreover we have (cf. e.g. [1] and [12, page 67])

$$n = \lambda_1(\mathbb{S}^n(1)) \neq \lambda_1(\mathbb{RP}^n(1)) = 2(n+1),$$

the Main Theorem thus implies that $\mathbb{RP}^n(1)$ never admits a best way of living in any Euclidean space regardless of codimension.

On the other hand, it is easy to verify that the inclusion map of $\mathbb{S}^n(1) \subset \mathbb{E}^{n+1}$ is an ideal embedding of $\mathbb{S}^n(1)$ in $\mathbb{E}^{n+1}$.

References

[1] Berger M., Gauduchon, P. and Mazet, E., *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Mathematics vol. **194**, Springer, Berlin 1971.

[2] Chen B.-Y., *Some Pinching and Classification Theorems for Minimal Submanifolds*, Arch. Math. **60** (1993) 568–578.

[3] Chen B.-Y., *A Riemannian Invariant and its Applications to Submanifold Theory*, Results Math. **27** (1995) 17–26.

[4] Chen B.-Y., *Can One Live in a Best World Without Tension?*, Colloquium Talk at Katholieke Universiteit Leuven, Leuven 1997.

[5] Chen B.-Y., *Strings of Riemannian Invariants, Inequalities, Ideal Immersions and Their Applications*, In: The Third Pacific Rim Geometry Conference (Seoul, 1996), J. Choe (Ed), International Press, Cambridge 1998, pp. 7–60.

[6] Chen B.-Y., *Some New Obstruction to Minimal and Lagrangian Isometric Immersions*, Japan. J. Math. **26** (2000) 105–127.

[7] Chen B.-Y., *Riemannian Submanifolds*, In: Handbook of Differential Geometry, Volume I, F. Dillen and L. Verstraelen (Eds), North-Holland, Amsterdam 2000, pp. 187–418.

[8] Chen B.-Y., *δ-Invariants, Inequalities of Submanifolds and their Applications*, in: Topics in Differential Geometry, A. Mihai, I. Mihai and R. Miron (Eds), Ed. Acad. Române, Bucharest 2008, pp. 29–155.

[9] Chen B.-Y., *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, World Scientific, Hackensack 2011.

[10] Chen B.-Y., *On Ideal Hypersurfaces of Euclidean 4-Space*, Arab J. Math. Sci. **19** (2013) 129–144.

[11] Chen B.-Y., *A Tour through δ-Invariants: From Nash's Embedding Theorem to Ideal Immersions, Best Ways of Living and Beyond*, Publ. Inst. Math. **94** (2013) 67–80.

[12] Chen B.-Y., *Total Mean Curvature and Submanifolds of Finite Type*, 2$^{\text{nd}}$ Edn, World Scientific, Hackensack 2015.

[13] Helgason S., *Differential Geometry, Lie groups, and Symmetric Spaces*, Academic Press, New York 1978.

[14] Kobayashi S. and Nomizu, K., *Foundations of Differential Geometry* vol. I, Interscience Publishers, New York 1963.

[15] Nash J. , *The Imbedding Problem for Riemannian Manifolds*, Ann. Math. **63** (1956) 20–63.

[16] Yoshiji K., *The First Eigenvalues of Finite Riemannian Covers*, Tohoku Math. J. **52** (2000) 261–270.

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