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KÄHLERIAN STRUCTURES AND $\mathcal{D}$-HOMOTHETIC BI-WARPING

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Abstract. We introduce the notion of $\mathcal{D}$-homothetic bi-warping and starting from a Sasakian manifold M, we construct a family of Kählerian structures on the product $\mathbb{R} \times M$. After, we investigate conditions on the product of a cosymplectic or Kenmotsu manifold and the real line to be a family of conformal Kähler manifolds. We construct several examples.

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1. Introduction

To study manifolds with negative curvature, Bishop and O'Neill introduced the notion of warped product as a generalization of Riemannian product [1].

In 1985, using the warped product, Oubiña showed that there is a one-to-one correspondence between Sasakian and Kählerian structures [14].

Recently, building on the work of Tanno [16] (the homothetic deformation on contact metric manifold), Blair [6] introduced the notion of $\mathcal{D}$-homothetic warping. He used it for generating further results and examples of various structures. In particular, he showed in another way that there is a one-to-one correspondence between Sasakian and Kählerian structures.

Here by generalizing the $\mathcal{D}$-homothetic warping and following what made Blair in [6], we exceed this correspondence and we show that every Sasakian manifold M generates a one-parameter family of Kählerian manifolds, thereby generalizing the results of Oubiña [14] and Blair [6]. On the other hand, we define a two-parameter family of conformally Kähler manifolds structures on the product manifold $\mathbb{I} \times M$ of an open interval and a cosymplectic or Kenmotsu manifold M (Theorem 7), which is the first main result of the present paper. This text is organized in the following way.

Section 2 is devoted to the background of the structures which will be used in the sequel.

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In Section 3 we introduce the notion of $\mathcal{D}$-homothetic bi-warping and prove some basic properties.

Finally in Section 4 we give an application to some questions of the characterization of certain geometric structures with examples.

2. Review Of Needed Notions

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (M, J, g) we thus have

$$J^2 = -1, \qquad g(JX, JY) = g(X, Y).$$

An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes, with

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y].$$

For an almost Hermitian manifold (M, J, g), we define the fundamental Kähler form Ω as

$$\Omega(X, Y) = g(X, JY).$$

The triple (M, J, g) is then called almost Kähler if Ω is closed, i.e., $\mathrm{d}\Omega = 0$. An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $\mathrm{d}\Omega = 0$ and $N_J = 0$. One can prove that both these conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

Definition 1 ([13]). *A Hermitian manifold (M, J, g) is called locally conformal Kähler (conformally Kähler) manifold if there exists a closed (exact) one-form θ (called the Lee form) such that*

$$\mathrm{d}\Omega = \theta \wedge \Omega.$$

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ tensor field φ, a vector field ξ (called the structure vector field) and a one-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (1)$$

for any vector fields X,Y on M. In particular, in an almost contact metric manifold we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a contact metric manifold if $\mathrm{d}\eta = \Phi$, where $\Phi(X,Y) = g(X,\varphi Y)$ is called the fundamental two-form of M.

On the other hand, the almost contact metric structure of M is said to be normal if

$$N_{\varphi}(X,Y) = [\varphi,\varphi](X,Y) + 2\mathrm{d}\eta\,(X,Y)\xi = 0 \tag{2}$$

for any X, Y, where $[\varphi,\varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y].$$

An almost contact metric structures (φ,ξ,η,g) on M is said to be

$$\begin{array}{l} a)\ \text{Sasaki} \;\Leftrightarrow\; \Phi = \mathrm{d}\eta \text{ and } (\varphi,\xi,\eta) \text{ is normal} \\ b)\ \text{Cosymplectic} \;\Leftrightarrow\; \mathrm{d}\Phi = \mathrm{d}\eta = 0 \text{ and } (\varphi,\xi,\eta) \text{ is normal} \\ c)\ \text{Kenmotsu} \;\Leftrightarrow\; \mathrm{d}\eta = 0,\ \mathrm{d}\Phi = 2\Phi\wedge\eta \text{ and } (\varphi,\xi,\eta) \text{ is normal} \end{array} \tag{3}$$

where d denotes the exterior derivative. These manifolds can be characterized through their Levi-Civita connection, by requiring

$$\begin{array}{l} 1)\ \text{Sasaki} \;\Leftrightarrow\; (\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X \\ 2)\ \text{Cosymplectic} \;\Leftrightarrow\; \nabla\varphi = 0 \\ 3)\ \text{Kenmotsu} \;\Leftrightarrow\; (\nabla_X\varphi)Y = g(\varphi X,Y)\xi - \eta(Y)\varphi X \end{array} \tag{4}$$

(see [2], [3], [9] and [17]).

3. $\mathcal{D}$-Homothetic Bi-Warping

Let (M,φ,ξ,η,g) be an almost contact metric manifold with $\dim M = 2n+1$. The equation $\eta = 0$ defines a $2n$-dimensional distribution $\mathcal{D}$ on M. By an $2n$-homothetic deformation or $\mathcal{D}$-homothetic deformation [16] we mean a change of structure tensors of the form

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = a\eta, \qquad \overline{\xi} = \frac{1}{a}\xi, \qquad \overline{g} = ag + a(a-1)\eta\otimes\eta$$

where a is a positive constant. If (M,φ,ξ,η,g) is a contact metric structure with contact form η, then $(\overline{M},\overline{\varphi},\overline{\xi},\overline{\eta},\overline{g})$ is also a contact metric structure [16].

The idea works equally well for almost contact metric structures, the deformation

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = \lambda\eta, \qquad \overline{\xi} = \frac{1}{\lambda}\xi, \qquad \overline{g} = \alpha^2 g + \beta^2\eta\otimes\eta$$

is again an almost contact metric structure if $\lambda^2 = \alpha^2 + \beta^2$.

Putting $\alpha^2 = a^2$ and $\beta^2 = a^2(b^2-1)$ where $\lambda = ab \neq 0$, we get the deformation

$$\overline{\varphi} = \varphi, \qquad \overline{\eta} = ab\eta, \qquad \overline{\xi} = \frac{1}{ab}\xi, \qquad \overline{g} = a^2 g + a^2(b^2-1)\eta\otimes\eta.$$

Definition 2. *Let (M', g') be a Riemannian manifold and let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold, and f, h be two smooth functions on M'. The $\mathcal{D}$-homothetically bi-warped metric on $\tilde{M} = M' \times M$ is defined by*

$$\tilde{g} = g' + f^2 g + f^2(h^2 - 1)\eta \otimes \eta$$

where $fh \neq 0$ everywhere.

In particular, if $h = \pm 1$ then we have a warped product metric and if $h = \pm f$ we get the $\mathcal{D}$-homothetically warped metric [6].

Using the Koszul formula for the Levi-Civita connection of a Riemannian metric, one can obtain the following

Proposition 3. *Let ∇', ∇ and $\tilde{\nabla}$ denote the Riemannian connections of $g', g,$ and $\tilde{g}$ respectively. For all X', Y' vector fields on M' and independent of M and X, Y vector fields on M, we have the relations*

$$\begin{aligned}
\tilde{\nabla}_{X'} Y' &= \nabla'_{X'} Y' \\
\tilde{g}(\tilde{\nabla}_{X'} Y, Z) &= \tilde{g}(\tilde{\nabla}_Y X', Z) = -\tilde{g}(\tilde{\nabla}_Y Z, X') \\
&= fX'(f) g(Y, Z) + f\Big((h^2 - 1)X'(f) + fhX'(h)\Big)\eta(Y)\eta(Z) \\
\tilde{g}(\tilde{\nabla}_X Y, Z) &= \tilde{g}(\nabla_X Y, Z) + f^2(h^2 - 1)\Big(\frac{1}{2}\big(g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\big)\eta(Z) \\
&\quad + \mathrm{d}\eta(X, Z)\eta(Y) + \mathrm{d}\eta(Y, Z)\eta(X)\Big).
\end{aligned}$$

Let σ denotes the second fundamental form of M in $M' \times M$ and while $f,\ h$ are two functions on M', for emphasis we denote their gradients by $\mathrm{grad}' f$ and $\mathrm{grad}' h$ respectively. Then we have the following Theorem.

Theorem 4. *For an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ and a $\mathcal{D}$-homothetically bi-warped metric on $\tilde{M} = M' \times M$ we have the following assertions*: 1) *M' is a totally geodesic submanifold.* 2) *If* $\mathrm{gard} f\ \mathrm{grad}'(h - f) = 0$ *then M is a quasi-umbilical submanifold and its second fundamental form is given by*

$$\sigma(X, Y) = -\frac{1}{2}\Big(g(X, Y) + (h^2 + fh - 1)\eta(X)\eta(Y)\Big)\mathrm{grad}' f^2.$$

3) *The mean curvature vector of M in $M' \times M$ is*

$$\mathcal{H} = -\mathrm{grad}'\Big(\frac{(2n + h^2)f^2}{2(2n + 1)}\Big).$$

4) *M is minimal if and only if*

$$h^2 = \frac{c}{f^2} - 2n$$

where $c > 0$ in which case M is quasi-umbilical and its second fundamental form is given by

$$\sigma(X,Y) = \frac{1}{2}\Big(g(X,Y) - (2n+1)\eta(X)\eta(Y)\Big)\mathrm{grad}' f^2.$$

5) *If* $\mathrm{d}\eta(\xi, X) = 0$ *for every X on M (equivalently the integral curves of ξ are geodesics), then the Reeb vector field ξ is $\tilde{g}$-Killing if and only if it is g-Killing.*

Proof: Recall that any submanifold N in $\tilde{M}$ is a quasi-umbilical submanifold if its second fundamental form ω has the following form

$$\omega(X,Y) = \alpha g(X,Y)\rho' + \beta\eta(X)\eta(Y)\rho'$$

where α, β are two scalars, X, Y are two vectors fields on N and ρ' is a normal vectors field.

- If $\alpha = 0$ then N is cylindrical.
- If $\beta = 0$ then N is umbilical.
- If $\alpha = \beta = 0$ then N is geodesic.

1. Let σ' denotes the second fundamental form of M'. Since we have $\tilde{\nabla}_{X'}Y' = \nabla'_{X'}Y'$ then

$$\sigma' = \tilde{\nabla}_{X'}Y' - \nabla'_{X'}Y' = 0.$$

2. From Proposition 3 we have

$$\begin{aligned}\tilde{g}\big(\tilde{\nabla}_X Y, Z'\big) &= -fZ'(f)g(X,Y) - f\Big((h^2-1)Z'(f) + fhZ'(h)\Big)\eta(X)\eta(Y) \\ &= -fg'\Big(g(X,Y)\mathrm{grad}' f + \big((h^2-1)\mathrm{grad}' f \\ &\quad + fh\ \mathrm{grad}' h\big)\eta(X)\eta(Y), Z'\Big)\end{aligned}$$

since $\tilde{g}\big(\nabla_X Y, Z'\big) = 0$ and knowing that $\sigma = \tilde{\nabla}_X Y - \nabla_X Y$ one ends with

$$\sigma(X,Y) = -\frac{1}{2}g(X,Y)\mathrm{grad}' f^2 - \frac{1}{2}\big((h^2-1)\mathrm{grad}' f^2 + f^2\ \mathrm{grad}' h^2\big)\eta(X)\eta(Y). \quad (*)$$

If grad $'h = \mathrm{grad}' f$ then we obtain

$$\sigma(X,Y) = -f\big(g(X,Y) + (h^2 + fh - 1)\eta(X)\eta(Y)\big)\mathrm{grad}' f.$$

3. Knowing that the mean curvature vector of M in $M' \times M$ is given by

$$\mathcal{H} = \frac{1}{2n+1}\mathrm{tr}_g\, \sigma = \frac{1}{2n+1}\sum_{i=1}^{2n+1}\sigma(e_i, e_i)$$

where $\{e_i\}_{i=1,2n+1}$ is an orthonormal basis on M so

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2n+1}\sum_{i=1}^{i=2n+1}\sigma(e_i, e_i)\\
&= -\frac{f}{2n+1}\sum_{i=1}^{i=2n+1}\Big((2n+1)\mathrm{grad}' f + \big((h^2-1)\mathrm{grad}' f + fh\ \mathrm{grad}' h\big)\Big)\\
&= -\frac{1}{2(2n+1)}\mathrm{grad}'\big((2n+h^2)f^2\big).
\end{aligned}$$

4. The submanifold M is minimal, i.e., the mean curvature $\mathcal{H}$ is zero, using the result (3), we get

$$h^2 = \frac{c}{f^2} - 2n.$$

Now replacing $\mathrm{grad}' h^2 = -\frac{c}{f^4}\mathrm{grad}' f^2$ in $(*)$ we find

$$\sigma(X, Y) = \frac{1}{2}\Big(g(X,Y) - (2n+1)\eta(X)\eta(Y)\Big)\mathrm{grad}' f^2.$$

5. For every two vectors fields $\tilde{X} = X' + X$ and $\tilde{Y} = Y' + Y$ on $\tilde{M}$ we have that

$$\xi \quad \text{is} \quad \tilde{g} - \text{Killing} \Leftrightarrow \tilde{g}(\tilde{\nabla}_{\tilde{X}}\xi, \tilde{Y}) + \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\xi, \tilde{X}) = 0$$

and

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_{\tilde{X}}\xi, \tilde{Y}) + \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\xi, \tilde{X}) &= \tilde{g}(\tilde{\nabla}_{X'+X}\xi, Y'+Y) + \tilde{g}(\tilde{\nabla}_{Y'+Y}\xi, X'+X)\\
&= \tilde{g}(\tilde{\nabla}_{X'}\xi, Y) + \tilde{g}(\tilde{\nabla}_{X}\xi, Y') + \tilde{g}(\tilde{\nabla}_{X}\xi, Y) \qquad (**)\\
&\quad + \tilde{g}(\tilde{\nabla}_{Y'}\xi, X) + \tilde{g}(\tilde{\nabla}_{Y}\xi, X') + \tilde{g}(\tilde{\nabla}_{Y}\xi, X).
\end{aligned}$$

Suppose $\mathrm{d}\eta(\xi, X) = 0$, i.e., $\xi\eta(X) = \eta\big(\nabla_\xi X\big)$ then, we can easily check the following assertions

$$\tilde{g}(\tilde{\nabla}_{X'}\xi, Y) = \frac{1}{2}X'(f^2h^2)\eta(Y)$$

$$\tilde{g}(\tilde{\nabla}_X\xi, Y') = -\frac{1}{2}Y'(f^2h^2)\eta(X)$$

$$\tilde{g}(\tilde{\nabla}_X\xi, Y) = \tilde{g}(\nabla_X\xi, Y) + f^2(h^2-1)\mathrm{d}\eta(X,Y).$$

When replacing in $(**)$ we obtain also

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_{\tilde{X}}\xi, \tilde{Y}) + \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\xi, \tilde{X}) &= \tilde{g}(\nabla_X\xi, Y) + \tilde{g}(\nabla_Y\xi, X)\\ &= f^2\big(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\big).\end{aligned}$$

This completes the proof. ■

4. Application to Geometric Structures

In the remaining part of the paper, we consider the case where $M' = \mathbb{R}$, M is an almost contact metric manifolds and the metric

$$\tilde{g} = \mathrm{d}t^2 + f^2g + f^2(h^2-1)\eta\otimes\eta \tag{5}$$

where f, h are functions on $\mathbb{R}$. For brevity we denote the unit tangent field to $\mathbb{R}$ by ∂_t. In this case the proposition (3) becomes

Proposition 5. *Let (M,φ,ξ,η,g) be an almost contact metric manifold. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of $g,$ and $\tilde{g}$ respectively. For all X, Y vector fields tangent to M and independent of $\mathbb{R}$, we have*

$$\tilde{g}(\tilde{\nabla}_{\partial_t}Y, Z) = \tilde{g}(\tilde{\nabla}_Y\partial_t, Z) = -\tilde{g}(\tilde{\nabla}_Y Z, \partial_t) = ff'g(\varphi Y, \varphi Z) + fh(fh)'\eta(Y)\eta(Z)$$

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_X Y, Z) &= \tilde{g}(\nabla_X Y, Z) + f^2(h^2-1)\Big(\frac{1}{2}\big(g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\big)\eta(Z)\\ &\quad + \mathrm{d}\eta(X,Z)\eta(Y) + \mathrm{d}\eta(Y,Z)\eta(X)\Big).\end{aligned}$$

Next, we introduce a class of almost complex structure $\tilde{J}$ on manifold $\tilde{M}$

$$\tilde{J}(\partial_t, X) = \Big(fh\eta(X)\partial_t\ ,\ \varphi X - \frac{1}{fh}\xi\Big) \tag{6}$$

for any vector filds X of M where $f, h,$ are functions on $\mathbb{R}$ and $fh \neq 0$ everywhere. That $J^2 = -I$ is easily checked and for all $\tilde{X} = (a\partial_t, X), \tilde{Y} = (b\partial_t, Y)$ on $\tilde{M}$ we can see that $\tilde{g}$ is almost Hermitian with respect to $\tilde{J}$, i.e.,

$$\tilde{g}\big(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}\big) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

On the other hand, the fundamental two-form $\tilde{\Omega}$ of $(\tilde{J}, \tilde{g})$ is

$$\tilde{\Omega}\Big(\big(a\frac{\partial}{\partial t}, X\big), \big(b\frac{\partial}{\partial t}, Y\big)\Big) = \tilde{g}\Big(\big(a\frac{\partial}{\partial t}, X\big), \tilde{J}\big(b\frac{\partial}{\partial t}, Y\big)\Big)$$

we can check that is very simply as follows

$$\tilde{\Omega} = f(2h\ \mathrm{d}t \wedge \eta + f\Phi) \tag{7}$$

we have immediately that

$$\mathrm{d}\tilde{\Omega} = f(-2h\ \mathrm{d}t \wedge \mathrm{d}\eta + 2f'\mathrm{d}t \wedge \Phi + f\mathrm{d}\Phi). \tag{8}$$

For the special cases we have the following

$$\begin{array}{ll} 1) & \text{contact metric} \quad \mathrm{d}\tilde{\Omega} = -2f(h - f')\mathrm{d}t \wedge \Phi \\ 2) & \text{almost cosymplectic} \quad \mathrm{d}\tilde{\Omega} = 2ff'\mathrm{d}t \wedge \Phi \\ 3) & \text{almost Kenmotsu} \quad \mathrm{d}\tilde{\Omega} = 2f(f'\mathrm{d}t + f\eta) \wedge \Phi. \end{array} \tag{9}$$

We note that $\tilde{\Omega}$ is closed in the contact metric case if and only if $h = f'$ and in the almost cosymplectic case if and only if f is constant. In the Kenmotsu case cannot be closed; it would force f to be zero.

Now, putting $h = f'$, the structure $(\tilde{g}, \tilde{J})$ (see (5), (6)) becomes

$$\tilde{g} \ = \ \mathrm{d}t^2 + f^2 g + f^2(f'^2 - 1)\eta \otimes \eta \tag{10}$$

$$\tilde{J}(a\frac{\partial}{\partial t}, X) \ = \ \Big(ff'\eta(X)\frac{\partial}{\partial t}\ ,\ \varphi X - \frac{a}{ff'}\xi\Big) \tag{11}$$

where $ff' \neq 0$ on M everywhere, and X any vector field of M.

We denote by $N_{\tilde{J}}$ the Nijenhuis tensor of the almost complex structure $\tilde{J}$. Then from (11) we have

$$N_{\tilde{J}}\big((0, X), (0, Y)\big) \ = \ \Big(ff'N^{(2)}_{\varphi}(X, Y)\frac{\partial}{\partial t}\ ,\ N^{(1)}_{\varphi}(X, Y)\Big)$$

$$N_{\tilde{J}}\Big(\big(\frac{\partial}{\partial t}, 0\big), (0, X)\Big) \ = \ \Big(N^{(4)}_{\varphi}(X)\frac{\partial}{\partial t}, \frac{1}{ff'}N^{(3)}_{\varphi}(X)\Big)$$

for any vector fields X, Y of M. We denote by $N^{(1)}_{\varphi}, N^{(2)}_{\varphi}, N^{(3)}_{\varphi}$ and $N^{(4)}_{\varphi}$ the following tensor fields on M defined respectively by

$$\begin{aligned} N^{(1)}_{\varphi}(X, Y) \ &= \ [\varphi, \varphi](X, Y) + 2\mathrm{d}\eta(X, Y)\xi \\ N^{(2)}_{\varphi}(X) \ &= \ \big(L_{\varphi X}\big)(Y) - \big(L_{\varphi_Y}\big)(X) \\ N^{(3)}_{\varphi}(X) \ &= \ -\big(L_{\xi}\varphi\big)(X), \qquad N^{(4)}_{\varphi}(X) = \big(L_{\xi}\eta\big)(X). \end{aligned}$$

Proposition 6 ([3]) . *For an almost contact manifold $M = (M, \varphi, \xi, \eta)$ the vanishing of the tensor field $N_\varphi^{(1)}$ implies the vanishing of the tensor fields $N_\varphi^{(2)}, N_\varphi^{(3)}$ and $N_\varphi^{(4)}$.*

From the above proposition, we see that an almost contact metric manifold $M = (M, \varphi, \xi, \eta)$ is normal if and only if $N_\varphi^{(1)}$ vanishes everywhere on M ([3], p.81). Therefore, summing up the arguments above, we have the following main theorem

Theorem 7.

1. *The almost contact metric structure on M is a contact metric structure if and only if the almost Hermitian structure $(\tilde{g}, \tilde{J})$ is almost Kähler (i.e., $\mathrm{d}\tilde{\Omega} = 0$) for all function f on $\mathbb{R}$ such that $ff' \neq 0$. In addition, the structure on M is Sasakian if and only if the structure $(\tilde{g}, \tilde{J})$ on $\tilde{M}$ is Kählerian.*

2. *The almost contact metric structure on M is almost cosymplectic if and only if the almost Hermitian structure $(\tilde{g}, \tilde{J})$ satisfies $\mathrm{d}\tilde{\Omega} = 2ff'(\mathrm{d}t \wedge \Phi)$ in which case the structure is conformally almost Kähler. In addition, the structure on M is cosymplectic if and only if the structure $(\tilde{g}, \tilde{J})$ on $\tilde{M}$ is conformally Kähler.*

3. *The almost contact metric structure on M is almost Kenmotsu if and only if the almost Hermitian structure $(\tilde{g}, \tilde{J})$ satisfies $\mathrm{d}\tilde{\Omega} = 2f(f'\mathrm{d}t + f\eta) \wedge \Phi$ in which case the structure is conformally almost Kähler if and only if η is exact. In addition, if the structure on M is Kenmotsu then the structure $(\tilde{g}, \tilde{J})$ on $\tilde{M}$ is conformally Kähler if and only if η is exact. Moreover, if $\eta = -\mathrm{d}\beta$ for some $\beta \in C^\infty(\tilde{M})$ then $\mathrm{e}^{2(\beta - \ln|f|)}\tilde{g}$ will be a Kähler metric on $\tilde{M}$.*

Proof: The necessity was observed above for both cases (see (3)). For the sufficiency, first observe that from equation (8) where $h = f'$ we have

1) $$\mathrm{d}\tilde{\Omega}\Big(\big(\frac{\partial}{\partial t}, 0\big), \big(0, X\big), \big(0, Y\big)\Big) = 2ff'\big(\Phi - \mathrm{d}\eta\big)(X, Y). \tag{12}$$

If $\mathrm{d}\tilde{\Omega} = 0$, then the equation (12) gives $\Phi = \mathrm{d}\eta$ and we have a contact metric structure.

So, if M is Sasakian then the structure (g, J) is Kählerian.

2) If $\mathrm{d}\tilde{\Omega} = 2ff'(\mathrm{d}t \wedge \Phi)$, then the equation (12) gives $\mathrm{d}\eta = 0$ and applying d to $\mathrm{d}\tilde{\Omega} = 2ff'(\mathrm{d}t \wedge \Phi)$ we have $\mathrm{d}\Phi = 0$ and hence an almost cosymplectic structure on M.

Now consider the metric $\overline{g} = \frac{1}{f^2}\tilde{g}$, it is almost Hermitian with respect to $\tilde{J}$ and its fundamental two-form $\overline{\Omega} = \frac{1}{f^2}\tilde{\Omega}$. Then

$$\begin{aligned} \mathrm{d}\overline{\Omega} &= \frac{-2f'}{f^3}\mathrm{d}t \wedge \tilde{\Omega} + \frac{1}{f^2}\mathrm{d}\tilde{\Omega} \\ &= \frac{-2f'}{f^3}\mathrm{d}t \wedge f(2f'\mathrm{d}t \wedge \eta + f\Phi) + \frac{1}{f}(-2f'\mathrm{d}t \wedge \mathrm{d}\eta + 2f'\mathrm{d}t \wedge \Phi + f\mathrm{d}\Phi) \\ &= 0 \end{aligned}$$

giving a conformally almost Kähler structure.

3) If $\mathrm{d}\tilde{\Omega} = 2f(f'\mathrm{d}t + f\eta) \wedge \Phi$, then the equation (12) gives $\mathrm{d}\eta = 0$ and applying d to $\mathrm{d}\tilde{\Omega} = 2f(f'\mathrm{d}t + f\eta) \wedge \Phi$ we get

$$(f'\mathrm{d}t + f\eta) \wedge \mathrm{d}\Phi = 2f'\mathrm{d}t \wedge \eta \wedge \Phi$$

so that $\mathrm{d}\Phi = 2\eta \wedge \Phi$, and hence an almost Kenmotsu structure on M.

Using (7) and (9,3) with $h = f'$ we get

$$\mathrm{d}\tilde{\Omega} = 2\big(\mathrm{d}(\ln|f|) + \eta\big) \wedge \tilde{\Omega}. \tag{13}$$

From definition (1), it is obvious that $\tilde{M}$ is conformally Kähler if and only if η is exact. Now, consider the metric $\widehat{g} = \mathrm{e}^{2(\beta - \ln|f|)}\tilde{g}$ with $\beta \in C^\infty(\tilde{M})$. This metric is Hermitian with respect to $\tilde{J}$ and its fundamental two-form $\widehat{\Omega} = \mathrm{e}^{2(\beta - \ln|f|)}\tilde{\Omega}$. Then, by straightforward calculations, using (13) and $\eta = -\mathrm{d}\beta$ we obtain $\mathrm{d}\widehat{\Omega} = 0$ and this completes the proof. ■

Special cases:

- For $f = t$ with $h = f'$, where $t > 0$ and by (10) we get the metric cone (see [14])

$$\tilde{g} = \mathrm{d}t^2 + t^2 g, \qquad \tilde{J}(a\frac{\partial}{\partial t}, X) = \Big(t\eta(X)\frac{\partial}{\partial t},\ \varphi X - \frac{a}{t}\xi\Big).$$

- For $f = h = e^t$, and by (10) we get the $\mathcal{D}$-homothetic warping (see [6])

$$\begin{aligned} \tilde{g} &= \mathrm{d}t^2 + \mathrm{e}^{2t}g + \mathrm{e}^{2t}(\mathrm{e}^{2t} - 1)\eta \otimes \eta \\ \tilde{J}(a\frac{\partial}{\partial t}, X) &= \Big(\mathrm{e}^{2t}\eta(X)\frac{\partial}{\partial t}\ ,\ \varphi X - a\mathrm{e}^{-2t}\xi\Big). \end{aligned}$$

Example 8. *We denote the Cartesian coordinates in a three-dimensional Euclidean space E^3 by (x, y, z) and define a symmetric tensor field g by*

$$g = \begin{pmatrix} \rho^2 + \tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix}$$

where ρ and τ are functions on E^3 such that $\rho \neq 0$ everywhere.
Further, we define an almost contact metric (φ, ξ, η) on E^3 by

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -\tau & 0 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \eta = (-\tau, 0, 1).$$

The fundamental one-form η and the two-form Φ have the forms

$$\eta = \mathrm{d}z - \tau \mathrm{d}x \qquad \textit{and} \qquad \Phi = -2\rho^2 \mathrm{d}x \wedge \mathrm{d}y$$

and hence

$$\mathrm{d}\eta = \tau_2 \mathrm{d}x \wedge \mathrm{d}y + \tau_3 \mathrm{d}x \wedge \mathrm{d}z, \qquad \mathrm{d}\Phi = -4\rho_3\rho \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.
Knowing that the components of the Nijenhuis tensor N_φ in (2) *can be written as*

$$N^i_{kj} \;=\; \varphi^l_k(\partial_l \varphi^i_j - \partial_j \varphi^i_l) - \varphi^l_j(\partial_l \varphi^i_k - \partial_k \varphi^i_l) + \eta_k(\partial_j \xi^i) - \eta_j(\partial_k \xi^i)$$

where the indices i, j, k and l run over the range $1, 2, 3$, then by a direct computation we can verify that

$$N^i_{kj} = 0, \qquad \textit{for all} \qquad i, j, k$$

implying that the structure (φ, ξ, η, g) is normal. From definitions in (3)*, the structure (φ, ξ, η, g) is a*

1) *Sasaki, when $\tau_2 = -2\rho^2$ and $\tau_3 = 0$*

2) *cosymplectic, when $\rho_3 = 0$, $\tau_2 = 0$ and $\tau_3 = 0$*

3) *Kenmotsu, when $\rho_3 = \rho$, $\tau_2 = 0$ and $\tau_3 = 0$.*

Using the above cases and Theorem (7)*, the manifold $(\mathbb{R} \times E^3, \tilde{g}, \tilde{J})$ is*

1) *Kählerian, when $\tau_2 = -2\rho^2$ and $\tau_3 = 0$*

2) *conformally Kählerian, when $\rho_3 = 0$, $\tau_2 = 0$ and $\tau_3 = 0$*

3) *conformally Kählerian, when $\rho_3 = \rho$, $\tau_2 = 0$ and $\tau_3 = 0$.*

Note that

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f^2(\rho^2 + f'^2\tau^2) & 0 & -\tau f^2 f'^2 \\ 0 & 0 & f^2\rho^2 & 0 \\ 0 & -\tau f^2 f'^2 & 0 & f^2 f'^2 \end{pmatrix}, \quad \tilde{J} = \begin{pmatrix} 0 & -\tau f f' & 0 & f f' \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{f f'} & 0 & -\tau & 0 \end{pmatrix}.$$

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