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# A RIGOROUS FRAMEWORK FOR THE LANDAU AND LIFSHITZ APPROACH TO THOMSON ELECTROSTATICS

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**Abstract.** Landau and Lifshitz [7] proposed a novel formulation of the famous Thomson theorem, also known as the Thomson variational principle. In an attempt to explain, rather than postulate, the distribution of electrical charge exclusively on the surface of the conductor, Landau and Lifshitz allow the admissible variations in the electrical charge to penetrate the interior of the conductor. This is a valuable generalization of their predecessors' work, as well as a step towards basing more of the analysis on first principles.

Landau and Lifshitz' approach has not received the attention it deserves because it was not formulated as a rigorous technique, but rather as a slight of hand to arrive at a known result. In this paper, we construct a rigorous mathematical framework based on the Landau and Lifshitz idea. In particular, we prove that surface distribution of charges corresponds to the absolute minimum of electrostatic energy.

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## 1. The Thomson Principle

The Thomson principle, also known as the variational principle of electrostatics of conductors, has played in important role in a broad range of disciplines from electromagnetism to applied mathematics [1, 2, 7–9]. There are many different forms of the Thomson principle. It was formulated for conductors and, in its original form, states that the equilibrium distribution of electric charges on the surface of an electric conductor minimizes the total electrostatic energy. The principle has since been generalized in a number of ways. For instance, according to one of the alternative formulations [2], at equilibrium, the conductor's boundary is an equipotential surface.

One extension of the Thomson principle was suggested by Landau and Lifshitz [7]. Until Landau and Lifshitz, only admissible variations in the surface charge density were considered. In other words, only those variations that kept all electrical

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charges on the conductor's surface. Landau and Lifshitz, on the other hand, additionally considered the redistribution of the electric charges into the conductor's interior.

To quote Landau and Lifshitz [7]: *The energy of the electrostatic field of conductors has a certain extremum property, though this property is more formal than physical. To derive it, let us suppose that the charge distribution on the conductors undergoes an infinitesimal change* (*the total charge on each conductor remaining unaltered*), *in which the charges may penetrate into the conductors; we ignore the fact that such a charge distribution cannot in reality be stationary.*

Here is the essence of the argument presented in [7]. The total electrostatic energy $U$ is given by the integral

$$U = \frac{1}{8\pi}\int E^2 \mathrm{d}V \tag{1}$$

which extends over the entire space, including the interior of the conductor, since the virtual displacement of charges results in a nonzero electric field $\mathbf{E}$. In what follows, we continue to use the integral sign $\int$ without specifying the domain to denote integration over the entire space, possibly with the exclusion of some singular interfaces.

The field $\mathbf{E}$ is related to the electrostatic potential $\varphi$ by the familiar identity

$$\mathbf{E} = -\nabla\varphi. \tag{2}$$

The energy variation $\delta U$ induced by redistribution of charges is given by

$$\delta U = -\frac{1}{4\pi}\int \nabla\varphi \cdot \delta\mathbf{E}\mathrm{d}V \tag{3}$$

where $\delta\mathbf{E}$ is the corresponding variation in the electric field $\mathbf{E}$. By the product rule, the energy variation can be written as

$$\delta U = -\frac{1}{4\pi}\int \nabla(\varphi\delta\mathbf{E})\mathrm{d}V + \frac{1}{4\pi}\int \varphi\nabla\cdot(\delta\mathbf{E})\,\mathrm{d}V. \tag{4}$$

Since the electrostatic potential vanishes sufficiently fast at infinity, the first integral in (4) is zero by the divergence theorem. The divergence of the electric field variation $\delta\mathbf{E}$ is given by

$$\nabla\cdot(\delta\mathbf{E}) = 4\pi\delta\rho \tag{5}$$

where $\delta\rho$ is the variation in the electric charge density. Therefore, the energy variation $\delta U$ is given by

$$\delta U = -\int \varphi\delta\rho\mathrm{d}V. \tag{6}$$

If $\varphi$ is the equilibrium electrostatic potential then the integral in (6) vanishes for the following reasons: a) outside the conductor, there are no charges and therefore $\delta\rho = 0$, b) inside the conductor, $\varphi$ is constant and can therefore be factored out, and c) once $\varphi$ is factored out, the remaining integral of $\delta\rho$ vanishes since the total charge is preserved. Therefore, the total energy corresponding to the equilibrium configuration represents a local stationary point.

This idea is a valuable extension to the classical formulation. However, it has not received the attention it deserves (if any at all) since its publication. The lack of attention can be explained by the fact that the proposed approach somewhat vague and unsystematic and cannot satisfy the demand for rigor commensurate with the standards of mathematical physics and, in particular, classical electrostatics. For instance, the infinitesimal change $\delta\rho$ in the spatially distributed charge will necessarily induce an infinitesimal change $\delta\tau$ in the surface charge distribution, leading to discontinuities in the electrical field across the interface $S$. Thus, an application of the divergence theorem in the entire space, without regard to the interface, is invalid. It is possible that this approach can be remedied by considering additional boundary terms or by introducing some concept of generalized derivatives. However, this was not discussed by Landau and Lifshitz.

It is equally important to note that Landau and Lifshitz did not perform the second energy variation analysis, necessary in order to show at least a *weak* minimum of the electrostatic energy [3]. Thus, there is no evidence that the equilibrium charge distribution indeed delivers the minimum of the accumulated electrostatic energy. Interestingly, the Russian original of [7] claimed that there exists a simple proof of the minimum property without actually presenting the proof. Nor did a proof appear in any other publication. Furthermore, the very claim of an existing proof is absent in the English translation [7] referenced throughout this paper. This gap is essential from the standpoint of mathematical physics and precludes this beautiful idea from being widely used. In this paper, we show that the Landau and Lifshitz approach is indeed correct and present a proof of not just a *weak*, but an *absolute*, minimum.

Consider a conductor $\Omega$ enclosed by the fixed surface $S$. Let $Q$ be the fixed total electrical charge distributed on $S$ with surface density $\tau$ and over $\Omega$ with the volume density $\rho$ subject to the single constraint

$$\int_\Omega \rho \mathrm{d}\Omega + \int_S \tau \mathrm{d}S = Q. \tag{7}$$

The electrostatic potential $\varphi$ is governed by Poisson's equation

$$\nabla_i \nabla^i \varphi = -4\pi\rho \tag{8}$$

subject to the boundary conditions across the interface $S$

$$[\varphi]_S = 0, \qquad N_i \left[\nabla^i \varphi\right]_S = 4\pi\tau \tag{9}$$

where $[\cdot]_S$ indicates the jump across the interface in the enclosed quantity and $N_i$ is the exterior unit normal. Finally, $\varphi$ vanishes (sufficiently fast) at infinity.

## 2. Demonstration of the Landau-Lifshitz Version of the Thomson Theorem

Consider two different distributions of the electric charge. Let the first distribution correspond the situation where the entire charge is distributed along the surface and let $\tau_0$ be the corresponding equilibrium surface distribution. It is well known that the corresponding electrostatic potential $\varphi_0$ is uniform over $\Omega$.

In the second configuration, the total charge is divided between surface charges and volume charges. The individual total charges $Q_\tau$ and $Q_\rho$

$$Q_\tau = \int_S \tau \mathrm{d}S, \qquad Q_\rho = \int_\Omega \rho \mathrm{d}\Omega \tag{10}$$

add up to the total charge $Q$

$$Q = Q_\tau + Q_\rho. \tag{11}$$

The electrostatic potential corresponding to the second configuration is denoted by $\varphi$ without a subscript.

In order to prove the Landau and Lifshitz version of the Thomson theorem, introduce the positive quantity

$$\frac{1}{8\pi} \int \nabla_i \left(\varphi_0 - \varphi\right) \nabla^i \left(\varphi_0 - \varphi\right) \mathrm{d}\Omega \tag{12}$$

where is integral is calculated over the entire space excluding the interface $S$. Multiplying out the integrand, we find

$$\frac{1}{8\pi} \int \left(\nabla_i \varphi \nabla^i \varphi - 2\nabla_i \varphi \nabla^i \varphi_0 + \nabla_i \varphi_0 \nabla^i \varphi_0\right) \mathrm{d}\Omega > 0. \tag{13}$$

In a moment, we will present a proof of the fact that

$$\int \nabla_i \varphi \nabla^i \varphi_0 \mathrm{d}\Omega = \int \nabla_i \varphi_0 \nabla^i \varphi_0 \mathrm{d}\Omega. \tag{14}$$

With the help of (14) equation (13) yields

$$\frac{1}{8\pi} \int \left(\nabla_i \varphi \nabla^i \varphi - \nabla_i \varphi_0 \nabla^i \varphi_0\right) \mathrm{d}\Omega > 0 \tag{15}$$

which is equivalent to

$$\frac{1}{8\pi}\int \nabla_i\varphi\nabla^i\varphi \mathrm{d}\Omega > \frac{1}{8\pi}\int \nabla_i\varphi_0\nabla^i\varphi_0. \tag{16}$$

Thus the electrostatic energy associated with the equilibrium surface distribution is smaller than that for the alternative configuration that combines surface and volume charges.

Now, onto the proof of (14). By the product rule

$$\int \nabla_i\varphi\nabla^i\varphi_0 \mathrm{d}\Omega = \int \left(\nabla^i\left(\varphi_0\nabla_i\varphi\right) - \varphi_0\nabla_i\nabla^i\varphi\right)\mathrm{d}\Omega. \tag{17}$$

The first term in the integrand in analyzed by the divergence theorem. Since the potentials $\varphi_0$ and $\varphi$ are assumed to vanish sufficiently fast at infinity, the outer boundary term vanishes. However, due to the discontinuities at the interface $S$, we find

$$\int \nabla_i\left(\varphi_0\nabla^i\varphi\right)\mathrm{d}\Omega = \int_S N_i\left[\varphi_0\nabla^i\varphi\right]_S \mathrm{d}S \tag{18}$$

where, once again, $[\cdot]_S$ denotes the jump in the enclosed quantity. Since the equilibrium potential $\varphi_0$ is continuous across $S$ and is constant on $S$ it can be factored out of the surface integral in (18)

$$\int \nabla_i\left(\varphi_0\nabla^i\varphi\right)\mathrm{d}\Omega = \varphi_0\int_S N_i\left[\nabla^i\varphi\right]_S \mathrm{d}S. \tag{19}$$

And, since according to the boundary condition (9), $N_i\left[\nabla^i\varphi\right]_S = 4\pi\tau$, we obtain

$$\int_S \mathrm{d}SN_i\left[\varphi_0\nabla^i\varphi\right]_S = 4\pi\varphi_0\int_S \tau\mathrm{d}S = 4\pi\varphi_0 Q_\tau. \tag{20}$$

In the second term of equation (17), we find $\nabla_i\nabla^i\varphi$, the Laplacian of $\varphi$, which, according to equation (8), equals $-4\pi\rho$. Furthermore, $\varphi_0$ is constant inside the conductor, which is the only part of the entire space where $\rho$ is nonzero. Therefore, the integral over the entire space reduces to the integral over $\Omega$

$$\int \varphi_0\nabla^i\nabla_i\varphi\mathrm{d}\Omega = -4\pi\varphi_0\int_\Omega \rho\mathrm{d}\Omega = -4\pi\varphi_0 Q_\rho. \tag{21}$$

Combining (20) and (21), we find that

$$\int \nabla_i\varphi\nabla^i\varphi_0\mathrm{d}\Omega = 4\pi\varphi_0 Q_\tau + 4\pi\varphi_0 Q_\rho = 4\pi\varphi_0 Q. \tag{22}$$

Since $Q$ is the total charge for each of the two configurations we are comparing, we can express it as the surface integral of the jump in the electrostatic potential $\varphi_0$

$$Q = \frac{1}{4\pi}\int_S N_i \left[\nabla^i \varphi_0\right]_S \mathrm{d}S. \tag{23}$$

Thus

$$\int \nabla_i \varphi \nabla^i \varphi_0 \mathrm{d}\Omega = 4\pi\varphi_0 Q = \varphi_0 \int_S N_i \left[\nabla^i \varphi_0\right]_S \mathrm{d}S = \int_S N_i \left[\varphi_0 \nabla^i \varphi_0\right]_S \mathrm{d}S \tag{24}$$

where, in the last step, we were able to insert $\varphi_0$ inside the discontinuity symbol since $\varphi_0$ is continuous across the interface $S$.

Next, apply the divergence theorem to the surface integral in (24)

$$\int_S N_i \left[\varphi_0 \nabla^i \varphi_0\right]_S \mathrm{d}S = \int \nabla_i \left(\varphi_0 \nabla^i \varphi_0\right) \mathrm{d}\Omega \tag{25}$$

where we have once again used the fact that $\varphi_0$ decays to zero at infinity sufficiently fast. Combining (24) and (25), we find

$$\int \nabla_i \varphi \nabla^i \varphi_0 \mathrm{d}\Omega = \int \nabla_i \left(\varphi_0 \nabla^i \varphi_0\right) \mathrm{d}\Omega.$$

Finally, recall that $\varphi_0$ is characterized by the vanishing Laplacian at all points in space away from the interface $S$. Thus, by the product rule, we find

$$\int \nabla_i \varphi \nabla^i \varphi_0 \mathrm{d}\Omega = \int \nabla_i \varphi_0 \nabla^i \varphi_0 \mathrm{d}\Omega. \tag{26}$$

This is the relationship we set out to prove in order to demonstrate (16) which shows that the equilibrium surface distribution of charges delivers smaller energy than any alternative distribution with spatially distributed charges. This completes the overall proof.

## 3. Discussion

The principle that we have demonstrated is stronger than Landau and Lifshitz' original claim. Landau and Lifshitz described a weak minimum of the electrostatic energy with respect to the distribution of charges, that is one with respect to infinitesimal perturbations of charges. We showed that the equilibrium configuration in which all of the charge is distributed on the surface is an absolute minimum. The presented proof is consistent with the standards of mathematical physics and classical electrostatics. At the same time, we would like to emphasize that the "infinitesimal" (variational) approach of [7] permits much more useful and straightforward generalization than our approach but our approach has the advantage of consistency, rigor, and clarity. Various other generalizations of the Thomson principle can be found in [4–6] and references therein.

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