



ON MATRIX REPRESENTATIONS OF GEOMETRIC (CLIFFORD) ALGEBRAS

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Abstract. Representations of geometric (Clifford) algebras with real square matrices are reviewed by providing the general theorem as well as examples of lowest dimensions. New definitions for isometry and norm are proposed. Direct and indirect isometries are identified respectively with automorphisms and antiautomorphisms of the geometric algebra, while the norm of every element is defined as the n^{th} -root of the absolute value of the determinant of its matrix representation of order n . It is deduced in which geometric algebras direct isometries are inner automorphisms (similarity transformations of matrices). Indirect isometries need reversion too. Finally, the most common isometries are reviewed in order to write them in this way.

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1. Introduction

The matrix representations of associative algebras have played a very important role in software programming, and this is not an exception for geometric algebras. Since the best way to compute rotations and other isometries in the Euclidean space is by means of geometric algebra, the technological applications of its matrix representations are obvious for programmers. The appearance of quantum mechanics and the discovery of the relativistic wave equation of the electron by Dirac also brought forth the matrix representation of the space-time geometric algebra to the foreground of the physics, as stressed by Hestenes in his recently re-edited book [12]. The development of particle physics has also enhanced the modelling of the fundamental interactions by means of groups of matrices, which are many times representations of geometric algebras. Therefore, since potentially interested readers of this paper can come from different disciplines, we will review, in Sections 2-5, some fundamental facts about matrix representations that can be trivial for those with good mathematical training. On the other hand, in [10] we have defined the norm of every element of a geometric algebra as the n^{th} -root of the determinant of its matrix representation of order n . Of course, this statement must be wrapped by a complete and consistent theory of isometries. Since we have considered only direct isometries there, we now give a more general definition of isometries for also including indirect isometries. Sections 6-8 are devoted to these new definitions and the theorems that follow from them.

2. Fundamentals of Matrix Representations

A matrix representation $M(A)$ of an associative algebra $A[\mathbb{F}]$ over a division algebra¹ \mathbb{F} is a square matrix algebra or subalgebra with entries in \mathbb{F} that is homomorphic to $A[\mathbb{F}]$, that is, for every pair of elements $a, b \in A[\mathbb{F}]$ there are two corresponding matrices $M(a), M(b)$ for which their linear combination and multiplication have the corresponding representations

$$M(\lambda a + \mu b) = \lambda M(a) + \mu M(b), \quad M(ab) = M(a)M(b), \quad \lambda, \mu \in \mathbb{F}. \quad (1)$$

Let us recall that a similarity transformation of matrices is the transformation given by

$$M' = S^{-1}MS, \quad \det M \neq 0. \quad (2)$$

Then, a matrix representation is determined up to a similarity transformation of matrices because:

$$S^{-1}M(\lambda a + \mu b)S = \lambda S^{-1}M(a)S + \mu S^{-1}M(b)S \quad (3)$$

owing to (1) and the distributive property. According to (2), the former equation can be written as

$$M'(\lambda a + \mu b) = \lambda M'(a) + \mu M'(b). \quad (4)$$

In the same way

$$S^{-1}M(ab)S = S^{-1}M(a)S S^{-1}M(b)S \quad (5)$$

which can be written as

$$M'(ab) = M'(a)M'(b). \quad (6)$$

Then, if a given set of matrices represents a set of elements of the algebra $A[\mathbb{F}]$, another set of matrices obtained from a similarity transformation represents them as well, and they are considered to be the same matrix representation. Therefore, what defines a matrix representation is not the specific matrices but their invariants under similarity transformations. One already sees that similarity transformations

¹According to Frobenius' theorem, the division associative algebras are only the real numbers \mathbb{R} , the complex numbers \mathbb{C} and quaternions \mathbb{H} or any other algebras isomorphic to them.

(2) play a very important role in the theory of matrix representations, which is enhanced by some new geometric considerations.

A matrix representation is said to be *faithful* if different elements are always represented by different matrices

$$a \neq b \quad \Rightarrow \quad M(a) \neq M(b). \quad (7)$$

The faithful representations are the only ones being of interest since the others collapse different elements into the same matrices, which is not suitable for any kind of technical application. The homomorphism becomes an isomorphism for faithful representations. Their dimensions must be high enough for representations to be faithful. In this way, the following inequality is obvious

$$M(A) \text{ faithful} \quad \Rightarrow \quad \dim A[\mathbb{F}] \leq \dim M(A) \quad (8)$$

since $M(A)$ are matrices with entries in \mathbb{F} . The most interesting representations are those being faithful and having the minimal dimension. The representation obtained from the multiplication rule of the elements of the algebra $A[\mathbb{F}]$ is called *regular*. For instance, let us deduce the regular representation of quaternions $q = a + bi + cj + dk$ ($a, b, c, d, \in \mathbb{R}$) from multiplication by the four units

$$\begin{aligned} 1 q &= a + bi + cj + dk \\ i q &= -b + ai - dj + ck \\ j q &= -c + di + aj - bk \\ k q &= -d - ci + bj + ak. \end{aligned} \quad (9)$$

Therefore, the regular representation of quaternions is [2]

$$a + bi + cj + dk = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}. \quad (10)$$

Sometimes, the regular representation is not faithful, but if the algebra has the unity element, then its regular representation is always faithful. Since every associative algebra can be embedded in an associative algebra with unity element, every associative algebra has a faithful matrix representation [2].

3. Representations of Geometric Algebras with Real Square Matrices

From now on, only matrix representations of geometric algebras with entries in \mathbb{R} will be considered, although we will also make use of some results about representations over complex numbers and quaternions.

It is well known that a geometric (Clifford) algebra $Cl(E_n)$ of a geometric vector space E_n with dimension n has dimension 2^n

$$\dim(E_n) = n \quad \Rightarrow \quad \dim Cl(E_n) = 2^n \quad (11)$$

independently of its signature ($n = p + q$). Let us indicate $Cl_n = Cl(E_n)$. Every faithful representation of a Clifford algebra with real square matrices of order k must satisfy

$$M_{k \times k}(Cl_n) \quad \text{faithful} \quad \Rightarrow \quad 2^n \leq k^2. \quad (12)$$

On the other hand, the matrix representation of geometric algebras always has order $k = 2^l$ (see theorem 1 below) so that we have

$$M_{k \times k}(Cl_n) \quad \text{faithful} \quad \Rightarrow \quad 2^n \leq 2^{2l}. \quad (13)$$

Therefore, the following inequality is satisfied

$$n \leq 2l. \quad (14)$$

For instance, we have $l = 2$ for the geometric algebra $Cl_{3,0}$, that is, the faithful matrix representation with minimal order has order $2^2 = 4$. Quaternions are the geometric algebra $Cl_{0,2}$ and a subalgebra of $Cl_{3,0}$. Their regular representation (10) also has order four.

Since the regular representations are obtained from the multiplication of the basis elements of the algebra, and $\dim(Cl_n) = 2^n$, the regular representation will be formed by matrices with this order

$$M(Cl_n) \quad \text{regular} \quad \Rightarrow \quad \text{order } M(Cl_n) = 2^n \quad (15)$$

while a faithful representation only needs that

$$M(Cl_n) \quad \text{faithful} \quad \Rightarrow \quad \text{order } M(Cl_n) \geq 2^{n/2}. \quad (16)$$

Therefore, faithful representations with order lower than that of the regular representation are usually taken.

4. Representations of Lower-Dimensional Geometric Algebras

Let us review the matrix representations of geometric algebras having the lowest dimensions.

4.1. Complex Numbers ($\mathbb{C} = Cl_{0,1}$)

Complex numbers are given by the rule of multiplication

$$(a + b i)(c + d i) = a c - b d + i (a d + b c), \quad a, b, c, d \in \mathbb{R}. \quad (17)$$

Their regular representation is obtained from

$$1 (a + b i) = a + b i, \quad i (a + b i) = -b + a i \quad (18)$$

whence

$$a + b i = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \quad (19)$$

4.2. Hyperbolic Numbers ($\mathcal{H} = Cl_{1,0}$)

Hyperbolic numbers are generated by a unit e_1 with positive square. Then, their multiplication rule is

$$(a + b e_1)(c + d e_1) = a c + b d + e_1 (a d + b c), \quad a, b, c, d \in \mathbb{R}. \quad (20)$$

Their regular representation is obtained from

$$1 (a + b e_1) = a + b e_1, \quad e_1 (a + b e_1) = b + a e_1 \quad (21)$$

whence

$$a + b e_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}. \quad (22)$$

4.3. The Geometric Algebra of the Euclidean Plane $Cl_{2,0}$

This algebra is generated by two unitary orthogonal vectors in the plane having positive square

$$e_1^2 = e_2^2 = 1, \quad e_1 e_2 = -e_2 e_1 = e_{12}, \quad Cl_{2,0} = \langle 1, e_1, e_2, e_{12} \rangle. \quad (23)$$

Its minimal faithful representation is

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (24)$$

Every multivector of $Cl_{2,0}$ is then represented by the following matrix

$$a + b e_1 + c e_2 + d e_{12} = \begin{pmatrix} a + b & c + d \\ c - d & a - b \end{pmatrix}. \quad (25)$$

$Cl_{2,0}$ has two subalgebras: the hyperbolic numbers and the complex numbers. The complex numbers are given by

$$z \in \mathbb{C} \quad \iff \quad z = a + b e_{12} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \quad (26)$$

The hyperbolic numbers are given by

$$t \in \mathcal{H} \quad \iff \quad t = a + b e_1 = \begin{pmatrix} a + b & 0 \\ 0 & a - b \end{pmatrix}. \quad (27)$$

As said above, a matrix representation is determined up to a similarity transformation. Therefore, the representation (27) is as good as (22) and is considered the same matrix representation, since it is only determined by its invariants under similarity transformation of matrices, trace and determinant

$$\text{tr}(t) = 2a, \quad \det(t) = a^2 - b^2. \quad (28)$$

The advantage of using (27) instead of (22) is the fact that (27) is a diagonal matrix, which makes computations easier.

4.4. The Geometric Algebra of the Hyperbolic Plane ($Cl_{1,1}$)

It is generated by two unit vectors with distinct square

$$e_1^2 = 1, \quad e_2^2 = -1, \quad e_1 e_2 = -e_2 e_1 = e_{12}, \quad Cl_{1,1} = \langle 1, e_1, e_2, e_{12} \rangle. \quad (29)$$

Its minimal faithful representation is

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (30)$$

Notice that the difference between (24) and (30) is only an exchange of matrices between e_2 and e_{12} . Therefore $Cl_{2,0}$ and $Cl_{1,1}$ have the same representation and are isomorphic: $Cl_{2,0} \simeq Cl_{1,1}$. The elements of the generator vector space $E = \langle e_1, e_2 \rangle$ are called hyperbolic vectors because of their Lorentzian (hyperbolic) norm

$$(a e_1 + b e_2)^2 = a^2 - b^2. \quad (31)$$

Notice that the square of their norm and their determinant are opposite

$$\det(a e_1 + b e_2) = \begin{vmatrix} a & b \\ -b & -a \end{vmatrix} = -a^2 + b^2. \quad (32)$$

This is not a casual coincidence, which will be explained later on. On the other hand, elements of the form $a + b e_{12}$ are the even subalgebra of the hyperbolic numbers already found in (22). Summarizing, the matrix algebra $M_{2 \times 2}$ is the representation of both geometric algebras of the Euclidean and hyperbolic planes, which are isomorphic. The only difference between $Cl_{2,0}$ and $Cl_{1,1}$ is which vector space is taken as generator of the geometric algebra.

4.5. Quaternions ($\mathbb{H} = Cl_{0,2}$)

It is the algebra generated by two units having a negative square (imaginary units)

$$e_1^2 = e_2^2 = -1, \quad e_1 e_2 = -e_2 e_1. \quad (33)$$

Then, their product is also an imaginary unit

$$e_{12}^2 = -e_1^2 e_2^2 = -1 \quad (34)$$

and we cannot distinguish one unit from the others. Since e_{12} anticommutes with e_1 and e_2 and

$$e_1 e_{12} = -e_2, \quad e_2 e_{12} = e_1. \quad (35)$$

We can rename the units as $i = e_1$, $j = e_2$ and $k = e_{12}$ to see that $Cl_{0,2}$ is the algebra of quaternions discovered by Hamilton [11] on October 16th, 1843

$$i^2 = j^2 = k^2 = -1 \quad (36)$$

$$ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j \quad (37)$$

$$Cl_{0,2} = \mathbb{H} = \langle 1, i, j, k \rangle. \quad (38)$$

Hamilton deduced that the norm of a quaternion $\|q\|$ is a generalization of the norm of complex numbers

$$\|a + bi + cj + dk\|^2 = a^2 + b^2 + c^2 + d^2. \quad (39)$$

Gibbs pointed out [8] that the norm of a quaternion can be obtained as the square root of the determinant of its matrix representation (10), although it is really its 4th root

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2. \quad (40)$$

This is not a simple coincidence, which caught my attention and led me to outline a new definition of norm.

4.6. The Geometric Algebra of the Three-Dimensional Euclidean Space ($Cl_{3,0}$)

Its geometric algebra $Cl_{3,0}$ is defined by

$$e_1^2 = e_2^2 = e_3^2 = 1, \quad e_i e_j = -e_j e_i, \quad i \neq j. \quad (41)$$

Then $Cl_{3,0} = \langle 1, e_1, e_2, e_3, e_{12}, e_{23}, e_{31}, e_{123} \rangle$. It is easy to see that the subalgebra of the elements of even grade (called even subalgebra) are quaternions $\mathbb{H} = \langle 1, e_{12}, e_{23}, e_{31} \rangle$. In fact, Hamilton himself [11] deduced quaternions as quotients of two vectors $q = v/w$, meaning $v w^{-1}$. However, a small detail must be taken into account when dealing with quaternions. Hamilton defined quaternions through the anticommutation rule (37). However, the multiplication of the anticommuting bivector units e_{ij} yields a somewhat different result

$$e_{23}e_{31} = -e_{12}, \quad e_{31}e_{12} = -e_{23}, \quad e_{12}e_{23} = -e_{31}. \quad (42)$$

Then, one must take into account a change of sign between Hamilton's and geometric algebra units

$$i = -e_{23}, \quad j = -e_{31}, \quad k = -e_{12}. \quad (43)$$

Therefore, a rotation through an angle θ in the plane of the unitary bivector n is written with Hamilton units as [13, p.130]

$$q' = \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} n \right) q \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} n \right), \quad q \equiv i, j, k, \quad \|n\| = 1 \quad (44)$$

while in geometric algebra

$$q' = \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} n \right) q \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} n \right), \quad q \equiv e_{ij}, \quad \|n\| = 1. \quad (45)$$

If we write $r = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} n$, the rotation can be written as

$$q' = r^{-1} q r \quad (46)$$

which resembles a similarity transformation of matrices (2). This is not by chance again as we will see later.

4.7. Pauli Matrices

Pauli matrices were introduced to explain the spin of the electron. Let us recall them

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (47)$$

Notice that

$$\sigma_i^2 = 1 \quad \text{and} \quad \sigma_i \sigma_j = -\sigma_j \sigma_i, \quad i \neq j \quad (48)$$

which shows that they are a complex representation of $Cl_{3,0}$

$$\sigma_x = e_1, \quad \sigma_y = e_2, \quad \sigma_z = e_3. \quad (49)$$

The Kronecker product by the real representation of complex numbers (26) yields

$$e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (50)$$

The quaternion units are obtained from the products of these units

$$e_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad e_{23} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad e_{31} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (51)$$

Comparison with equation (10) leads to the identification $e_{12} = i$, $e_{23} = k$ and $e_{31} = j$. Of course, one can also use $e_{12} = -k$, $e_{23} = -i$ and $e_{31} = -j$, which means a conjugation plus a rotation. Anyway, the matrix representation is the same and the precaution consists of taking a set of matrices that are consistent with the rule of geometric product.

4.8. Dirac Matrices and Spacetime Geometric Algebra

Dirac matrices γ_i were introduced by Dirac in 1928 [7] when dealing with the relativistic wave equation of the electron. They are

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (52)$$

Another matrix sometimes called γ_4 or γ_5 is introduced

$$\gamma_4 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (53)$$

Dirac matrices form a set of anticommuting units

$$\gamma_0^2 = \gamma_4^2 = 1, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \quad \gamma_i \gamma_j = -\gamma_j \gamma_i, \quad i \neq j. \quad (54)$$

The matrices $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ generate the geometric algebra $Cl_{1,3}$ and have the Lorentzian metric of the Minkowski spacetime. Majorana [16] discovered that,

after a suitable choice of the matrices, the relativistic wave equation for the electron becomes real

$$\left[\frac{1}{c} \frac{\partial}{\partial t} - (\alpha, \nabla) + \beta' \mu \right] \psi = 0 \quad (55)$$

where $\mu = 2\pi mc/h$, and α_i and β' are [7]

$$\begin{aligned} \alpha_x &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \alpha_y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \alpha_z &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \beta' &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (56)$$

Multiplication by β' on the left yields

$$\left[\beta' \frac{1}{c} \frac{\partial}{\partial t} - (\beta' \alpha, \nabla) - \mu \right] \psi = 0. \quad (57)$$

Since the correspondence between energy-momentum operators and partial derivatives is

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} \quad (58)$$

we have

$$\begin{aligned} e_0 = \beta' &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & e_1 = \beta' \alpha_x &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ e_2 = \beta' \alpha_y &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & e_3 = \beta' \alpha_z &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \end{aligned} \quad (59)$$

They are a set of anticommuting matrices

$$e_i e_j = -e_j e_i, \quad i \neq j \quad (60)$$

which have the metric of the Minkowski spacetime

$$e_0^2 = -1, \quad e_1^2 = e_2^2 = e_3^2 = 1 \quad (61)$$

and are generators of $Cl_{3,1}$. The faithful matrix representation of $Cl_{3,1}$ with the lowest dimension is the algebra of all real 4×4 matrices

$$Cl_{3,1} \simeq M_{4 \times 4}[\mathbb{R}]. \quad (62)$$

Therefore, the matrix algebra $M_{4 \times 4}[\mathbb{R}]$ is the real geometric algebra of spacetime.

5. Representation of Geometric Algebras of any Dimension

How the representations of geometric algebras with square matrices are is a well-known result stated in the following theorem that we recall

Theorem 1 (Square matrix representations of Clifford algebras [1]). *Let $Cl_{p,q}$ be a geometric algebra then*

- 1) *If $p - q \not\equiv 1 \pmod{4}$ then $Cl_{p,q}$ is a simple algebra of dimension 2^n , $n = p + q$, isomorphic with a full matrix algebra $M_{2^k \times 2^k}[\mathbb{K}]$ where $k = q - r_{q-p}$ and \mathbb{K} is a division (associative) algebra (\mathbb{R} , \mathbb{C} or \mathbb{H}), and r_i is the Radon-Hurwitz number.*
- 2) *If $p - q \equiv 1 \pmod{4}$ then $Cl_{p,q}$ is a semisimple algebra of dimension 2^n , $n = p + q$, isomorphic to $M_{2^{k-1} \times 2^{k-1}}[\mathbb{K}] \oplus M_{2^{k-1} \times 2^{k-1}}[\mathbb{K}]$ where $k = q - r_{q-p}$ and \mathbb{K} is \mathbb{R} or \mathbb{H} depending on whether $p - q \equiv 1 \pmod{8}$ or $p - q \equiv 5 \pmod{8}$. The Radon-Hurwitz number is defined by recursion as $r_{i+8} = r_i + 4$ and these initial values: $r_0 = 0$, $r_1 = 1$, $r_2 = r_3 = 2$, $r_4 = r_5 = r_6 = r_7 = 3$.*

Corollary 2 (Real square matrix representations). *The reduced faithful representation of a geometric algebra $Cl_{p,q}$ with real square matrices is a full matrix algebra $M_{2^k \times 2^k}$ or a subalgebra of this matrix algebra such that $n = p + q \leq 2k$.*

Corollary 3 (Central simple algebra). *If $p - q$ is even, $Cl_{p,q}$ is a central simple algebra. If $p - q \equiv 3 \pmod{4}$ then $Cl_{p,q}$ is a simple algebra but it is no longer central.*

Proof: For $p - q$ (or $n = p + q$) even the pseudoscalar $e_{1\dots n}$ anticommutes with all one-vectors, whence it cannot belong to the centre of $Cl_{p,q}$, and the centre is only \mathbb{R} . Then, $Cl_{p,q}$ is a central simple algebra. For $p - q$ odd, the pseudoscalar $e_{1\dots n}$ commutes with all the elements of the geometric algebra, that is, it belongs to the centre of the $Cl_{p,q}$, an algebra over \mathbb{R} . Therefore $Cl_{p,q}$ is no longer central. ■

Therefore, all the representations of geometric algebras are matrices of order 2^n with entries in \mathbb{R} , \mathbb{C} , \mathbb{H} (quaternions), ${}^2\mathbb{R}$ (hyperbolic numbers) or ${}^2\mathbb{H}$ [6, p.81], [15, pp.215-217], yielding the classification of geometric algebras given in Table 1 for $p + q < 8$. Representations of geometric algebras of higher dimension are obtained from the periodicity theorem [6, p.81], [15, pp.215-217]

$$Cl_{p+8,q} \simeq Cl_{p,q+8} \simeq Cl_{p,q} \otimes \mathbb{R}(16) \quad (63)$$

where $\mathbb{R}(16)$ represents the real square matrices of order 16. From Table 1, it is easy to obtain the real matrix representation of any geometric algebra by doing the corresponding Kronecker product with the representation of \mathbb{C} (see equation (26)) or \mathbb{H} (see equation (10)) if needed.

5.1. Isomorphisms of Clifford Algebras

As Table 1 shows, some geometric algebras have the same matrix representation because they are isomorphic. The isomorphisms of Clifford algebras were studied by Cartan [3, p.464], and Lounesto reviewed them in [15]. They are $Cl_{p,q} \simeq Cl_{q+1,p-1}$ if $p \geq 1$, $Cl_{p,q} \simeq Cl_{p-4,q+4}$ if $p \geq 4$ and $Cl_{p,q} \otimes Cl_{1,1} \simeq Cl_{p+1,q+1}$.

6. New Definitions and Theorems About Isometries

A main question in the theory of transformations among elements of geometric algebra is what we understand by *isometry*. An intuitive definition is that an *isometry* is a transformation preserving the norm (length) of one-vectors. However, if we regard complex numbers or quaternions we will say that an *isometry* is a transformation preserving the norm, but then we will define their norm as the square root of the product by its conjugate. Then, the concept of isometry is linked to the concept of *norm*, which is not trivial except for some particular cases. A point of view from matrix representation can be an advantage since it allows us to give general definitions of all these concepts. Let us see some of them

Table 1. $\mathbb{F}(d)$ means the real algebra of square matrices $M_{d \times d}[\mathbb{F}]$ with entries in the ring $\mathbb{R}, \mathbb{C}, \mathbb{H}, {}^2\mathbb{R}$ or ${}^2\mathbb{H}$. E.g., ${}^2\mathbb{R}$ are diagonal matrices with real entries, which are the matrix representation of hyperbolic numbers, and ${}^2\mathbb{H}$ are diagonal matrices having quaternions as entries.

Matrix representations of Clifford algebras $Cl_{p,q}$ [15].

$q \backslash p$	0	1	2	3	4	5	6	7
0	\mathbb{R}	${}^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
1	\mathbb{C}	$\mathbb{R}(2)$	${}^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	
2	\mathbb{H}	$\mathbb{C}(2)$	$\mathbb{R}(4)$	${}^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$		
3	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$			
4	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$				
5	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$					
6	$\mathbb{R}(8)$	$\mathbb{C}(8)$						
7	${}^2\mathbb{R}(8)$							

Definition 4 (Complete geometric algebra). A complete geometric algebra is a geometric algebra whose representation is the full real matrix algebra with real entries of a given order.

Since the dimension of a geometric algebra $Cl_{p,q}$ is 2^n , where $n = p + q$, and its real matrix representation is the matrix algebra of order 2^k or some of its subalgebras, if $Cl_{p,q}$ is a complete geometric algebra then

$$2^n = (2^k)^2 \quad \Rightarrow \quad k = \frac{n}{2}. \tag{64}$$

Therefore, the complete geometric algebras are represented by real matrix algebras of order $2^{n/2}$. Examples of complete geometric algebras are $Cl_{2,0} \simeq Cl_{1,1} \simeq M_{2 \times 2}(\mathbb{R})$ or $Cl_{3,1} \simeq Cl_{2,2} \simeq M_{4 \times 4}(\mathbb{R})$.

Definition 5 (Norm). *The norm $\|a\|$ of every element a of a geometric algebra is the n^{th} -root of the absolute value of the determinant of its matrix representation $M(a)$ of order n*

$$\|a\| = \sqrt[n]{|\det(M_{n \times n}(a))|}. \quad (65)$$

This extends what was appraised by Gibbs [8] for quaternions (40). As an example of the usefulness of this definition, let us calculate the norm of a bivector w of the space-time algebra $Cl_{3,1}$ from Majorana's representation (59) [9, p.12]

$$\begin{aligned} w &= a e_{01} + b e_{02} + c e_{03} + d e_{23} + f e_{31} + g e_{12}, \quad a, b, c, d, f, g \in \mathbb{R} \\ \det(w) &= (a^2 + b^2 + c^2 - d^2 - e^2 - f^2)^2 + 4(a f + b g + c h)^2 \quad (66) \\ \|w\| &= \sqrt[4]{(a^2 + b^2 + c^2 - d^2 - e^2 - f^2)^2 + 4(a f + b g + c h)^2}. \end{aligned}$$

If w is the electromagnetic field (E being the electric field, B the magnetic field and c is the speed of light)

$$w = E_x e_{01} + E_y e_{02} + E_z e_{03} + c (B_x e_{23} + B_y e_{31} + B_z e_{12}) \quad (67)$$

it follows that

$$a^2 + b^2 + c^2 - d^2 - e^2 - f^2 = E_x^2 + E_y^2 + E_z^2 - c^2(B_x^2 + B_y^2 + B_z^2) \quad (68)$$

is the true invariant of the electromagnetic field and

$$a f + b g + c h = c(E_x B_x + E_y B_y + E_z B_z) \quad (69)$$

is its pseudoinvariant, which is preserved under the Lorentz transformations but changes the sign under inversion of the coordinate axes [14, p.172].

Corollary 6. *The norm of a product of elements of a geometric algebra is equal to the product of the norms of each one of them, i.e.,*

$$\|a b c \dots\| = \|a\| \|b\| \|c\| \dots \quad a, b, c \in Cl_{p,q}. \quad (70)$$

Therefore, we have $\|a^n\| = \|a\|^n$. Since an isometry must preserve the norm, a plausible definition of isometry can be the following one.

Definition 7 (Isometries). *Isometries are defined as those linear transformations T that commute with the power of every element of the geometric algebra, that is*

$$T(a^n) = [T(a)]^n, \quad T(\lambda a + \mu b) = \lambda T(a) + \mu T(b), \quad a, b \in Cl_{p,q}, \quad \lambda, \mu \in \mathbb{R}.$$

Theorem 8. *Isometries preserve the characteristic polynomial of the matrix representation of every element of geometric algebra.*

Proof: Let $p(x)$ be the characteristic polynomial of a matrix A

$$p(x) = \det(A - x \text{Id}) = \sum_{i=1}^k a_i x^i. \quad (71)$$

According to the Hamilton-Cayley theorem, the substitution of the matrix A into its own characteristic polynomial $p(x)$ yields zero

$$p(A) = \det(A - A \text{Id}) = \sum_{i=1}^k a_i A^i = 0, \quad a_i \in \mathbb{R}. \quad (72)$$

If we apply an isometry to this equality we have

$$T(p(A)) = T\left(\sum_{i=1}^k a_i A^i\right) = \sum_{i=1}^k a_i [T(A)]^i = 0 \quad (73)$$

that is, $T(A)$ is also a root of $p(x)$, or equivalently $p(x)$ is also the characteristic polynomial of $T(A)$. ■

Corollary 9. *Isometries preserve the norm of all the elements of a geometric algebra.*

Proof: Since isometries preserve the characteristic polynomial of their matrix representation, and the determinant is its independent term, it follows that the determinant and the norm are preserved. ■

Corollary 10. *If T is an isometry of a geometric algebra then $\|a\| = \sqrt{\|a T(a)\|}$.*

This is an immediate consequence of the Corollary 9. Examples are the norm of a complex number or a quaternion. Since the conjugate q^* is an isometry of q and $q q^* > 0$ then we have

$$q \in \mathbb{H} \quad \|q^*\| = \|q\| \quad \Rightarrow \quad \|q\| = \sqrt{q q^*}. \quad (74)$$

Another example is the norm of a space-time bivector (66). By doing the square of the bivector w we obtain

$$w^2 = a^2 + b^2 + c^2 - d^2 - e^2 - f^2 + 2(a d + b f + c g)e_{0123} \quad (75)$$

where one takes into account that $e_{01}^2 = e_{02}^2 = e_{03}^2 = 1$ and $e_{23}^2 = e_{31}^2 = e_{12}^2 = -1$. Since $e_{0123}^2 = -1$, w^2 is a complex number whose norm is well known. Then, according to (26), $\det(w^2)$ is just given by the determinant (66), whence $\|w\|$ follows.

Corollary 11. *Isometries leave invariant the identity and real numbers.*

Proof: It is a consequence of the application of the Definition 7 of isometry to a power of the identity Id

$$\begin{aligned} \text{Id}^2 = \text{Id} \Rightarrow T(\text{Id}^2) = T(\text{Id}) &\iff [T(\text{Id})]^2 = T(\text{Id})\text{Id} \\ &\iff T(\text{Id}) = \text{Id}. \end{aligned} \quad (76)$$

Notice that the last simplification needs the existence of $[T(\text{Id})]^{-1}$, that is, it needs that $\det[T(\text{Id})] \neq 0$, which is satisfied because $\det[T(\text{Id})] = \det(\text{Id}) = 1$. The last simplification could not be carried on for idempotents, whose determinant is null. Then idempotents can be changed by isometries. By linearity we see that isometries also preserve real numbers

$$T(\lambda \text{Id}) = \lambda T(\text{Id}) = \lambda \text{Id}, \quad \lambda \in \mathbb{R}. \quad (77)$$

■

Definition 7 implies that $T(a^2) = [T(a)]^2$, but what happens with $T(a b)$? We have two options giving two definitions.

Definition 12 (Direct isometries). *Direct isometries T of a geometric algebra $Cl_{p,q}$ are defined as its automorphisms*

$$T(a b) = T(a)T(b), \quad T(\lambda a + \mu b) = \lambda T(a) + \mu T(b), \quad a, b \in Cl_{p,q}, \quad \lambda, \mu \in \mathbb{R}$$

Definition 13 (Indirect isometries). *Indirect isometries T of a geometric algebra $Cl_{p,q}$ are defined as its antiautomorphisms*

$$T(ab) = T(b)T(a), \quad T(\lambda a + \mu b) = \lambda T(a) + \mu T(b), \quad a, b \in Cl_{p,q}, \quad \lambda, \mu \in \mathbb{R}.$$

Definition 14 (Units). *A unit of a geometric algebra is a matrix with square equal to $\pm \text{Id}$ (the identity) obtained either from Kronecker products of the four units of $Cl_{2,0} \simeq Cl_{1,1}$ or from isometries of these Kronecker products.*

Let us recall the four units (24) or (30)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (78)$$

For instance, a unit of the geometric algebra $Cl_{3,1} = Cl_{2,0} \otimes Cl_{2,0}$ would be

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (79)$$

However, the following matrix, despite having as square $\pm \text{Id}$, will not be considered a unit

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (80)$$

because it cannot be obtained as Kronecker products of (78). How units look like are clearly shown in all the matrices (not only γ_i) considered by Dirac [7]. Despite being complex, they also fit the definition 14. Here we only consider real matrices. On the other hand, it is not needed to have only 0 and 1 as entries. For instance, an isometric matrix of the second unit of (78) must have the same trace $\text{Tr}M=0$ and determinant $\det M = -1$, and it can be

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta - \cos \theta \end{pmatrix} = \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sin \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (81)$$

It is, in fact, a linear combination of the second and four units in (78), which are isometric.

Theorem 15 (Pseudo-Pythagorean). *Let $\{u_i\}$ be a set of anticommuting units. Then they satisfy*

$$\left\| \sum_i \lambda_i u_i \right\| = \sqrt{\left| \sum_i \lambda_i^2 \chi_i \right|}, \quad \lambda_i \in \mathbb{R}, \quad \chi_i = u_i^2 = \pm 1. \quad (82)$$

If all the units have the same square, it becomes the Pythagorean theorem.

Proof: Let us calculate the square of a linear combination of anticommuting units $a = \sum_i \lambda_i u_i$

$$a^2 = \left(\sum_i \lambda_i u_i \right)^2 = \sum_i \lambda_i^2 u_i^2 = \sum_i \lambda_i^2 \chi_i \in \mathbb{R} \quad (83)$$

since crossed terms cancel owing to the anticommutativity of the units. Then, the determinant and the norm are

$$\det(a^2) = (\det(a))^2 = \left(\sum_i \lambda_i^2 \chi_i \text{Id} \right)^n \Rightarrow \|a\| = \sqrt{\left| \sum_i \lambda_i^2 \chi_i \right|}. \quad (84)$$

The absolute value arises because n (the order of the matrix representation of a geometric algebra) is always an even number. ■

Notice that the only supposition is that they are anticommuting units, but there are no considerations about whether they are generators or have the same grade.

Theorem 16. *Isometries transform a set of anticommuting units into another set of anticommuting units.*

Proof: Let $\{u_k\}$ be a set of anticommuting units. Then, for each pair u_i and u_j of them, we have

$$i \neq j \quad u_i u_j + u_j u_i = 0 \quad \iff \quad T(u_i)T(u_j) + T(u_j)T(u_i) = 0. \quad (85)$$

This equality is satisfied for $T(ab) = T(a)T(b)$ as well as for $T(ab) = T(b)T(a)$. Then $\{T(u_i)\}$ is a set of orthogonal units. ■

Notice again that no consideration about grades is present, and only anticommutativity is taken into account.

Corollary 17. *Those isometries preserving grades transform orthogonal vectors into orthogonal vectors, since they are anticommuting.*

Definition 18. *Two elements are said to be isometric if one can be obtained from the other through any isometry.*

Proposition 19. *Isometric units always have the same square.*

Proof: Denoting by $\chi_i = u_i^2 = \pm 1$ as above, we have

$$u_j = T(u_i) \quad \Rightarrow \quad u_j^2 = \chi_j = T(u_i^2) = \chi_i \quad (86)$$

since isometries preserve real numbers according to the Corollary 11. ■

Proposition 20 (Equivalence). *To be isometric is an equivalence relation (indicated with \sim), and two isometric elements are equivalent.*

Proof: Let us check the three properties of equivalence relations

1. Reflexive property. Each element is isometric to itself by the identity.
2. Symmetric property. If $b = T(a)$ then $a = T^{-1}(b)$. Let us verify that T^{-1} is also an isometry.

$$[T^{-1}(T(a))]^n = a^n = T^{-1}(T(a^n)) = T^{-1}([T(a)]^n). \quad (87)$$

3. Transitive property. If $b = T(a)$ and $c = S(b)$ then $c = S(T(b))$. Let us check that $S \circ T$ is also an isometry

$$(S \circ T)(a^n) = S(T(a^n)) = S([T(a)]^n) = [S(T(a))]^n = [(S \circ T)(a)]^n. \quad (88)$$

The linearity in properties 2 and 3 is also easily checked. ■

Theorem 21. *A set of anticommuting units that have the same square are equivalent.*

Proof: Let u_1 and u_2 be two anticommuting units that have the same square. Then

$$u_2^2 = u_1^2 \quad \Rightarrow \quad u_2 = u_1 u_1 u_2^{-1} = u_1 u_{12} \chi \quad (89)$$

where $\chi = u_1^2 = u_2^2 = \pm 1$. The fact that both units have the same square implies that $u_{12}^2 = -1$, that is, u_{12} behaves as the imaginary unit i . Let us separate two cases depending on the value of χ

1) If $\chi = +1$ then we have

$$\begin{aligned} u_2 = u_1 u_{12} &= u_1 \left(\cos \frac{\pi}{4} + u_{12} \sin \frac{\pi}{4} \right)^2 \\ &= \left(\cos \frac{\pi}{4} - u_{12} \sin \frac{\pi}{4} \right) u_1 \left(\cos \frac{\pi}{4} + u_{12} \sin \frac{\pi}{4} \right) = t^{-1} u_1 t. \end{aligned} \quad (90)$$

2) If $\chi = -1$ then we have

$$\begin{aligned} u_2 = -u_1 u_{12} &= u_1 \left(\cos \frac{3\pi}{4} + u_{12} \sin \frac{3\pi}{4} \right)^2 \\ &= \left(\cos \frac{3\pi}{4} - u_{12} \sin \frac{3\pi}{4} \right) u_1 \left(\cos \frac{3\pi}{4} + u_{12} \sin \frac{3\pi}{4} \right) = t'^{-1} u_1 t'. \end{aligned} \quad (91)$$

In both cases, each unit is obtained from the other by means of a similarity transformation, which is a direct isometry. In fact, these isometries are rotations in the Euclidean plane u_{12} through an angle $\pm\pi/2$. Owing to the transitivity of the equivalence relation, this result also applies to a set of any number of anticommuting units whenever they have the same square. Then, in $Cl_{p,q}$, all the p units of square $+1$ are equivalent among them, and so are the q units of square -1 . ■

7. Direct Isometries

Many direct isometries such as rotations or axial symmetries can be written as a similarity transformation of matrices

$$T(v) = t^{-1} v t, \quad \det(t) \neq 0. \quad (92)$$

This operator is general and can be applied to every element of a geometric algebra. But, can all direct isometries be written as similarity transformation of matrices? The Skolem-Noether theorem gives us the answer.

Theorem 22 (Skolem-Noether [17]). *Let R, S be finite dimensional algebras, R simple and S central simple. If $f, g : R \rightarrow S$ are homomorphisms then there is an element $s \in S$ such that, for all $r \in R$, $g(r) = s^{-1} f(r) s$.*

Corollary 23. *Every automorphism of a central simple algebra is an inner automorphism.*

Proof: If $R = S$, then f, g are automorphisms. Take f as the identity, then for every automorphism $g(r)$, there exists $s \in S$ such that $g(r) = s^{-1} r s$. ■

Theorem 24. *In the geometric algebras $Cl_{p,q}$ with $n = p + q$ even (which include complete geometric algebras), every direct isometry can be written as a similarity transformation of matrices*

$$\left. \begin{array}{l} (p+q) \bmod 2 = 0 \\ a, b \in Cl_{p,q} \quad T(ab) = T(a)T(b) \\ \lambda, \mu \in \mathbb{R} \quad T(\lambda a + \mu b) = \lambda T(a) + \mu T(b) \end{array} \right\} \Rightarrow T(a) = t^{-1} a t. \quad (93)$$

Proof: According to Corollary 3, geometric algebras $Cl_{p,q}$ with $p+q$ even are central simple algebras, where the Skolem-Noether Theorem 22 and Corollary 23 applies to. Therefore, every direct isometry (automorphism) is an inner automorphism, that is, a similarity transformation of matrices. ■

7.1. Axial Symmetries

Under axial symmetries, the component of a vector with the direction of the given axis is preserved, while the components orthogonal to this axis are reversed. Their operator is simply the direction vector d of the axis

$$v' = d^{-1} v d = \frac{d v d}{d^2}. \quad (94)$$

In the particular case that v is a one-vector, it can be resolved into proportional and perpendicular components, and then

$$v' = d^{-1} v d = \frac{d (v_{\parallel} + v_{\perp}) d}{d^2} = v_{\parallel} - v_{\perp}. \quad (95)$$

7.2. Reflections

A reflection in a $(n-1)$ -hyperplane of $Cl_{p,q}$ ($n = p+q$) preserves all the components except the one being perpendicular to the hyperplane. If e_n is taken as the direction perpendicular to the $(n-1)$ -hyperplane, the operator should then be $e_{1\dots n-1}$

$$v' = e_{1\dots n-1}^{-1} v e_{1\dots n-1}. \quad (96)$$

If v is a one-vector, we can resolve it into coplanar and perpendicular components v_{\parallel} and v_{\perp} respectively

$$v' = e_{1\dots n-1}^{-1} (v_{\parallel} + v_{\perp}) e_{1\dots n-1}. \quad (97)$$

Let us suppose that n is an even number. There are $n - 1$ permutations of v_{\perp} with e_i in order to exchange v_{\perp} and $e_{1\dots n-1}$, which yields a change of sign since $n - 1$ is odd

$$v_{\perp} e_{1\dots n-1} = -e_{1\dots n-1} v_{\perp}. \quad (98)$$

On the other hand, the coplanar component v_{\parallel} can be resolved into components of the vectors in the hyperplane

$$v_{\parallel} = \sum_{i=1}^{n-1} \lambda_i e_i, \quad \lambda_i \in \mathbb{R} \quad (99)$$

which commute with $e_{1\dots n-1}$

$$v_{\parallel} e_{1\dots n-1} = \sum_{i=1}^{n-1} \lambda_i e_i e_{1\dots n-1} = e_{1\dots n-1} \sum_{i=1}^{n-1} \lambda_i e_i = e_{1\dots n-1} v_{\parallel} \quad (100)$$

since one e_i is the same, and there are only an even number $n - 2$ of vectors in $e_{1\dots n-1}$ with which e_i anticommutes. Gathering both results we have

$$v' = e_{1\dots n-1}^{-1} (v_{\parallel} + v_{\perp}) e_{1\dots n-1} = v_{\parallel} - v_{\perp}, \quad n \bmod 2 = 0 \quad (101)$$

which is the sought result for a reflection in a hyperplane perpendicular to the e_n direction. If the hyperplane is perpendicular to any given direction d , then the reflection operator is $e_{1\dots n} d$ and

$$v' = (e_{1\dots n} d)^{-1} v e_{1\dots n} d, \quad n \bmod 2 = 0. \quad (102)$$

Therefore, reflections can always be written as similarity transformations in geometric algebras having an even number of generators, including complete geometric algebras. Reflections cannot be written as similarity transformations in a geometric algebra with an odd number of generators, which includes the important case of Cl_3 , the geometric algebra of the Euclidean space. In fact, by the same reasoning one obtains

$$e_{1\dots n-1}^{-1} (v_{\parallel} + v_{\perp}) e_{1\dots n-1} = -v_{\parallel} + v_{\perp}, \quad n \bmod 2 = 1. \quad (103)$$

This means that, in order to describe reflections, different operators must be applied to elements with different grade

$$a' = (-1)^k e_{1\dots n-1}^{-1} \langle a \rangle_k e_{1\dots n-1}, \quad n \bmod 2 = 1 \quad (104)$$

raising the trouble of knowing which is the reflection operator for elements with mixed grade, such as spinors. This clearly shows the incompleteness of geometric algebras of odd dimension n such as, for instance, Cl_3 . The general operator for reflections that allows to write reflections in a plane of the Euclidean space as a similarity transformation is a real matrix that has the same order 4 as the matrix representation of Cl_3 , but it is simply not included in Cl_3 . Programmers have an immediate application here. They are using the same matrix algebra, and must only take the suitable matrix for the reflection operator. They are not exiting the matrix algebra although they are going out of Cl_3 , but this is not a trouble for the program, which is really working in $M_{4 \times 4}(\mathbb{R}) \simeq Cl_{3,1}$.

7.3. Rotations

Rotations are isometries that change the components in a plane of the geometric algebra, not necessarily of two vectors. Let u_1 and u_2 be two anticommuting units of the geometric algebra. Then, rotations in the plane u_{12} through an angle ξ are given by

$$v' = \exp\left(-\frac{\xi u_{12}}{2}\right) v \exp\frac{\xi u_{12}}{2}, \quad \xi \in \mathbb{R}. \quad (105)$$

The operator $\exp(\xi u_{12}/2)$ is usually called a *rotor*. Now, we have two cases:

1) $u_1^2 = u_2^2$. Since $u_{12}^2 = -1$, the exponential is complex and the plane is Euclidean

$$v' = \left(\cos\frac{\xi}{2} - u_{12} \sin\frac{\xi}{2}\right) v \left(\cos\frac{\xi}{2} + u_{12} \sin\frac{\xi}{2}\right) \quad (106)$$

v' is a periodic function of the circular angle ξ with period 4π instead of 2π ! This is a well-known fact in quantum mechanics [4, p. 167], [18, p.11]. The true identity is only achieved after a rotation through 4π . Although an exact mathematical identity is obtained for $\xi = 2\pi$, this is not the case for neighbouring values and therefore there is no identity in quantum mechanics owing to the uncertainty principle.

2) $u_1^2 = -u_2^2$. Since $u_{12}^2 = 1$, the exponential and the plane are hyperbolic

$$v' = \left(\cosh\frac{\psi}{2} - u_{12} \sinh\frac{\psi}{2}\right) v \left(\cosh\frac{\psi}{2} + u_{12} \sinh\frac{\psi}{2}\right). \quad (107)$$

If space-time is considered, the hyperbolic rotation is called *Lorentz transformation*, and the rotation operator is called a *boost*. The relation between the argument of the hyperbolic rotation ψ and the relative velocity V of both inertial frames is given by

$$\tanh \psi = \frac{V}{c} \quad (108)$$

where c is the speed of light.

The rotation (105) leaves invariant all the components orthogonal to the plane u_{12} , i.e., if the unit u_3 anticommutes with u_1 and u_2 then commutes with u_{12} and does not change under rotation: $u'_3 = u_3$. This also applies to every linear combination of units orthogonal to the plane u_{12} .

7.4. Duality

Exterior and geometric algebras are graded algebras, whose elements of grade k called k -multivectors are obtained from successive exterior products of elements of the generator space E , called *geometric vectors* or simply one-vectors. Duality is the inversion of grades. The dual of a k -multivector is defined as the $(n - k)$ -multivector obtained from multiplication by the pseudoscalar $e_{1\dots n}$

$$v' = v e_{1\dots n}. \quad (109)$$

This is the so called *Hodge duality*. However, this expression is not a similarity transformation, and then another expression must be sought. In order to split the Hodge duality operator into two parts like (90), one needs the pseudoscalar $e_{1\dots n}$ to anticommute with all the e_i , the generator units of the geometric algebra

$$e_1 e_{1\dots n} = -e_{1\dots n} e_i \quad (110)$$

and to have negative square

$$e_{1\dots n}^2 = -1. \quad (111)$$

When is this last condition satisfied for $Cl_{p,q}$? Let us calculate the square of the pseudoscalar

$$e_{12\dots n}^2 = (-1)^{n-1+n-2+\dots+1} e_1^2 e_2^2 \dots e_n^2 = (-1)^{\frac{n(n-1)}{2}+q} = (-1)^{\frac{n^2+q-p}{2}} \quad (112)$$

where $n = p+q$ as usual. The anticommutation (110) of the pseudoscalar with one-vectors only takes place for geometric algebras with an even number of generators $n = 2k$ because there is then an odd number $n - 1$ of permutations (e_i repeated once). In this case

$$n = 2k \quad \Rightarrow \quad e_{12\dots n}^2 = (-1)^{\frac{q-p}{2}}. \quad (113)$$

Then, the pseudoscalar has a negative square if $p - q = 2 \pmod{4}$ [10]. Complete geometric algebras always have an even number of generators, but sometimes they do not satisfy the condition $p - q = 2 \pmod{4}$. In this way let us see the following isomorphism theorem

Theorem 25 (Binary isomorphism). $Cl_{p,q} \simeq Cl_{p+1,q-1}$ if $p - q = 0 \pmod{4}$ and $q \geq 1$.

Proof: If $q - p = 0 \pmod{4}$, $e_{12\dots n}^2 = 1$ and $n = p + q$ is even. Since the pseudoscalar $e_{12\dots n}$ anticommutes with e_i for n even, it can be renamed as $e'_n = e_{12\dots n}$ and taken as generator. If $e_i^2 = 1$, there is no change of signature and one obtains the same algebra. If $e_i^2 = -1$ then $\{e_1, e_2, \dots, e'_n\}$ generate $Cl_{p+1,q-1}$, and $e'_{12\dots n} = (-1)^q e_n$ is the new pseudoscalar with negative square. ■

Therefore, each Clifford algebra with $p - q = 0 \pmod{4}$ is isomorphic to another Clifford algebra with $p - q = 2 \pmod{4}$. This isomorphism applies to all the Clifford algebras with $n = p + q$ even. Some examples are $Cl_{1,1} \simeq Cl_{2,0}$, $Cl_{2,2} \simeq Cl_{3,1}$, $Cl_{3,3} \simeq Cl_{4,2}$, $Cl_{0,4} \simeq Cl_{1,3}$, $Cl_{1,5} \simeq Cl_{2,4}$, etc. An example are Dirac's matrices γ_i , generators of $Cl_{1,3} \simeq Cl_{0,4}$. Now, the role that γ_4 (53) plays with regard to the set of generators $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ becomes clear. It is the pseudoscalar of the algebra multiplied by i that can exchange its role and also act as generator.

Complete geometric algebras always have an even number of generator units, and the binary isomorphism (Theorem 25) assures us that they have a Clifford structure with $p - q = 2 \pmod{4}$ in which the pseudoscalar is an imaginary unit. Let us suppose that this is the case. Then, we can write the Hodge duality for a vector v as

$$\begin{aligned} p - q = 2 \pmod{4} \quad \Rightarrow \quad v' &= v e_{1\dots n} = v \frac{(1 + e_{1\dots n})^2}{2} \\ &= \frac{1 - e_{1\dots n}}{\sqrt{2}} v \frac{1 + e_{1\dots n}}{\sqrt{2}} \end{aligned} \quad (114)$$

where de Moivre's identity $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^2$ has been applied considering $e_{1\dots n} = i$ as the imaginary unit. Therefore, duality can be written as a similarity transformation

$$v' = t^{-1} v t \quad t = \frac{1 + e_{1\dots n}}{\sqrt{2}} \quad \text{or simply} \quad t = 1 + e_{1\dots n} \quad (115)$$

for $p - q = 2 \pmod{4}$, which includes all the complete geometric algebras if a suitable choice of generators is taken. This case includes the important case of the space-time algebra $Cl_{3,1}$.

Warning. The operator (115) acts exactly as Hodge's dual for grades $1, \dots, n-1$ but it leaves scalars and pseudoscalars invariant. For instance, we have

$$e'_1 = e_{2\dots n}. \quad (116)$$

Then we expect that

$$1' = (e'_1)^2 = (e_1)^2 = e_{2\dots n}^2 = 1. \quad (117)$$

This is ensured by a similarity transformation. Then, it is not right and makes no sense to say that the dual of scalars are pseudoscalars and vice versa. Notice that if instead of $\{e_i\}$ you take as generators of the complete geometric algebra their dual units $\{e_{1\dots i-1, i+1, \dots n}\}$, their product generates the same pseudoscalar. Then, duality does not change scalars and pseudoscalars as commonly argued in the literature, and its right operator is (115).

Let us now see whether the three main involutions can be written as similarity transformations of matrices.

7.5. Grade Involution

It consist of changing the direction of one-vectors. For geometric algebras with an even number of generator units, the pseudoscalar anticommutes with vectors, which suffices to state the grade involution operator

$$v' = (e_{1\dots n})^{-1} v e_{1\dots n} = -v, \quad n \bmod 2 = 0. \quad (118)$$

As usual, the similarity transformation operator applies to all the elements of geometric algebra. Then two-vectors do not change, three-vectors change the sign and so on. The grade involution is clearly a direct isometry.

8. Indirect Isometries

Let us recall the Definition 13 of indirect isometries: Indirect isometries are the antiautomorphisms of a geometric algebra, which change the order of the factors when applied to a product of elements. An example of antiautomorphism (and involution) of geometric algebras is reversion.

8.1. Reversion

Reversion R is the transformation that changes the order of the basis vectors in any given multivector [5]

$$R(e_{i\dots k}) = e_{k\dots i}. \quad (119)$$

The number of permutations of adjacent vectors that is necessary to reverse the unit $e_{1\dots k}$ is $k(k-1)/2$

$$e_{1\dots k} = (-1)^{k-1+k-2+\dots+1} e_{k\dots 1} = (-1)^{\frac{k(k-1)}{2}} e_{k\dots 1}. \quad (120)$$

Depending on the value of $k \pmod 4$ we have the following cases

$$k \pmod 4 = \begin{cases} 0, 1 & \Rightarrow \frac{k(k-1)}{2} \text{ even} \Rightarrow R\langle e \rangle_k = \langle e \rangle_k \\ 2, 3 & \Rightarrow \frac{k(k-1)}{2} \text{ odd} \Rightarrow R\langle e \rangle_k = -\langle e \rangle_k \end{cases} \quad (121)$$

where $\langle e \rangle_k$ indicates not only $e_{1\dots k}$ but any other unit of grade k . Then scalars and one-vectors do not change the sign, two-vectors and three-vectors change the sign and so on. Reversion is an involutory antiautomorphism

$$a_i \in Cl_{p,q}, \quad \begin{cases} R(a_1 \cdots a_n) = R(a_n) \cdots R(a_1) \\ R(R(a_1 \cdots a_n)) = a_1 \cdots a_n. \end{cases} \quad (122)$$

If a_i are one-vectors then reversion simply changes their order

$$u_i \in \langle Cl_{p,q} \rangle_1, \quad R(u_1 \cdots u_n) = u_n \cdots u_1 \quad (123)$$

which is a consequence of (121) owing to the distributive property of the geometric product. Let us see the following theorem about indirect isometries.

Theorem 26 (Inner antiautomorphism). *Any indirect isometry of a geometric algebra $Cl_{p,q}$ with $n = p + q$ even (including complete geometric algebras) is the composition of reversion with an inner automorphism (similarity transformation of matrices).*

Proof: Let S be an indirect isometry, R reversion and $T = S \circ R$. Then we have

$$\begin{aligned} S(uv) = S(v)S(u) &\Rightarrow T(uv) = S(R(uv)) = S(R(v)R(u)) \\ &= S(R(u))S(R(v)) = T(u)T(v). \end{aligned} \quad (124)$$

On the other hand, S and R are linear and therefore T is also linear. In consequence, T is an automorphism and a direct isometry. Since R is an involution, we have $T \circ R = S \circ R \circ R = S$. According to Theorem 24 every direct isometry T can be written as a similarity transformation of matrices for $n = p + q$ even, so that we finally prove that S is the composition of reversion with an inner automorphism

$$\left. \begin{array}{l} u, v \in Cl_{p,q}, \quad S(uv) = S(v)S(u) \\ (p+q) \pmod{2} = 0 \end{array} \right\} \Rightarrow S(u) = t^{-1}R(u)t. \quad (125)$$

■

Let us review some indirect isometries now.

8.2. Clifford Conjugation

Clifford conjugation is the composition of reversion and grade involution. Therefore, Clifford conjugation is an involutory antiautomorphism and an indirect isometry

$$u' = (e_{1\dots n})^{-1}R(u)e_{1\dots n}. \quad (126)$$

Chisholm and Farwell [5] described how the signs of multivectors change for reversion, grade involution and Clifford conjugation. Their Table 27.1.3 is summarized in Table 2.

8.3. Transposition or Hermitian Conjugation

Transposition (usually denoted with T) is the transformation consisting of exchanging rows and columns in the real matrix representation of every element of a geometric algebra. Transposition is an antiautomorphism because every pair of matrices A and B satisfy $(AB)^T = B^T A^T$. Let us see that complex numbers become conjugate under transposition

Table 2. Below the sign “+” means that $\langle e \rangle_k$ is preserved, and the “-” means that its sign is changed under the involution: $\langle e \rangle'_k = -\langle e \rangle_k$.

Sign changes of $\langle e \rangle_k$ under the main involutions of Clifford algebras $Cl_{p,q}$.

$0 \leq k \leq n = p + q$	Reversion	Grade involution	Clifford conjugation
$k = 0 \pmod 4$	+	+	+
$k = 1 \pmod 4$	+	-	-
$k = 2 \pmod 4$	-	+	-
$k = 3 \pmod 4$	-	-	+

$$(a + b i)^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a - b i = (a + b i)^*. \quad (127)$$

The Hermitian conjugate (usually represented with \dagger) of a matrix with complex entries is defined as the transpose matrix with complex conjugate entries. If a complex matrix is expanded by means of a Kronecker product with the real representation of complex numbers given above, the Hermitian conjugate corresponds to the transposition of the expanded real matrix. Let us see how units change under transposition.

Theorem 27 (Symmetry of unit matrices). *The real representations of units with positive or negative square are respectively symmetric or skew-symmetric matrices.*

Proof: 1) Case $u^2=1$. Let u be a unit having positive square, and let v be an eigenvector (column matrix) of u with eigenvalue λ . Then

$$\left. \begin{array}{l} u^2 = \text{Id} \\ u v = \lambda v \end{array} \right\} \Rightarrow u^2 v = v = \lambda^2 v \Rightarrow \lambda^2 = 1. \quad (128)$$

Multiplying $u v$ by its transpose we have

$$v^T u^T u v = v^T \lambda^2 v = v^T v \Rightarrow u^T u = \text{Id}. \quad (129)$$

But u is a unit such that $u^2 = \text{Id}$. Then

$$u^T u = \text{Id} = u^2 \Rightarrow u^T = u^2 u^{-1} = u. \quad (130)$$

That is, the real representation of a unit having positive square is a symmetric matrix

$$u^2 = \text{Id} \quad \Rightarrow \quad u^T = u. \quad (131)$$

2) Case $u^2 = -1$. Let us now suppose that the unit u has negative square, and let v be an eigenvector (column matrix) of u with eigenvalue λ

$$\left. \begin{array}{l} u^2 = -\text{Id} \\ u v = \lambda v \end{array} \right\} \Rightarrow u^2 v = -v = \lambda^2 v \Rightarrow \lambda^2 = -1. \quad (132)$$

Since the eigenvalues are imaginary, $\lambda = \pm i$, the invariant subspaces for v have dimension 2. Let v_1 and v_2 be two associated real vectors that are the real and imaginary components of the complex eigenvectors of two conjugate eigenvalues

$$u(v_1 + iv_2) = i(v_1 + iv_2), \quad u(v_1 - iv_2) = -i(v_1 - iv_2). \quad (133)$$

Both equations are resolved into two equations involving only real numbers

$$u v_1 = -v_2, \quad u v_2 = v_1. \quad (134)$$

Multiplying $u v_1$ by its transpose we have

$$v_1^T u^T u v_1 = v_2^T v_2 = c v_1^T v_1, \quad c \in \mathbb{R}^+ \quad (135)$$

$v_1^T v_1$ and $v_2^T v_2$ are real positive numbers because they are sums of squares. Their quotient is the positive constant c . Comparison of the left and right sides of this equation implies

$$u^T u = c \text{Id} \quad \Rightarrow \quad u^T = -c u \quad (136)$$

because $u^{-1} = -u$ for unities with negative square. The fact that the determinants of a matrix and its transpose are equal implies $c = 1$

$$\det(u^T) = c^k \det u \quad \Rightarrow \quad c = 1, \quad u^T = -u \quad (137)$$

where we take into account that the order k of the matrix representation of u is always even. Therefore, the real representation of units with negative square are skew-symmetric matrices

$$u^2 = -\text{Id} \quad \Rightarrow \quad u^T = -u. \quad (138)$$

■

In fact, the Hermitian conjugation has already been defined from its action under the units of a geometric algebra [5]

$$\begin{aligned} e_{i\dots k}^\dagger = e_{i\dots k} &\iff e_{i\dots k}^2 = \text{Id} \\ e_{i\dots k}^\dagger = -e_{i\dots k} &\iff e_{i\dots k}^2 = -\text{Id}. \end{aligned} \quad (139)$$

The operator of Hermitian conjugation must discriminate units with positive square from units with negative square, and it seems that in $Cl_{p,q}$ the unit $e_{1\dots p}$ could be the suitable operator. Let us see how it is exchanged with vector units

$$\begin{aligned} p \text{ odd} &\Rightarrow \begin{cases} e_i e_{1\dots p} = e_{1\dots p} e_i & \text{if } i \leq p \\ e_i e_{1\dots p} = -e_{1\dots p} e_i & \text{if } i > p \end{cases} \\ p \text{ even} &\Rightarrow \begin{cases} e_i e_{1\dots p} = -e_{1\dots p} e_i & \text{if } i \leq p \\ e_i e_{1\dots p} = e_{1\dots p} e_i & \text{if } i > p \end{cases} \end{aligned} \quad (140)$$

where $1 \leq i \leq n = p + q$. On the other hand, we also have

$$\begin{aligned} q \text{ odd} &\Rightarrow \begin{cases} e_i e_{p+1\dots p+q} = -e_{p+1\dots p+q} e_i & \text{if } i \leq p \\ e_i e_{p+1\dots p+q} = e_{p+1\dots p+q} e_i & \text{if } i > p \end{cases} \\ q \text{ even} &\Rightarrow \begin{cases} e_i e_{p+1\dots p+q} = e_{p+1\dots p+q} e_i & \text{if } i \leq p \\ e_i e_{p+1\dots p+q} = -e_{p+1\dots p+q} e_i & \text{if } i > p. \end{cases} \end{aligned} \quad (141)$$

The suitable anticommutation with unit vectors having negative square (Hermitian conjugation) is only possible for p odd or q even. All geometric algebras admit one of both possibilities. If p is even and q is odd, then the isomorphism $Cl_{p,q} \simeq Cl_{q+1,p-1}$ (valid for all geometric algebras [15]) changes the parity of p and q . Then, the general operator of Hermitian conjugation is

$$a \in Cl_{p,q} \begin{cases} p \text{ odd} & a^\dagger = (e_{1\dots p})^{-1} R(a) e_{1\dots p} \\ q \text{ even} & a^\dagger = (e_{p+1\dots p+q})^{-1} R(a) e_{p+1\dots p+q}. \end{cases} \quad (142)$$

Of course, reversion must also be applied since Hermitian conjugation is the same as transposition of real matrices, which is an antiautomorphism. In the important case of space-time geometric algebra $Cl_{3,1}$, Hermitian conjugation is obtained through the operator e_{123}

$$a^\dagger = (e_{123})^{-1} R(a) e_{123}, \quad v \in Cl_{3,1}. \quad (143)$$

For instance

$$e_0^\dagger = (e_{123})^{-1} e_0 e_{123} = - (e_{123})^{-1} e_{123} e_0 = -e_0 \quad (144)$$

$$e_{123}^\dagger = (e_{123})^{-1} R(e_{123}) e_{123} = (e_{123})^{-1} e_{321} e_{123} = (e_{123})^{-1} = -e_{123} \quad (145)$$

since $e_{123}^2 = -1$.

From the Hermitian conjugation (142), we can isolate reversion to get a useful formula for reversion

$$v \in Cl_{p,q} \begin{cases} p & \text{odd} & R(v) = e_{1\dots p} v^\dagger (e_{1\dots p})^{-1} \\ q & \text{even} & R(v) = e_{p+1\dots p+q} v^\dagger (e_{p+1\dots p+q})^{-1}. \end{cases} \quad (146)$$

The equalities (146) yield a fast algorithm for computing reversion with matrix representation. If the matrices are real, the Hermitian conjugation consists of the very easy algorithm of taking the transpose matrix. Then, after multiplication by the matrix representation of the product of the units with positive or negative square depending on the case, we have reversed the matrix very quickly. For instance, in $Cl_{3,1} \simeq M_{4 \times 4}$, using the Majorana representation (59) we have for every element represented by a matrix A

$$R(A) = e_{123} A^T (e_{123})^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A^T \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (147)$$

without resolving A into multivectors.

Since every square matrix can be resolved into an addition of a symmetric matrix and a skew-symmetric matrix in a unique way, the linear space of square matrices of order m is the direct sum of the linear space of skew-symmetric matrices of the same order, which has dimension $m(m-1)/2$, and the linear space of symmetric matrices, which has dimension $m(m+1)/2$. Therefore, the total number w of units with positive and negative square in the basis of a complete geometric algebra $Cl_{p,q}$ is

$$w_+ = 2^{\frac{n}{2}-1} (2^{\frac{n}{2}} + 1), \quad w_- = 2^{\frac{n}{2}-1} (2^{\frac{n}{2}} - 1), \quad n = p + q \quad (148)$$

because $m = 2^{\frac{n}{2}}$. For instance $Cl_{3,1} \simeq M_{4 \times 4}(\mathbb{R})$ has $n = 4$, $w_+ = 10$ and $w_- = 6$.

9. Conclusions

Matrix representations of geometric algebras carry additional information that cannot be obtained from their pure algebraic definition. We have defined isometries of a geometric algebra as those linear transformations that commute with the power of every element $T(a^n) = [T(a)]^n$. Since isometries preserve the characteristic polynomial, whose independent term is the determinant, the norm of every element of a geometric algebra is defined as the n^{th} -root of the absolute value of the determinant of its real square matrix representation of order n . Direct isometries of a geometric algebra are then defined as its automorphisms, while indirect isometries are defined as its antiautomorphisms. Since the geometric algebras $Cl_{p,q}$ with $p + q$ even are central simple algebras, their direct isometries are inner automorphisms (according to the Skolem-Noether theorem) and can be written as similarity transformations of matrices $T(a) = t^{-1} a t$, while their indirect isometries can be written as the composition of reversion R with similarity transformations of matrices. In this way, rotations, reflections, axial symmetries, duality, Clifford conjugation, transposition and Hermitian conjugation are reviewed and written in the canonical forms either $T(a) = t^{-1} a t$ or $S(a) = t^{-1} R(a) t$. These canonical forms of isometries are general and can be applied to every element of a geometric algebra, with either homogeneous or heterogeneous grade, which has not only theoretical significance but also practical usefulness since it simplifies algorithms for programmers. On the other hand, it is proven that units with square $+1$ are a basis of symmetric matrices, while units with square -1 are a basis of skew-symmetric matrices. That is, the signature of the Clifford algebra is closely related to the symmetry of the matrix representation.

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