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TAUB-NUT AS BERTRAND SPACETIME WITH MAGNETIC FIELDS

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Abstract. Based on symmetries Taub-NUT shares with Bertrand spacetime, we cast it as the latter with magnetic fields. Its nature as a Bianchi-IX gravitational instanton and other related geometrical properties are reviewed. We provide an easy derivation and comparison between the spatial Killing-Yano tensors deduced from first-integrals and the corresponding hyperkähler structures and finally verify the existence of a graded Lie-algebra structure via Schouten-Nijenhuis brackets.

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Contents

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1. Introduction

The Taub-NUT [26] is an exact solution of Einstein's equations, found by Abraham Huskel Taub (1951), and extended to a larger manifold by E. Newman, T. Unti and L. Tamburino (1963). It is a gravitational anti-instanton with corresponding $\mathrm{SU}(2)$ gauge fields, frequently studied for its geodesics which approximately describe the motion of well seperated monopole-monopole interactions. As a dynamical system it exhibits spherically symmetry, with geodesics admitting Kepler-type symmetry, implying first-integrals such as the angular momentum and Runge-Lenz vectors respectively. Witten's prescription [45] realized Taub-NUT space as a hyper-Kahler quotient using T-duality. This construction has a natural interpretation in terms of D-branes [12], serving as an important example in string theory.

Bertrand spacetime, formulated by Perlick [41] is also spherically symmetric

$$\mathrm{d}s^2 = h(\rho)^2 \mathrm{d}\rho^2 + \rho^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big) - \frac{\mathrm{d}t^2}{\Gamma(\rho)} \tag{1}$$

derived from Bertrand's Theorem, describing stable and closed geodesics with periodic orbits. Upon comparison, Euclidean Bertrand spaces and Taub-NUT spaces appear similar apart from magentic monopole and dipole interaction of the Taub-NUT. This implies dynamical similarities due to similar first-integrals characterizing motion, and that Taub-NUT possibly exhibits Kepler-Hooke duality.

Consequently, we try to find first-integrals similar to those associated with central-force motion under potentials involved in Bertrand's Theorem: the angular momentum and Laplace-Runge-Lenz vector. Since we are interested in the dynamical aspects of Taub-NUT spaces, our attention is directed toward geodesics and Killing tensors. Naturally, we will be looking at Killing tensors affiliated with Runge-Lenz-like vector. They obey the equation

$$\nabla_{(a}K_{b_1)b_2\ldots b_n} = 0. \tag{2}$$

Such tensors are the Killing-Stäckel tensors which are symmetric under index permutation and the Killing-Yano tensor. The Killing-Yano tensors are antisymmetric under index permutation, and their square gives the Stäckel tensor, like the antisymmetric tensor whose square gives the Runge-Lenz-like quantity as we shall see. Such Killing tensors exhibit quaternionic algebra, implying a connection to Hyperkähler structures associated with the metric.

We start in Section 2, with preliminaries on dynamics with magnetic field interactions, then compute first-integrals similar to the angular momentum and the Laplace-Runge-Lenz vector for the Taub-NUT. We deduce such first-integrals first by from equations of motion and then by a momentum polynomial expansion.

In Section 3, we compare Taub-NUT metric to Euclidean Bertrand spacetime with magnetic monopoles and dipoles. Demonstrating such a similarity allows the intensely studied Bertrand spacetimes to share many important properties, and conversely extend properties of the Taub-NUT to Bertrand spaces with magnetic fields. This helps us identify symmetries and conserved quantities of Taub NUT and employ its curvature properties for Bertrand spacetimes. The last subsection covers the conserved quantity called the Fradkin tensor under Bohlin-Arnold-Vassiliev transformation which are bound to have such Killing tensors embedded.

In Section 4 we derive the Taub-NUT from the self-dual Bianchi-IX metric described by the classical Darboux-Halphen system. Then we geometrically analyze it, computing curvature and confirming its self-duality as a gravitational instanton

$$R_{\mu\nu\rho\sigma} = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\gamma}R_{\lambda\gamma\rho\sigma}. \tag{3}$$

This helps us study the metric as an integrable system. Finally, we compute topological invariants shared with comparable Bertrand spacetime with magnetic fields.

In Section 5, after introducing Killing Stäckel and Yano tensors, we will focus on the latter. After a brief review of their properties, we will show how to find them embedded within conserved quantities, and see if they exhibit a graded Lie-algebra structure that allows construction of higher order Killing-Yano tensors.

Finally, in Section 6 we derive hyperkähler structures of the Taub-NUT and compare them to the Killing-Yano tensors to see if they also exhibit quaternion algebra.

In the last section we conclude our work, discussing possibilities of further research along the line of the present article. The Appendix contains detailed computation regarding Killing tensors following Holten's algorithm, a review of Bohlin's transformation and the double derivative of Killing Yano tensors.

2. Conserved Quantities

In classical mechanics it is important to identify constants of motion called conserved quantities or first-integrals of the system. In the theory of integrable systems, all first-integrals are in involution or commute with each other within the Poisson brackets, with at least one integral definitely being available.

In Hamiltonian mechanics, a conserved quantity Q commutes with H, a first-integral resulting from time-translation invariance, within the Poisson brackets

$$\big\{Q, H\big\} = 0. \tag{4}$$

However, this prescription is not gauge covariant for systems with gauge interactions. To understand why, consider the following metrics as examples.

For spacetime with scalar potential $U(\boldsymbol{x})$ only, the metric where t is cyclical is

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j - \frac{1+2U(\boldsymbol{x})}{m}\mathrm{d}t^2.$$

Under the parameterization $t = \tau$, the lagrangian, Hamiltonian and Hamilton's dynamical equations for particles in presence of scalar potentials is given by

$$L_{\dot{t}=1} = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x})$$

the Legendre transform giving the Hamiltonian $H = \sum_{k\neq t}\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k - L$

$$H = \frac{1}{2m}\boldsymbol{p}^2 + U(\boldsymbol{x}) \qquad \Rightarrow \qquad \begin{cases} \dot{\boldsymbol{x}} = \dfrac{\partial H}{\partial \boldsymbol{p}} = \dfrac{\boldsymbol{p}}{m} \\ \dot{\boldsymbol{p}} = -\dfrac{\partial H}{\partial \boldsymbol{x}} = -\boldsymbol{\nabla}U(\boldsymbol{x}). \end{cases}$$

For this system without magnetic fields, the fundamental brackets are

$$\big\{x^i, p_j\big\} = \delta^{ij}, \qquad \big\{x^i, x^j\big\} = \big\{p_i, p_j\big\} = 0. \tag{5}$$

Now, for charged particles in U(1) gauge fields from magnetic dipoles and monopole interactions, without scalar potential, the metric is

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j - \frac{1}{m}\big(\mathrm{d}t - A_k(\boldsymbol{x})\mathrm{d}x^k\big)^2 \tag{6}$$

so the corresponding Lagrangian and Hamiltonian for $\dot{t} = 1$ are given by

$$L = \frac{1}{2}\Big(m\dot{\boldsymbol{x}}^2 - \big(\dot{t} - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}}\big)^2\Big)$$

$$\therefore \quad H = \frac{1}{2m}\big(\mathbf{p} - q\boldsymbol{A}(\boldsymbol{x})\big)^2 + \frac{q^2}{2}, \qquad q = \left(\frac{\partial L}{\partial \dot{t}}\right)_{\dot{t}=1} = 1 - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}}.$$

Now let us consider a Kaluza-Klein modification of this spacetime, such that we include another cyclical co-ordinate ψ that is periodic along with magnetic field components coupled with it. This results in a $4+1$ spacetime from a $3+1$ one

$$\mathrm{d}s^2 = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \frac{1}{m}\big(\mathrm{d}\psi + A_k(\boldsymbol{x})\mathrm{d}x^k\big)^2 - \big(1 + 2U(\boldsymbol{x})\big)\mathrm{d}t^2$$

so the Lagrangian and Hamiltonian for $\dot{t} = 1$, ignoring constant additive terms are

$$\begin{aligned} L_{\dot{t}=1} &= \frac{1}{2}\Big[m\dot{\boldsymbol{x}}^2 + \big(\dot{\psi} + \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}}\big)^2\Big] - U(r) \\ \therefore \qquad H &= \frac{1}{2m}\big(\mathbf{p} - q\boldsymbol{A}(\boldsymbol{x})\big)^2 + U(r), \qquad q = \frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} - \boldsymbol{A}(\boldsymbol{x}).\dot{\boldsymbol{x}} \end{aligned} \tag{7}$$

where q is a conserved charge. The corresponding Hamilton's equations are

$$\begin{aligned} \dot{\boldsymbol{x}} &= \frac{\partial H}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p} - q\boldsymbol{A}}{m} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{x}} = \frac{q}{m}\big(\boldsymbol{\nabla}\boldsymbol{A}\big).\big(\boldsymbol{p} - q\boldsymbol{A}\big) - \boldsymbol{\nabla}U. \end{aligned}$$

Since the potentials are gauge dependent ($\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$), the momenta therefore must be so as well ($\boldsymbol{p} \to \boldsymbol{p} + q\boldsymbol{\nabla}\Lambda$). Then, we must write gauge invariant momenta and express the Hamiltonian in its gauge invariant form

$$H = \frac{\boldsymbol{\Pi}^2}{2} + V(r), \qquad \boldsymbol{\Pi} = \boldsymbol{p} - q\boldsymbol{A}.$$

Functions and partial derivative operators in gauge invariant forms are written as

$$\begin{aligned} &f(\boldsymbol{x}, \boldsymbol{p}) \longrightarrow f(\boldsymbol{x}, \boldsymbol{\Pi}) \\ &\frac{\partial}{\partial x^i} \quad \longrightarrow \quad \frac{\partial \Pi^j}{\partial x^i}\frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial x^i} = -q\partial_i A_j \frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial x^i} \\ &\frac{\partial}{\partial p^i} \quad \longrightarrow \quad \frac{\partial \Pi^j}{\partial p^i}\frac{\partial}{\partial \Pi^j} + \frac{\partial}{\partial p^i} = \frac{\partial}{\partial \Pi^i} \quad \text{(No explicit dependence on } \boldsymbol{p}\text{)} \end{aligned}$$

with which the fundamental brackets become

$$\big\{x^i, \Pi_j\big\} = \delta^{ij}, \qquad \big\{x^i, x^j\big\} = 0, \qquad \big\{\Pi_i, \Pi_j\big\} = -qF_{ij}. \tag{8}$$

Interestingly, the new Poisson Brackets (8) between gauge covariant momenta are non-zero, as opposed to (5). This is a classical analogue of Ricci-identity (in the absence of torsion). We can furthermore redefine the Poisson Brackets as

$$\{f, g\} = \frac{\partial f}{\partial \boldsymbol{x}} \cdot \frac{\partial g}{\partial \boldsymbol{\Pi}} - \frac{\partial f}{\partial \boldsymbol{\Pi}} \cdot \frac{\partial g}{\partial \boldsymbol{x}} - qF_{ij}\frac{\partial f}{\partial \boldsymbol{\Pi}} \cdot \frac{\partial g}{\partial \boldsymbol{\Pi}}. \tag{9}$$

Having redefined the Poisson Brackets to make Hamiltonian dynamics manifestly gauge invariant in the modified bracket, we can analyze the conserved quantities in a general gauge invariant form with the Holten Algorithm as shown in [24] and [38] discussed later as we shall see.

2.1. A Dynamical-Systems Description of Taub-NUT

The Euclidean Taub-NUT metric as shown in [26] is given by

$$\begin{gathered}\mathrm{d}s^2 = f(r)\big\{\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big)\big\} + g(r)\big(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\big)^2 \\ \text{where} \qquad f(r) = 1 + \frac{4M}{r}, \qquad g(r) = \frac{(4M)^2}{1+\frac{4M}{r}}.\end{gathered} \tag{10}$$

For later reference, taking $\mathrm{d}\tilde{s}^2 = \frac{\mathrm{d}s^2}{4M}$ we shall re-write the above metric into

$$\begin{gathered}\mathrm{d}\tilde{s}^2 = V(r)\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + V^{-1}(r)\big(\mathrm{d}\psi + \boldsymbol{A}.\mathrm{d}\boldsymbol{x}\big)^2 \\ \text{where} \qquad V(r) = \frac{1}{4M} + \frac{1}{r}, \qquad \boldsymbol{A}.\mathrm{d}\boldsymbol{x} = \cos\theta\,\mathrm{d}\phi.\end{gathered} \tag{11}$$

We now consider the geodesic flows of the generalized Taub-NUT metric given by (10), for which we can compose the Lagrangian

$$\mathcal{L} = \frac{1}{2}f(r)\big\{\dot{r}^2 + r^2\big(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\big)\big\} + \frac{1}{2}g(r)\big(\dot{\psi} + \cos\theta\,\dot{\phi}\big)^2. \tag{12}$$

We can further re-write the Lagrangian (12) into three-dimensional form with a potential, as in (7), independent of the ψ as

$$\mathcal{L} = \frac{1}{2}f(r)|\dot{\boldsymbol{x}}|^2 + \frac{1}{2}g(r)\big(\dot{\psi} + \boldsymbol{A}.\dot{\boldsymbol{x}}\big)^2 - U(r)$$

where the momentum can be written as

$$\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{x}}} = f(r)\dot{\boldsymbol{x}} + q\boldsymbol{A}, \qquad \boldsymbol{\Pi} = f(r)\dot{\boldsymbol{x}} = \boldsymbol{p} - q\boldsymbol{A}.$$

Spaces with the metric (10) exhibit $\mathrm{SU}(2)\times\mathrm{U}(1)$ isometry group. Since we have two cyclical variables ψ and ϕ, we will have four Killing vectors given by

$$\begin{aligned} D_0 &= \partial_\psi \\ D_1 &= -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\,\partial_\phi + \frac{\cos\phi}{\sin\theta}\,\partial_\psi \\ D_2 &= \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi + \frac{\sin\phi}{\sin\theta}\,\partial_\psi \\ D_3 &= \partial_\phi \end{aligned} \tag{13}$$

where D_0 commutes with the other three killing vectors D_1, D_2, D_3, which exhibit SU(2) Lie algebra: $\big[D_i, D_j\big] = -\varepsilon_{ij}{}^k D_k$. Since ψ is cyclic, we have a conserved quantity known as the relative electric charge

$$q = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = g(r)\big(\dot{\psi} + \cos\theta\, \dot{\phi}\big) = g(r)\big(\dot{\psi} + \boldsymbol{A}.\dot{\boldsymbol{x}}\big) = \text{const.}$$

The symplectic two-form ω and energy $\mathcal{H} = \mathcal{E}$ for the Taub- NUT are

$$\omega = \frac{1}{2}\big(\omega_0 + qF(\boldsymbol{x})\big)_{jk} \mathrm{d}x^j \wedge \mathrm{d}x^k = \sum_{i=1}^{3} \mathrm{d}\big(p_i - qA_i(\boldsymbol{x})\big) \wedge \mathrm{d}x^i = \sum_{i=1}^{3} \mathrm{d}\Pi_i \wedge \mathrm{d}x^i$$

$$\omega = \sum_{i=1}^{3} \mathrm{d}p_i \wedge \mathrm{d}x^i - \frac{q}{2r^3} \sum_{i,j,k} \varepsilon_{ijk} x^i\, \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{14}$$

$$\mathcal{H} = \frac{|\boldsymbol{\Pi}|^2}{2f(r)} + \frac{q^2}{2g(r)} + U(r) = \mathcal{E}, \qquad F_{ij}(\boldsymbol{x}) = -\sum_k \varepsilon_{ijk} \frac{x^k}{r^3}. \tag{15}$$

Consequently, the Hamilton's equations are given by

$$\dot{\boldsymbol{x}} = \big\{\boldsymbol{x}, \mathcal{H}\big\}_\theta = \frac{\boldsymbol{\Pi}}{f(r)}$$

$$\dot{\boldsymbol{\Pi}} = \big\{\boldsymbol{\Pi}, \mathcal{H}\big\}_\theta = \left[\frac{f'(r)}{2\big(f(r)\big)^2}|\boldsymbol{\Pi}|^2 + \frac{g'(r)}{2\big(g(r)\big)^2} - U'(r)\right]\frac{\boldsymbol{x}}{r} + \frac{q}{r^3 f(r)}\boldsymbol{x} \times \boldsymbol{\Pi}.$$

Using these equations, we find angular momentum in presence of magnetic fields

$$\left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \times \boldsymbol{\Pi} + \boldsymbol{x} \times \frac{\mathrm{d}\boldsymbol{\Pi}}{\mathrm{d}t}\right) = \frac{q}{r^3 f(r)}\big[\boldsymbol{x} \times \big(\boldsymbol{x} \times \boldsymbol{\Pi}\big)\big] = q\left[\frac{\big(\boldsymbol{x}.\dot{\boldsymbol{x}}\big)\boldsymbol{x}}{r^3} - \frac{\dot{\boldsymbol{x}}}{r}\right] = -q\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{x}}{r}\right)$$

$$\therefore \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{x} \times \boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}\right) = 0 \qquad \Rightarrow \qquad \boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}. \tag{16}$$

The cyclic variable allows reduction of the geodesic flow on $T(\mathbb{R}^4 - \{0\})$ to a system on $T(\mathbb{R}^3 - \{0\})$. The reduced system's rotational invariance implies a conserved energy, angular momentum and vector $\boldsymbol{K}$ analogous to the Laplace-Runge-Lenz vector

$$\mathcal{H} = \frac{1}{2}\frac{\boldsymbol{\Pi}^2}{f(r)} + \left(\frac{1}{2}\frac{q^2}{g(r)} + U(r)\right) \tag{17}$$

$$\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r} \tag{18}$$

$$\boldsymbol{K} = \frac{1}{2}K_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \boldsymbol{\Pi} \times \boldsymbol{J} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\boldsymbol{x}}{r}. \tag{19}$$

This concludes the detailing of conserved quantities of Taub-NUT from a dynamical systems perspective. Now we shall proceed to consider a systematic analytic process describing conserved quantities as power series expansions of momenta.

2.2. The Holten Algorithm Description

One way of obtaining conserved quantities that are polynomials in momenta is by writing them in a power series expansion involving the gauge invariant momenta

$$Q = C^{(0)}(\mathbf{r}) + C^{(1)}_i(\mathbf{r})\Pi^i + \frac{1}{2!}C^{(2)}_{ij}(\mathbf{r})\Pi^i\Pi^j + \frac{1}{3!}C^{(3)}_{ijk}(\mathbf{r})\Pi^i\Pi^j\Pi^k + ...$$

where all the coefficients of momenta power series are symmetric under index permutation. Applying this to eq (4), we can obtain the relations for each coefficient by matching the appropriate product series of momenta for both the terms

$$\{Q, H\} = \sum_n \Big[\big\{C^{(n)}_{\{i\}}\prod_{\{i\}}\Pi^k, \Pi^j\big\}\Pi_j + \big\{C^{(n)}_{\{i\}}\prod_{\{i\}}\Pi^k, V(r)\big\}\Big] = 0$$

$$\therefore \qquad \nabla_j C^{(n)}_{\{m\}}\prod_{\{m\}}\Pi^k = qC^{(n+1)}_{\{m\}i}\Big(F^{ij} + \partial_j V(r)\Big)\prod_{(\{m\}, k\neq i)}\Pi^k.$$

The equations we will get up to the third order setting $C^{(i)}_{\{m\}} = 0,\ i \geq 3$ are

$$\begin{aligned}
&\text{order 0:} && 0 = C^{(1)}_m \partial_m\big(V(r)\big)\\
&\text{order 1:} && \nabla_i C^{(0)} = qF_{ij}C^{(1)}_j + C^{(2)}_{ij}\partial_j\big(V(r)\big)\\
&\text{order 2:} && \nabla_i C^{(1)}_j + \nabla_j C^{(1)}_i = q\big(F_{im}C^{(2)m}{}_j + F_{jm}C^{(2)m}{}_i\big)\\
&\text{order 3:} && \nabla_i C^{(2)}_{jk} + \nabla_k C^{(2)}_{ij} + \nabla_j C^{(2)}_{ki} = 0.
\end{aligned} \tag{20}$$

Now we will turn our attention to some familiar conserved quantities.

2.2.1. Some Basic Killing Tensors

Using the above relations, we now look at some familiar conserved quantities studied in classical mechanics.

Angular Momentum

The conserved quantity deduced from first order term of the Holten series alone is

$$\begin{aligned} Q^{(1)} &= C_i^{(1)}\Pi^i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \Pi^i \\ \Rightarrow \qquad \boldsymbol{L}.\boldsymbol{\theta} &= -\big(\varepsilon_{ijk}\Pi^i x^j\big)\theta^k = \big(\boldsymbol{x}\times\boldsymbol{\Pi}\big).\boldsymbol{\theta} \\ \therefore \qquad \boldsymbol{L} &= \boldsymbol{x}\times\boldsymbol{\Pi}. \end{aligned}$$

This eventually becomes the conserved quantity known as the angular momentum.

Laplace-Runge-Lenz Vector

On the other hand, the conserved quantity from the second order term alone is

$$\begin{aligned} Q^{(2)} &= \frac{1}{2}C_{ij}^{(2)}\Pi^i\Pi^j = \big\{\big|\boldsymbol{\Pi}\big|^2\big(\boldsymbol{n}.\boldsymbol{x}\big) - \big(\boldsymbol{\Pi}.\boldsymbol{x}\big)\big(\boldsymbol{\Pi}.\boldsymbol{n}\big)\big\} \\ \Rightarrow \qquad \boldsymbol{N}.\boldsymbol{n} &= \big\{\big|\boldsymbol{\Pi}\big|^2\boldsymbol{x} - \big(\boldsymbol{\Pi}.\boldsymbol{x}\big)\boldsymbol{\Pi}\big\}.\boldsymbol{n} = \big\{\boldsymbol{\Pi}\times\big(\boldsymbol{x}\times\boldsymbol{\Pi}\big)\big\}.\boldsymbol{n} \\ \therefore \qquad \boldsymbol{N} &= \boldsymbol{\Pi}\times\big(\boldsymbol{x}\times\boldsymbol{\Pi}\big). \end{aligned}$$

This quantity is a term contained in another conserved quantity known as the Laplace-Runge-Lenz vector. Having found the two familiar types of conserved quantities, we can now proceed to see what it looks like for the Taub-NUT metric.

2.2.2. The Holten Algorithm for Taub-NUT

Now, for the Taub-NUT metric, we have (17) giving the Hamiltonian. This can be written in dimensionally reduced form as

$$\mathcal{H} = \frac{1}{2}|\boldsymbol{\Pi}|^2 + f(r)W(r), \qquad W(r) = U(r) + \frac{q^2}{2g(r)} + \frac{\mathcal{E}}{f(r)} - \mathcal{E}.$$

From this Hamiltonian, after setting all higher orders $C_{ij}^{(2)} = C_{ijk}^{(3)} = 0$, we get the modified first and second order equations to be the following

$$\begin{aligned} &\text{order 1:} \qquad \partial_i C^{(0)} = qF_{ij}C^{(1)j} \\ &\text{order 2:} \qquad \nabla_i C_j^{(1)} + \nabla_j C_i^{(1)} = 0. \end{aligned} \tag{21}$$

The constraint equation of the second order of (21) gives us

$$C_i^{(1)} = g_{im}(\boldsymbol{x})\varepsilon^m{}_{jk}\theta^j x^k$$

$$\partial_i C^{(0)} = \frac{q}{r^3}\varepsilon_{ijk}\,\varepsilon^j{}_{nm}x^k\theta^m x^n \quad \equiv \quad \frac{q}{r^3}\big[\boldsymbol{x}\times\big(\boldsymbol{\theta}\times\boldsymbol{x}\big)\big]_i = \frac{q}{r^3}\big[r^2\boldsymbol{\theta} - \big(\boldsymbol{x}.\boldsymbol{\theta}\big)\boldsymbol{x}\big]_i$$

$$\partial_i C^{(0)} = q\bigg(\frac{\theta_i}{r} - \frac{\big(\boldsymbol{x}.\boldsymbol{\theta}\big)x_i}{r^3}\bigg) \qquad \Rightarrow \qquad C^{(0)} = q\theta_i\frac{x^i}{r}\cdot$$

Thus, we have the overall solution, and the corresponding conserved quantity

$$Q \equiv J_k\theta^k = C^{(0)} + C_i^{(1)}\Pi^i = \bigg(- g_{im}(\boldsymbol{x})\varepsilon^m{}_{jk}x^j\Pi^i + q\frac{x_k}{r}\bigg)\theta^k$$

$$\therefore \quad \boldsymbol{J}.\boldsymbol{\theta} = \bigg(\boldsymbol{x}\times\boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}\bigg).\boldsymbol{\theta} \quad \Rightarrow \quad \boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{\Pi} + q\frac{\boldsymbol{x}}{r}\cdot \tag{22}$$

However, if we explore upto the second order, setting $C_{ij}^{(2)} \neq 0$, we will return to the equations (20). For the third order, the solution for $C_{ij}^{(2)}$ is given by (80), so

$$C_{ij}^{(2)} = \big(2g_{ij}(\boldsymbol{x})n_k - g_{ik}(\boldsymbol{x})n_j - g_{kj}(\boldsymbol{x})n_i\big)x^k.$$

Eventually the other co-efficients from (20) are given by

$$F_{ik}C_{kj}^{(2)} = -2\varepsilon_{ijn}\frac{x^n}{r^3}\underbrace{\big(n_m x^m\big)}_{\boldsymbol{n}.\boldsymbol{x}} + \underbrace{\varepsilon_{ikn}\frac{x^k x^n}{r^3}n_j}_{0} + \underbrace{\varepsilon_{ikn}n^k x^n}_{(\boldsymbol{n}\times\boldsymbol{x})_i}\frac{x_j}{r^3}$$

$$\nabla_i C_j^{(1)} + \nabla_j C_i^{(1)} = -q\bigg\{\nabla_j\bigg(\frac{\varepsilon_{ikm}n^k x^m}{r}\bigg) + \nabla_i\bigg(\frac{\varepsilon_{jkm}n^k x^m}{r}\bigg)\bigg\}.$$

Thus, we can easily see which term on the RHS corresponds to what on the LHS, allowing us to solve for the first order and zeroth order coefficients from (20)

$$C_i^{(1)} = -\frac{q}{r}g_{im}(\boldsymbol{x})\varepsilon^m{}_{jk}n^k x^j.$$

In the case of the generalised Taub-NUT metric, the most general potentials admitting a Runge-Lenz vector are of the form

$$U(r) = \frac{1}{f(r)}\bigg(\frac{q^2}{2r^2} + \frac{\beta}{r} + \gamma\bigg) - \frac{q^2}{2g(r)} + \mathcal{E}$$

$$\nabla_i C^{(0)} = \beta\bigg(\frac{n_i}{r} - \frac{\big(\boldsymbol{n}.\boldsymbol{x}\big)x_i}{r^3}\bigg), \qquad C^{(0)} = \beta n_i\frac{x^i}{r}\cdot$$

For integrability, we require the commutation relation

$$\big[\partial_i,\partial_j\big]C^{(0)} = 0 \quad \Rightarrow \quad \Delta\bigg(f(r)W(r) - \frac{q^2g^2}{2r^2}\bigg) = 0$$

$$\Rightarrow \qquad f(r)W(r) = \frac{q^2 g^2}{2r^2} + \frac{\beta}{r} + \gamma, \qquad \beta, \gamma \in \mathbb{R}.$$

Thus, this overall conserved quantity is given as

$$Q \equiv R_k \theta^k = C^{(0)} + C^{(1)}_i \Pi^i + C^{(2)}_{ij} \Pi^i \Pi^j$$

$$\boldsymbol{R}.\boldsymbol{n} = \left(\boldsymbol{\Pi} \times \big(\boldsymbol{x} \times \boldsymbol{\Pi}\big) - \frac{q}{r}\boldsymbol{x} \times \boldsymbol{\Pi} + \beta \frac{\boldsymbol{x}}{r}\right).\boldsymbol{n} \qquad \Rightarrow \qquad \boldsymbol{R} = \boldsymbol{\Pi} \times \boldsymbol{J} + \beta \frac{\boldsymbol{x}}{r}. \quad (23)$$

Now we will take a detour to look at some details regarding the Runge-Lenz vector.

3. Bertrand Spacetime Dualities

In Newtonian mechanics, there are only two potentials allowing stable, closed and periodic orbits: Hooke's Oscillator ($V(r) = ar^2 + b$), and Kepler's orbital motion ($\Gamma(r) = \frac{a}{r} + b$) potentials. There is a relativistic analogue, given by the corresponding metrics in [41], describing spherically symmetric and static spacetime, with bounded and periodic trajectories. The Taub- NUT is one example of a spherically symmetric spacetime. Naturally, one would ask how it compares with the Euclidean Bertrand spacetime (BST) metric with magnetic fields.

3.1. Bertrand Spacetimes with Magnetic Fields

If we take the Euclidean version of Bertrand spacetime metric (1) and include magnetic monopole and dipole interactions, the metric becomes like (6) as

$$\mathrm{d}s^2 = h(\rho)^2 \mathrm{d}\rho^2 + \rho^2 \big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\big) + \frac{1}{\Gamma(\rho)} \big(\mathrm{d}t + A_i \, \mathrm{d}x^i\big)^2. \qquad (24)$$

If we recall, the Taub-NUT metric was given by (10). To see how it compares to (24), we shall attempt a co-ordinate map.

$$f(r)\,\mathrm{d}r^2 = h(\rho)^2 \mathrm{d}\rho^2, \qquad f(r) r^2 = \rho^2, \qquad g(r) = \frac{1}{\Gamma(\rho)}, \qquad t = \psi + k$$

$$\Rightarrow \qquad \frac{\mathrm{d}r}{r} = \frac{h(\rho)\,\mathrm{d}\rho}{\rho} \qquad \Rightarrow \qquad r = r_0 \, e^{\int \mathrm{d}\rho \, \frac{h(\rho)}{\rho}}.$$

Thus, Taub-NUT resembles Bertrand spacetime with magnetic fields. We can go the other way, starting with the generalized Taub-NUT metric and proceeding towards Bertrand spacetime by applying appropriate potential power laws [27].

Therefore, like the BSTs, there are two Taub-NUT configurations

1. Hooke's Oscillator configuration

$$f_{\mathcal{O}}(r) = ar^2 + b, \qquad g_{\mathcal{O}}(r) = \frac{r^2\big(ar^2 + b\big)}{cr^4 + dr^2 + 1}$$

2. Kepler's orbital configuration

$$f_{\mathcal{K}}(r) = \frac{a + br}{r}, \qquad g_{\mathcal{K}}(r) = \frac{r\big(a + br\big)}{cr^2 + dr + 1}.$$

The duality between these two configurations of the metric can be demonstrated. To study Taub-NUT space duality, we confine motion to a cone ($\theta = \text{const}$) because of the conserved angular momentum (16), for which [26–28]

$$\boldsymbol{J}.\boldsymbol{e}^{\boldsymbol{r}} = \big|\boldsymbol{J}\big| \cos\theta = \text{const} \qquad \Rightarrow \qquad \theta = \text{const}. \tag{25}$$

This allows us to reduce the problem to two-dimensions by rendering θ a constant co-ordinate, allowing us to write the metric using $\alpha = \sin\theta, \beta = \cos\theta$

$$\mathrm{d}s^2 = f(r)\big(\mathrm{d}r^2 + r^2\alpha^2\, \mathrm{d}\phi^2\big) + g(r)\big(\mathrm{d}\psi + \beta\, \mathrm{d}\phi\big)^2. \tag{26}$$

We represent the co-ordinates as $z = x + \mathrm{i}y,\ \xi = X + \mathrm{i}Y$, where $|z| = r\cos\frac{\theta}{2}\cdot$ The complex co-ordinates in these spaces [14], where ($\theta = \text{const}$) (25) are

$$z = |z| \exp\left[\frac{\mathrm{i}}{2}\big(\psi + \phi\big)\right], \qquad \xi = |\xi| \exp\left[\frac{\mathrm{i}}{2}\big(\chi + \Phi\big)\right], \qquad Z, \xi \in \mathbb{C}$$

$$\begin{aligned} z \to \xi = z^2 \qquad &\Rightarrow \qquad |z|^2 \exp\big[\mathrm{i}\big(\psi + \phi\big)\big] = |\xi| \exp\left[\frac{\mathrm{i}}{2}\big(\chi + \Phi\big)\right] \\ &\Rightarrow \qquad \phi \to \Phi = 2\phi, \qquad \psi \to \chi = 2\psi \end{aligned} \tag{27}$$

So Bohlin's transformation (27) [4] on the Oscillator metric from (26) gives

$$\big(\mathrm{d}s^2\big)_{\mathcal{O}} = \big(a|z|^2 + b\big)|\mathrm{d}z|^2 + \frac{|z|^2(a|z|^2 + b)}{c|z|^4 + d|z|^2 + 1}\big(\mathrm{d}\psi + \beta\, \mathrm{d}\phi\big)^2$$

$$z \to \xi = z^2, \quad \phi \to \Phi = 2\phi, \quad \psi \to \chi = 2\psi$$

$$\big(\mathrm{d}s^2\big)_{\mathcal{O}} = \frac{1}{4}\left\{\frac{a|\xi| + b}{|\xi|}|\mathrm{d}\xi|^2 + \frac{|\xi|\big(a|\xi| + b\big)}{c|\xi|^2 + d|\xi| + 1}\big(\mathrm{d}\chi + \beta\, \mathrm{d}\Phi\big)^2\right\}. \tag{28}$$

Comparing (28) with the Kepler system in presence of magnetic fields

$$\big(\mathrm{d}s^2\big)_{\mathcal{K}} = \frac{b|z| + a}{|z|}|\mathrm{d}z|^2 + \frac{|z|\big(b|z| + a\big)}{c|z|^2 + d|z| + 1}\big(\mathrm{d}\psi + \beta\, \mathrm{d}\phi\big)^2$$

shows that aside from a factor of $\frac{1}{4}$, a variable swap $a \leftrightarrow b$ completes the transformation, and thus, the two configurations of Taub-NUT are related via Bohlin's transformation like Bertrand spacetime. For various settings of the constants, we get different configurations of spacetime, as shown in the following table.

Table 1. Systems for various settings, $\mathcal{K}$ - Kepler, $\mathcal{O}$ - Oscillator.

Type	a	b	c	d	$f(r)$	$g(r)$	**System Name**
$\mathcal{K}$	0	1	1	-2	1	$\frac{r^2}{(1-r)^2}$	MIC-Zwangier
$\mathcal{K}$	0	1	0	$-\frac{2k}{q^2}$	1	$\frac{r^2}{1-\frac{2k}{q^2}r}$	MIC-Kepler
$\mathcal{O}$	0	1	$\frac{k}{q^2}$	0	1	$\frac{r^2}{1+\frac{k}{q^2}r^4}$	MIC-Oscillator
$\mathcal{K}$	$4m$	1	0	$\frac{1}{4m}$	$\frac{4m+r}{r}$	$\frac{(4m)^2 r}{4m+r}$	Euclidean Taub-NUT

3.2. Kepler-Oscillator Duality

In the study of central force problem, we learn that the Kepler and Oscillator systems are dual to each other according a duality map demonstrated in [42] and [22]. This is summed up in Bertrand's theorem, describing them as the only systems with stable, closed and periodic orbits. Thus, curved Bertrand space-times are classified as Type I and Type II, representing Kepler and Oscillator systems respectively.

If we start with the two-dimensional simple harmonic oscillator equation $\ddot{x}^i = -\omega^2 x^i$, we are reminded of a conserved tensorial quantity, known as the Fradkin tensor

$$T^{ij} = p^i p^j + \kappa x^i x^j, \qquad i,j = 1,2. \tag{29}$$

Any conserved quantity can be obtained by contracting the Fradkin tensor (29) over its two indices by any chosen structure, i.e.,

$$Q = M_{ij} T^{ij}.$$

This quantity is symmetric under index permutation. Its complex counterpart is

$$T_{z^a z^b} = G^{ij}_{z^a z^b} T_{ij}, \qquad z^a = \{z, \bar{z}\}$$

$$G_{zz} = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}, \qquad G_{\bar{z}\bar{z}} = \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}, \qquad G_{z\bar{z}} = G_{\bar{z}z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

According to the Arnold-Vasiliev duality [1], a co-ordinate transformation and re-parametrization of the first two complex Fradkin tensors will give us the Laplace-Runge-Lenz vector

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} + \beta \frac{\boldsymbol{x}}{r} \cdot \tag{30}$$

In tensorial form, (30) is written as follows

$$A_i = \varepsilon_{ikl} \varepsilon^l{}_{jm} p^k x^j p^m + \frac{\beta}{r} \delta_{ij} x^j = x^j \Big\{ \big(\delta_{ij} \delta_{km} - \delta_{ik} \delta_{jm} \big) p^k p^m + \frac{\beta}{r} \delta_{ij} \Big\}.$$

Showing that the first term can be expressed as quadratic in momenta. Since it is a linear combination of Fradkin tensor components, we prefer it being symmetric in the momentum indices like its oscillator counterpart. Thus, we can write

$$A_i = x^j \left\{ \frac{1}{2} \big(2\delta_{ij} \delta_{km} - \delta_{ik} \delta_{jm} - \delta_{im} \delta_{jk} \big) p^k p^m + \frac{\beta}{r} \delta_{ij} \right\}. \tag{31}$$

Hence, to describe this first-integral of the Kepler system, we need tensors that are:

1. quadratic in momenta
2. symmetric under index permutation
3. conserved along geodesics.

We will explore such tensors in the next section.

4. A Review of Geometric Properties

An instanton or pseudo-particle is a concept in mathematical physics describing solutions to equations of motion of classical field theory on Euclidean spacetime. The first such solutions discovered were localized in spacetime, hence, named instanton or pseudoparticle. They are important in quantum field theory because

1. They are leading quantum corrections to classical equations in path integral
2. They are useful for studying tunneling in systems like Yang-Mills theory.

While considering the Taub-NUT metric defined on a four-dimensional Euclidean space, it is worth checking if it is an instanton. In this section, we will analyze its geometrical properties exhaustively, verify if Taub-NUT is an instanton from the curvature computed from the metric, and also look at its topological properties.

Taub-NUT (10) under variable transformation $m = 2M$ and $r \longrightarrow r - m$ is

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\mathrm{d}r^2 + 4m^2\frac{r-m}{r+m}\big(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\big)^2 + \big(r^2 - m^2\big)\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big). \tag{32}$$

This can be further recast into the form

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\mathrm{d}r^2 + 4m^2\frac{r-m}{r+m}\sigma_1^2 + \big(r^2 - m^2\big)\big(\sigma_2^2 + \sigma_3^2\big) \tag{33}$$

where the variables σ_i are essentially solid angle elements in four-dimensional Euclidean space obeying the following structure equation

$$\mathrm{d}\sigma^i = -\varepsilon^i{}_{jk}\,\sigma^j \wedge \sigma^k, \qquad \sigma^i = -\frac{1}{r^2}\eta^i_{\mu\nu}x^\mu \mathrm{d}x^\nu. \tag{34}$$

Having identified the vierbeins, we implement Cartan's method of computing spin connections and the Riemann curvature components. Embedded within them are the $\mathrm{SU}(2)$ gauge fields and their corresponding field strengths as we shall see.

4.1. Taub-NUT as a Darboux-Halphen System

The Taub-NUT is a special case of self-dual Bianchi-IX metrics [5], characterized by the classical Darboux-Halphen system. The self-dual Bianchi-IX metric is

$$\mathrm{d}\widetilde{s}^2 = \big(\Omega_1\Omega_2\Omega_3\big)\mathrm{d}\widetilde{r}^2 + \frac{\Omega_2\Omega_3}{\Omega_1}\big(\sigma_1\big)^2 + \frac{\Omega_3\Omega_1}{\Omega_2}\big(\sigma_2\big)^2 + \frac{\Omega_1\Omega_2}{\Omega_3}\big(\sigma_3\big)^2 \tag{35}$$

and its characteristic classical Darboux-Halphen system of equations are

$$\begin{aligned} \Omega_1' &= \Omega_2\Omega_3 - \Omega_1\big(\Omega_2 + \Omega_3\big) \\ \Omega_2' &= \Omega_3\Omega_1 - \Omega_2\big(\Omega_3 + \Omega_1\big) \\ \Omega_3' &= \Omega_1\Omega_2 - \Omega_3\big(\Omega_1 + \Omega_2\big), \qquad (\,)' = \frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}(\,) \end{aligned} \tag{36}$$

where Ω_i are parameters defined to re-write Bianchi-IX metric into the form (35) to write self-dual equations. One particular first integral of the this system [7] is

$$Q = \frac{(\Omega_1)^2}{(\Omega_3 - \Omega_1)(\Omega_1 - \Omega_2)} + \frac{(\Omega_2)^2}{(\Omega_1 - \Omega_2)(\Omega_2 - \Omega_3)} + \frac{(\Omega_3)^2}{(\Omega_2 - \Omega_3)(\Omega_3 - \Omega_1)}. \tag{37}$$

In case of the Taub-NUT, we need to set $\Omega_2 = \Omega_3 = \Omega \neq \Omega_1 = \Lambda$ in (35) and (36). This way, we will get the following metric and system of equations

$$\mathrm{d}\widetilde{s}^2 = \Omega^2\Lambda\,\mathrm{d}\widetilde{r}^2 + \Lambda\big(\big(\sigma_2\big)^2 + \big(\sigma_3\big)^2\big) + \frac{\Omega^2}{\Lambda}\big(\sigma_1\big)^2 \tag{38}$$

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}\widetilde{r}} = \Omega(\Omega - 2\Lambda), \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}\widetilde{r}} = -\Omega^2. \tag{39}$$

Under the limit $\Omega_2 \to \Omega_3 = \Omega$, the conserved quantity Q from (37) becomes

$$\begin{aligned}\left[\lim_{\Omega_2\to\Omega_3=\Omega} Q\right]_{\Omega_1=\Lambda} &= -\frac{\Lambda^2}{(\Lambda-\Omega)^2} + \frac{1}{\Lambda-\Omega}\left[\lim_{\Omega_2\to\Omega_3=\Omega}\left(\frac{(\Omega_2)^2}{\Omega_2-\Omega_3} - \frac{(\Omega_3)^2}{\Omega_2-\Omega_3}\right)\right] \\ &= -\frac{\Lambda^2}{(\Lambda-\Omega)^2} + \frac{2\Omega}{\Lambda-\Omega} = -1 - \left(\frac{\Omega}{\Lambda-\Omega}\right)^2. \end{aligned} \tag{40}$$

Rescaling the radius and solving (39) with suitable constants of integration gives

$$\mathrm{d}\widetilde{r} = -\frac{\mathrm{d}r}{2m\Omega^2}, \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}r} = \frac{1}{2m}, \qquad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\Lambda}{\Omega^2}\right) = -\frac{1}{\Omega^2}\frac{\mathrm{d}\Omega}{dr}$$

$$\Omega = \frac{r-m}{2m}, \qquad \Lambda = \frac{r^2-m^2}{4m^2}. \tag{41}$$

Rescaling the metric (38) as $\mathrm{d}s = 2m\ \mathrm{d}\widetilde{s}$ and applying (41) gives us tthe Taub-NUT (33)

$$\mathrm{d}s^2 = \frac{r+m}{r-m}\mathrm{d}r^2 + 4m^2\frac{r-m}{r+m}\left(\sigma_1\right)^2 + (r^2-m^2)\left[\left(\sigma_2\right)^2 + \left(\sigma_3\right)^2\right]$$

and the conserved quantity (40) becomes

$$\left[\lim_{\Omega_2\to\Omega_3=\Omega} Q\right]_{\Omega_1=\Lambda} = -1 - \left(\frac{\Omega}{\Lambda-\Omega}\right)^2 = -\frac{r^2-2mr+5m^2}{(r-m)^2}. \tag{42}$$

This concludes another possible symmetry of the Taub-NUT as a member of Bianchi-IX metrics or solutions to Darboux-Halphen systems.

4.2. Curvature and Anti-Self Duality

Now that we have identified the individual vierbeins, we shall proceed to compute the spin connections. We can describe the vierbeins as

$$e^0 = c_0(r)\,\mathrm{d}r, \qquad e^i = c_i(r)\sigma^i, \qquad i = 1, 2, 3.$$

Obviously, e^0 produces no connection terms $(\mathrm{d}e^0 = 0)$. Under torsion-free condition the first Cartan structure equation $(\mathrm{d}e^i = -\omega^i{}_j \wedge e^j)$ gives the spin connections.

$$\omega^i{}_0 = \frac{\partial_r c_i}{c_0}\,\sigma^i, \qquad \omega^i{}_j = -\varepsilon^i{}_{jk}\frac{c_i^2 + c_j^2 - c_k^2}{2c_ic_j}\,\sigma^k.$$

The elaborate form of the spin connections is used to keep it anti-symmetric. We therefore construct the spin-connection matrix as shown below

$$\omega = \begin{pmatrix} 0 & -\frac{2m^2}{(r+m)^2}\sigma^1 & -\left(1-\frac{m}{r+m}\right)\sigma^2 & -\left(1-\frac{m}{r+m}\right)\sigma^3 \\ \frac{2m^2}{(r+m)^2}\sigma^1 & 0 & -\frac{m}{r+m}\sigma^3 & \frac{m}{r+m}\sigma^2 \\ \left(1-\frac{m}{r+m}\right)\sigma^2 & \frac{m}{r+m}\sigma^3 & 0 & -\left(1-\frac{2m^2}{(r+m)^2}\right)\sigma^1 \\ \left(1-\frac{m}{r+m}\right)\sigma^3 & -\frac{m}{r+m}\sigma^2 & \left(1-\frac{2m^2}{(r+m)^2}\right)\sigma^1 & 0 \end{pmatrix}. \quad (43)$$

If we view spin connections as a linear combination of self dual and anti-self dual tensors, then we can accordingly seperate out the self and anti-self dual components as $\omega_{ij} = \omega_{ij}^{(+)} + \omega_{ij}^{(-)}$. To this end, we can split the spin connection matrix (43) into two separate components: the self dual and the anti-self dual parts

$$\begin{aligned} \omega^{(+)} &= -\frac{1}{2}\left(\sigma^1\eta_1 + \sigma^2\eta_2 + \sigma^3\eta_3\right) = -\frac{1}{2}\sigma^i\eta_i \\ \omega^{(-)} &= \left\{\left(\frac{1}{2} - \frac{2m^2}{(r+m)^2}\right)\sigma^1\bar{\eta}_1 - \left(\frac{1}{2} - \frac{m}{r+m}\right)\left(\sigma^2\bar{\eta}_2 - \sigma^3\bar{\eta}_3\right)\right\}. \end{aligned} \quad (44)$$

For reference, the t'Hooft symbol matrices $\eta^{(\pm)}$ exhibit the $\mathrm{SU}(2)$ Lie algebra

$$\left[\eta_i, \eta_j\right] = -2\varepsilon_{ij}{}^{k}\eta_k.$$

The curvature tensor can be decomposed into self and anti-self dual components $R_{ij} = R_{ij}^{(+)} + R_{ij}^{(-)}$, where by Cartan's second equation, $R = \mathrm{d}\omega + \omega \wedge \omega$. Thus, we can write the spin connections as a linear combination of self and anti-self dual t'Hooft symbols giving self-dual and anti-self dual spin connections described in (44). Consequently, according to (34), the self-dual curvature vanishes

$$R^{(+)} = \mathrm{d}\omega^{(+)} + \omega^{(+)} \wedge \omega^{(+)} = -\frac{1}{2}\left(\mathrm{d}\sigma^i + \varepsilon^i{}_{jk}\,\sigma^j \wedge \sigma^k\right)\eta_i = 0. \quad (45)$$

Only the anti-self dual curvature remains, reflecting the Taub-NUT's anti-self duality. We make our job easier by writing the spin connection as $\omega^{(-)} = \omega_1^{(-)} + \omega_2^{(-)}$

$$\omega^{(-)} = \frac{1}{2}\left(\sigma^1\bar{\eta}_1 - \sigma^2\bar{\eta}_2 + \sigma^3\bar{\eta}_3\right) + \left(-\frac{2m^2}{(r+m)^2}\sigma^1\bar{\eta}_1 + \frac{m}{r+m}\left(\sigma^2\bar{\eta}_2 - \sigma^3\bar{\eta}_3\right)\right)$$

where one can verify that $\omega_1^{(-)}$ will follow the same rule as $\omega^{(+)}$ in (45). This allows us to compute the anti-self-dual curvature given by

$$\begin{aligned} R^{(-)} = &\frac{2m}{(r+m)^3}\bar{\eta}_1\left(e^0 \wedge e^1 - e^2 \wedge e^3\right) \\ &+ \frac{m}{(r+m)^3}\left(-\bar{\eta}_2\left(e^0 \wedge e^2 - e^3 \wedge e^1\right) + \bar{\eta}_3\left(e^0 \wedge e^3 - e^1 \wedge e^2\right)\right) \end{aligned} \quad (46)$$

where we can see from the signs attached to the dual components that the curvature derived from Taub-NUT metric is clearly anti-self dual, as shown in [37]. This also confirms that it is an instanton. To elaborate, we can show that only $\mathrm{SU}(2)_-$ gauge fields are embedded within the spin-connection components as shown below

$$
\begin{aligned}
\omega^{(\pm)}_{\mu\nu} &= \eta^{(\pm)k}_{\mu\nu} A^{(\pm)}_k \qquad\qquad \Rightarrow \qquad\qquad A^{(\pm)i} = \frac{1}{4}\eta^{(\pm)i}_{\mu\nu}\omega_{\mu\nu} \\
A^{(+)1} &= -\frac{\sigma^1}{2}, \qquad A^{(-)1} = \left(1 - \frac{4m^2}{(r+m)^2}\right)\frac{\sigma^1}{2} \\
A^{(+)2} &= -\frac{\sigma^2}{2}, \qquad A^{(-)2} = -\frac{r-m}{r+m}\frac{\sigma^2}{2} \\
A^{(+)3} &= -\frac{\sigma^3}{2}, \qquad A^{(-)3} = \frac{r-m}{r+m}\frac{\sigma^3}{2}
\end{aligned}
\tag{47}
$$

while the field strengths are given by

$$
\begin{aligned}
R^{(-)}_{\mu\nu} &= \eta^{(-)k}_{\mu\nu} F^{(-)}_k \qquad\qquad \Rightarrow \qquad\qquad F^{(\pm)i} = \frac{1}{4}\eta^{(\pm)i}_{\mu\nu} R_{\mu\nu} \\
F^{(-)1} &= R_{01} = -R_{23} = \frac{2m}{(r+m)^3}\left(e^0 \wedge e^1 - e^2 \wedge e^3\right) \\
F^{(-)2} &= R_{02} = -R_{31} = -\frac{m}{(r+m)^3}\left(e^0 \wedge e^2 - e^3 \wedge e^1\right) \\
F^{(-)3} &= R_{03} = -R_{12} = \frac{m}{(r+m)^3}\left(e^0 \wedge e^3 - e^1 \wedge e^2\right)
\end{aligned}
\tag{48}
$$

where it is obvious that due to the absence of self-dual curvature, there are no $\mathrm{SU}(2)_+$ gauge fields, i.e., $F^{(+)i} = 0$ and thus field strengths are anti-self dual ($F = - * F$) which of course, coincide with the curvature tensor (46). In terms of two-forms, the independent components are given by

$$
\begin{aligned}
R^{(-)}_{0101} &= R^{(-)}_{2323} = -R^{(-)}_{0123} = \frac{2m}{(r+m)^3} \\
R^{(-)}_{0202} &= R^{(-)}_{1313} = R^{(-)}_{0213} = -\frac{m}{(r+m)^3} \\
R^{(-)}_{0303} &= R^{(-)}_{1212} = -R^{(-)}_{0213} = -\frac{m}{(r+m)^3}.
\end{aligned}
$$

This lets us compute the Ricci tensors and scalar in accordance with the formula:

$$
\begin{aligned}
\mathcal{R}_{ik} &= g^{jl}\mathcal{R}_{ijkl} = \delta^{jl}\mathcal{R}_{ijkl}, \qquad\qquad \mathcal{R} = \delta^{ik}\mathcal{R}_{ik} \\
\therefore \quad \mathcal{R}_{00} &= \mathcal{R}_{11} = \mathcal{R}_{22} = \mathcal{R}_{33} = 0, \qquad \mathcal{R} = 0.
\end{aligned}
$$

Since the Ricci tensors vanish, the Taub-NUT is clearly a vaccum solution of Einstein's equations.

4.3. Topological Invariants

Topological invariants are analogous to an overall charge distributed in the manifold. In the gravity side, there are two topological invariants associated with the Atiyah-Patodi-Singer index theorem for a four dimensional elliptic complex [3,13], the Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$, which can be expressed as integrals of four-manifold curvature.

Recall that in electromagnetic theory, the field action is given by

$$S = -\frac{1}{16\pi}\int \mathrm{d}\Omega\, F_{ij}F^{ij} = -\frac{1}{16\pi}\int F\wedge F$$

$$\text{where}\quad F = \frac{1}{2}F_{ij}\mathrm{d}x^i\wedge\mathrm{d}x^j \qquad \text{and} \qquad \varepsilon^{ijkl}\mathrm{d}\Omega = \mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k\wedge\mathrm{d}x^l.$$

The equations of motion are obtained by solving for minimum variation of the electromagnetic field action. We merely apply these equations to compute topological invariants as integrals analogous to action. We can write the general lagrangian

$$\mathcal{L} = c^{abcd}R_{ab}\wedge R_{cd} = c^{abcd}F^{(\pm)m}_{ab}F^{(\pm)n}_{cd}\eta^{(\pm)m}_{ij}\eta^{(\pm)n}_{kl}\varepsilon^{ijkl}\mathrm{d}\Omega$$

$$\mathcal{L} = \pm 2\mathrm{d}\Omega c^{abcd}F^{(\pm)m}_{ab}\partial_c A^{(\pm)m}_d \tag{49}$$

where $\varepsilon^{ijkl}\mathrm{d}\Omega = e^i\wedge e^j\wedge e^k\wedge e^l$. Applying Lagrange's equation to (49) gives the contracted Bianchi identity for curvature

$$\partial_c\left(\frac{\partial\mathcal{L}}{\partial(\partial_c A^{(\pm)m}_d)}\right) = \pm 2c^{abcd}\partial_c F^{(\pm)m}_{ab} = 0. \tag{50}$$

Conversely, the Bianchi identity for $\mathrm{SU}(2)_\pm$ gauge fields the root of topological invariance. One can verify this starting from (50) and work backwards to obtain the invariants.

Since the boundary integral vanishes, the overall invariant is computed only from the bulk part. For non-compact manifolds like Taub-NUT, there are additional boundary terms not separated into self-dual or anti-self-dual parts unlike the volume terms. They are the eta-invariant $\eta_S(\partial M)$, given for k self-dual gravitational instantons by [19]

$$\eta_S(\partial M) = -\frac{2\epsilon}{3k} + \frac{(k-1)(k-2)}{3k} \qquad \begin{cases}\epsilon = 0, & \text{ALE boundary conditions}\\ \epsilon = 1, & \text{ALF boundary conditions.}\end{cases}$$

Since Taub-NUT is an ALF hyper-kahler four-manifold it has a non-vanishing eta-invariant which is equal to $-\frac{2}{3}$. According to [39], upon applying curvature of

(46), the Euler characteristic χ and the Hirzebruch signature complex τ are

$$\chi(M) = \frac{1}{32\pi^2}\int_M \varepsilon^{abcd} R_{ab}\wedge R_{cd} = 1 \tag{51}$$

$$\begin{aligned} \tau_{bulk}(M) &= -\frac{1}{12\pi^2}\bigg(\int_M R_{ab}\wedge R_{ab}\bigg)_{a<b} = \frac{2}{3} \\ \therefore \qquad \tau(M) &= \tau_{bulk}(M) + \eta_S(\partial M) = 0. \end{aligned} \tag{52}$$

One could say that the form of topological invariants can enerally be written as

$$\begin{aligned} \mathcal{C}(M) &= \frac{1}{k\pi^2}\int_M c^{abcd} R_{ab}\wedge R_{cd} \\ &= \begin{cases} \dfrac{1}{k\pi^2}\int_M F_{ab}\big(*F^{ab}\big), \quad c^{abcd} = \varepsilon^{abcd} \ \big(\text{Euler Characteristic}\big) \\ \dfrac{1}{k\pi^2}\int_M F_{ab}F^{ab}, \quad c^{abcd} = g^{ac}g^{bd} \ \big(\text{Hirzebruch Signature}\big). \end{cases} \end{aligned} \tag{53}$$

where c^{abcd} is contracting tensor defined in respect to the relevant circumstances.

5. Killing-Yano Tensors and the Taub-NUT Metric

There are tensors quadratic in momenta and conserved along geodesics, expressed as a vector $\boldsymbol{K}$ whose components transform among themselves under three-dimensional rotations. They are similar to the Runge-Lenz vector in the Kepler problem with components

$$K^{(i)} = \frac{1}{2}K^{(i)\mu\nu}p_\mu p_\nu. \tag{54}$$

Provided that $J^0 \neq 0$, such vectors usually satisfy the following property

$$\boldsymbol{r}.\bigg(\boldsymbol{K} \pm \frac{H\boldsymbol{J}}{J^0}\bigg) = \frac{1}{2}\Big(\boldsymbol{J}^2 - \big(J^0\big)^2\Big) \tag{55}$$

where if $(J^0, \boldsymbol{J}, H, \boldsymbol{K})$ are all constant, the three-dimensional position vector $\boldsymbol{r}$ lies in a plane. Using (55) and the relation $J^0 = \dfrac{\boldsymbol{r}.\boldsymbol{J}}{r}$, we can see that

$$\boldsymbol{r}.\boldsymbol{K} = \mp rH + \frac{1}{2}\Big(\boldsymbol{J}^2 - \big(J^0\big)^2\Big). \tag{56}$$

In Taub-NUT geometry, there are also 4 antisymmetric Killing tensors known as Killing-Yano tensors. Three of these are complex structures, realizing quaternionic algebra since the Taub-NUT manifold is hyper-Kähler. The fourth is a scalar with a non- vanishing field strength, existing by virtue of the metric being of Petrov type D. Their existence is implied by a triplet of symmetric second rank Killing tensors called the Stäckel-Killing tensor satisfying

$$D_{(\lambda} K^{(i)}_{\mu\nu)} = 0. \tag{57}$$

We will examine properties of Killing-Yano tensors relevant for studying Taub-NUT symmetries after listing references that initiated the study of such symmetries.

Dynamical symmetries of the Kaluza Klein monopole were discussed in detail by Feher in [15]. The dynamics of two non-relativistic BPS monopoles was described using Atiyah-Hitchin metric (Taub-NUT being a special case), the corresponding $\mathrm{O}(4)/\mathrm{O}(3,1)$ symmetry discovered in [18], and applied to calculate the underlined motion group-theoretically in [16]. The symmetry was then extended to $\mathrm{O}(4,2)$ in [20] and [10]. In [20] Gibbons *et al* discussed dynamical symmetries of multi-centre metrics and applied the results to the scattering of BPS monopoles and fluctuations around them, giving a detailed account of the hidden symmetries of the Taub-NUT. The hidden symmetries in large-distance interactions between BPS monopoles and of the fluctuations around them are traced to the existence of a Killing-Yano tensor on the self-dual Taub-NUT. The global action on classical phase space of these symmetries was discussed in [21] and the quantum picture involving the"dynamical groups" $\mathrm{SO}(4)$, $\mathrm{SO}(4,1)$ and $\mathrm{SO}(4,2)$ was also given. A comprehensive review of the dynamical symmetry can be found in [15]. Supersymmetry and extension to spin has also been studied in [9,25].

5.1. Yano and Stäckel Tensors

We can construct these Killing-Yano tensors in terms of simpler objects known as Yano tensors that are antisymmetric rank two tensors satisfying the Killing like equation. Thus, the covariant derivative is antisymmetric over permutations of all possible pairs of indices. This allows us to write the covariant derivative of the Yano tensor in terms of the cyclic permutations as

$$f_{\mu\nu} = -f_{\nu\mu}, \qquad \nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\mu\lambda} = 0 \tag{58}$$

$$\nabla_\mu f_{\nu\lambda} = \nabla_\nu f_{\lambda\mu} = \nabla_\lambda f_{\mu\nu} = \nabla_{[\mu} f_{\nu\lambda]} = \frac{1}{3}\big(\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu}\big). \tag{59}$$

We can construct symmetric rank two Killing tensors by symmetrized multiplication

$$K^{(ab)}_{\mu\nu} = \frac{1}{2}\big(f^{(a)\lambda}_{\mu} f^{(b)}_{\lambda\nu} + f^{(b)\lambda}_{\mu} f^{(a)}_{\lambda\nu}\big) \equiv \frac{1}{2}\big(f^{(a)\lambda}_{\mu} f^{(b)}_{\lambda\nu} + f^{(a)\lambda}_{\nu} f^{(b)}_{\lambda\mu}\big) = K^{ab}_{(\mu\nu)} \tag{60}$$

These symmetric Killing tensors satisfy the Killing-Yano condition (57). The Taub-NUT manifold admits 4 such Killing-Yano tensors, given by a scalar f^0 and three components that transform as a vector $f^i \;\forall\; i = 1,2,3$. We can form triplets of symmetric Killing tensors as in (60), given by setting $a = 0$ and $b = i$

$$K^{(i)}_{\mu\nu} = K^{(0i)}_{\mu\nu} = \frac{1}{2}\big(f^{0\lambda}_{\mu} f^{i}_{\lambda\nu} + f^{i\lambda}_{\mu} f^{0}_{\lambda\nu}\big), \qquad i = 1,2,3. \tag{61}$$

Using (58) we can see how they obey (57) as follows

$$\nabla_\gamma K^{ij}_{(\mu\nu)} + \nabla_\mu K^{ij}_{(\nu\gamma)} + \nabla_\nu K^{ij}_{(\gamma\mu)} = 0$$

$$\nabla_{(\gamma} K^{ij}_{\mu\nu)} = 0 \qquad \Rightarrow \qquad \nabla_{(\gamma} K^{i}_{\mu\nu)} \equiv \nabla_{(\gamma} K^{0i}_{\mu\nu)} = 0. \tag{62}$$

Thus, we are assured that (57) is satisfied by this symmetric Killing tensor. This allows construction of tensors of (57) that are quadratic in momenta, showing how to get Stäckel tensors from Killing-Yano tensors. However, since Killing-Yano tensors are anti-symmetric, they cannot form polynomials with components of the same vector. They must be mixed products of components of different vectors, as in the case of angular momentum, a product with a position and a momentum component each. Applying Holten's algorithm yields the Killing equation in (58).

5.2. Euclidean Taub-NUT

The Taub-NUT metric [44] admits four Yano tensors written as two-forms

$$f^0 = 4\big(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\big) \wedge \mathrm{d}r + 2r\big(r \pm 1\big)\big(r \pm 2\big) \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi \tag{63}$$

$$f^i = \pm 4\big(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\big) \wedge \mathrm{d}x^i - \varepsilon^{i}{}_{jk} f(r)\,\mathrm{d}x^j \wedge \mathrm{d}x^k. \tag{64}$$

One can always find Killing tensors embedded within conserved quantities, as evident from the Poisson Brackets of any conserved quantity expanded ala Holten algorithm. The coefficient from Laplace-Runge-Lenz vector is analogous to the Killing-Stäckel tensor K_{ij}, so we can argue

$$Q^{(2)} = K_{ij}\Pi^i\Pi^j \equiv \frac{1}{2} C^{(2)}_{ij}\Pi^i\Pi^j.$$

Now the angular momentum co-efficients according to (22) are

$$C^{(0)} = q\, g_{jk}(\boldsymbol{x})\,\frac{x^j}{r}\theta^k, \qquad C^{(1)}_i = -g_{im}(\boldsymbol{x})\varepsilon^{m}{}_{jk}\theta^k x^j.$$

If we write $C^{(1)}_i = f_{ik}\theta^k$ (see Section 8.1), using Holten's Algorithm gives

$$\nabla_j C^{(1)}_i = \nabla_j f_{ik}\theta^k = -g_{im}(\boldsymbol{x})\varepsilon^{m}{}_{jk}\theta^k$$

$$\nabla_i C^{(1)}_j + \nabla_j C^{(1)}_i = 0 \qquad \Rightarrow \qquad \big(\nabla_i f_{jk} + \nabla_j f_{ik}\big)\theta^k = 0$$

which is the Killing equation (58). Thus, we can say that the Killing-Yano tensor is

$$\begin{aligned} f^0_{jk} &= g_{jk}(\vec{x}) \qquad &\Rightarrow \qquad f^j_{0k} &= \delta^j_k \\ f^i_{jk} &= \varepsilon^i{}_{jk} \qquad &\Rightarrow \qquad f^i &= \varepsilon^i{}_{jk} e^j \wedge e^k \end{aligned}$$

such that the square of it gives the Stäckel tensor

$$K^k_{ij} = f^0{}_{im} f^{km}{}_j.$$

This shows how Killing tensors are embedded within the conserved quantities. We can choose four combinations of three indices out of the available four. Since Taub-NUT can alternately be written in the form given by (10), the vierbeins of the metric are given by

$$e^0 = \frac{4\big(\mathrm{d}\psi + \boldsymbol{A}.\mathrm{d}\boldsymbol{x}\big)}{\sqrt{f(r)}}, \qquad e^i = \sqrt{f(r)}\,\mathrm{d}x^i. \tag{65}$$

So, according to our theory, we should have

$$\begin{aligned} f^i &= -\varepsilon^i{}_{jk}\, e^j \wedge e^k + \delta^i{}_k\, e^0 \wedge e^k \\ &= -\varepsilon^i{}_{jk} f(r)\, \mathrm{d}x^j \wedge \mathrm{d}x^k \pm 4\big(\mathrm{d}\psi + \boldsymbol{A}.\mathrm{d}\boldsymbol{x}\big) \wedge \mathrm{d}x^i. \end{aligned}$$

This result so far is comparable with the result (64), so we have a possible method for constructing Killing-Yano tensors from the coefficients of conserved quantities. Their covariant exterior derivatives and their properties are given by

$$\begin{aligned} Df^0 &= \nabla_\gamma f^0_{\mu\nu}\, \mathrm{d}x^\gamma \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = r\big(r \pm 2\big) \sin\theta\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \\ Df^i &= 0, \qquad\qquad i = 1, 2, 3. \end{aligned}$$

From above, we infer that covariant derivatives hold the following properties

$$\nabla_\gamma f^0_{\mu\nu} = \nabla_\mu f^0_{\nu\gamma} = \nabla_\gamma \nu f^0_{\gamma\mu}, \qquad \nabla_\gamma f^i_{\mu\nu} = 0, \qquad i = 1, 2, 3.$$

Showing that they obey the condition for covariant derivatives of Killing-Yano tensors. As shown in (61), these tensors can form a symmetric triplet or a vector of Killing tensors. They also exhibit the mutual anti-commutation property

$$f^i f^j = -\delta^{ij} + \varepsilon^{ij}{}_k f^k \qquad \begin{cases} \big\{f^i, f^j\big\} = f^i f^j + f^j f^i = -2\delta^{ij} \\ \big[f^i, f^j\big] = f^i f^j - f^j f^i = 2\varepsilon^{ij}{}_k f^k. \end{cases} \tag{66}$$

Proving that they are complex structures realizing the quaternion algebra. This implies that f^i are objects in quaternionic geometry and possibly hyperkähler structures, which we shall examine for the Taub-NUT in the next section.

5.3. Graded Lie-Algebra via Schouten-Nijenhuis Brackets

We will now see if Killing-Yano tensors of the Taub-NUT exhibit Lie algebra under the action of Schouten-Nijenhuis brackets. If they do, we could form higher order Killing-Yano tensors from lower order ones of rank above 1. It is noteworthy, in this context, that Kastor *et al* already found that Killing-Yano tensors on constant curvature spacetimes do form Lie algebras with respect to the Schouten-Nijenhuis Bracket (SNB) [29].

The SNB is a bracket operation between multivector fields which for two such fields $A = A^{i_1 i_2 \dots i_m} \bigwedge_{k=1}^{m} \partial_{i_k}$, $B = B^{j_1 j_2 \dots j_n} \bigwedge_{k=1}^{n} \partial_{j_k}$, is given by

$$\begin{aligned} C^{a_1 \dots a_{m+n-1}} &= \big[A, B\big]_{SN}^{a_1 \dots a_{m+n-1}} \\ &= mA^{c[a_1 \dots a_{m-1}} \nabla_c B^{a_m \dots a_{m+n-1}]} \\ &\qquad + n\big(-1\big)^{mn} B^{c[a_1 \dots a_{n-1}} \nabla_c A^{a_n \dots a_{m+n-1}]}. \end{aligned} \tag{67}$$

This new tensor is completely antisymmetric, fulfilling the first requirement to be considered a Killing-Yano tensor. All that remains is for its covariant derivative to exhibit the same Killing equation (59) relevant to such tensors. Now, we will use an important identity (see (87) in Appendix) for Killing-Yano tensors

$$\nabla_a \nabla_b K_{c_1 c_2 \dots c_n} = (-1)^{n+1} \frac{n+1}{2} R_{[bc_1|a|}{}^{d} K_{c_2 c_3 \dots c_n]d}. \tag{68}$$

Upon applying (68) to the covariant derivative of (67), we get

$$\begin{aligned} \nabla_b C_{a_1 \dots a_{m+n-1}} = &- \big(m+n\big)\big(\nabla_c A_{[ba_1 \dots a_{m-1}}\big) \nabla^c B_{a_m \dots a_{m+n-1}]} \\ &- \big(m+n\big) A^c{}_{[a_1 \dots a_{m-1}} R_{|bd|ca_m} B_{a_{m+1} \dots a_{m+n-1}]}{}^{d}. \end{aligned} \tag{69}$$

The first term shows anti-symmetry of index b with other indices, but the second term exhibits it only under certain circumstances. One could say that by symmetry of the curvature tensor, in maximally symmetric spaces it could be expressed as

$$R_{abcd}(\boldsymbol{x}) = f(\boldsymbol{x}) g_{ij}(\boldsymbol{x}) \varepsilon^i{}_{ab} \varepsilon^j{}_{cd} = f(\boldsymbol{x}) \big\{ g_{ac}(\boldsymbol{x}) g_{bd}(\boldsymbol{x}) - g_{ad}(\boldsymbol{x}) g_{bc}(\boldsymbol{x}) \big\}.$$

So, for cases of constant curvature $f(x) = k$, we could write

$$\big(R_{abcd}\big)_{\text{const}} = k \big\{ g_{ac}(\boldsymbol{x}) g_{bd}(\boldsymbol{x}) - g_{ad}(\boldsymbol{x}) g_{bc}(\boldsymbol{x}) \big\}. \tag{70}$$

Thus, upon applying the constant curvature formula of (70) to (69), we get

$$\begin{aligned} \nabla_b C_{a_1 \dots a_{m+n-1}} = &- \big(m+n\big)\big[\big(\nabla_c A_{[ba_1 \dots a_{m-1}}\big) \nabla^c B_{a_m \dots a_{m+n-1}]} \\ &- k A_{[a_1 \dots a_{m-1}} B_{a_m \dots a_{m+n-1} b]}\big] = \nabla_{[b} C_{a_1 \dots a_{m+n-1}]}. \end{aligned} \tag{71}$$

This is similar to equation (59), showing that it is also a Killing-Yano tensor. So the SNB of any two Killing-Yano tensors in constant curvature space is also a Killing-Yano tensor. However, as evident from (46), the curvature of the Taub-NUT is not constant, allowing us to conclude that its Killing-Yano tensors do not exhibit Lie algebra under Schouten-Nijenhuis brackets. Thus, we cannot produce higher order Killing-Yano tensors using the lower order ones for the Taub-NUT as in [43], limiting us to the four available rank two Killing-Yano tensors.

6. Hyperkähler Structure and the Killing-Yano Tensors

Let M be a complex manifold. A Riemannian metric on M is called Hermitian if it is compatible with the complex structure J of M, $\big(\langle JX, JY\rangle = \langle X, Y\rangle\big)$. Then the associated differential two-form ω defined by

$$\omega(X, Y) = \langle JX, Y\rangle$$

is called the Kähler form, where ω is closed if and only if J is parallel. Then M is called a Kähler manifold.

The connection between the metric g and the Kähler form ω is

$$\omega_{\mu\nu} = J_\mu{}^\lambda . g_{\lambda\nu} = \big(Jg\big)_{\mu\nu}$$

where J is the complex structure, for which $J^2 = -\mathbb{I}$.

Definition 1 (Hyperkähler manifold) *A hyperkähler manifold is a $\mathbb{C}^\infty$ Riemannian manifold together with three covariantly constant orthogonal endomorphisms I, J and K of the tangent bundle which satisfy the quaternionic relations*

$$I^2 = J^2 = K^2 = IJK = \mathbb{I}.$$

Define three symplectic forms

$$\omega_1(v, w) = g(Iv, w), \omega_2(v, w) = g(Jv, w), \omega_3(v, w) = g(Kv, w)$$

for $v, w \in TM$. It is the same as the Kähler manifold except with more than one type of complex structures. This implies a corresponding number of individual Hyperkähler two-forms, given by

$$\omega^i_{\mu\nu} = J^{i\,\lambda}_{\mu} . g_{\lambda\nu} = \big(J^i g\big)_{\mu\nu} \tag{72}$$

$g_{\lambda\nu}$ being the hyper-hermitian metric and $J^i_{\mu\lambda}$ the almost complex structure exhibiting quaternion algebra

$$J_\alpha J_\beta = -\delta_{\alpha\beta}\mathbb{I} + \varepsilon_{\alpha\beta}{}^\gamma J_\gamma$$

and thus, we can see that the hyperkähler structures exhibit the same algebra

$$\big(J^i J^j\big)_{\mu\nu} = J^i_{\mu\rho}\, g^{\rho\sigma} J^j_{\sigma\nu} \qquad \big[J^i, J^j\big]_{\mu\nu} = 2\varepsilon^{ij}{}_k J^k_{\mu\nu}$$

$$\Big(\omega^i\omega^j\Big)_{\mu\nu} = \omega^i_{\mu\gamma}\omega^{j\gamma}{}_\nu = \Big(J^{i\,\rho}_{\mu}.g_{\rho\gamma}\Big)g^{\gamma\lambda}\Big(J^{j\,\sigma}_{\lambda}.g_{\sigma\nu}\Big) = J^{i\,\rho}_{\mu}J^{j\,\sigma}_{\rho}\, g_{\sigma\nu} = \big(J^i J^j g\big)_{\mu\nu}$$

$$\therefore \qquad \big[\omega^i, \omega^j\big]_{\mu\nu} = \Big(\big[J^i, J^j\big]g\Big)_{\mu\nu} = 2\big(\varepsilon^{ij}{}_k J^k g\big)_{\mu\nu} = 2\varepsilon^{ij}{}_k \omega^k_{\mu\nu}.$$

These complex structures originate from t'Hooft symbols which have three self dual and three anti-self dual components, meaning six different symplectic two-forms. The almost complex J^i can be represented by t'Hooft symbols, themselves given by

$$J^i_{jk} = \varepsilon^i{}_{jk} \pm \frac{1}{2}\big(\delta^0{}_j\delta^i{}_k - \delta^0{}_k\delta^i{}_j\big).$$

Thus, we can argue that hyper-kähler structures given by (72) are

$$\omega^i_{jk} = \big(J^i g\big)_{jk} = g_{jn}(\boldsymbol{x})\Big[\varepsilon^{in}{}_k \pm \frac{1}{2}\big(\delta^{0n}\delta^i{}_k - \delta^0{}_k\delta^{in}\big)\Big]. \tag{73}$$

As shown in (11) and following [17] we take a different form of the Taub-NUT

$$\mathrm{d}s^2 = V(r)\,\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + V^{-1}(r)\big(\mathrm{d}\tau + \boldsymbol{\sigma}.\mathrm{d}\boldsymbol{x}\big)^2$$

for which, the vierbeins, in a similar fashion to (65) are given by

$$e^0 = \frac{4\big(\mathrm{d}\tau + \boldsymbol{\sigma}.\mathrm{d}\boldsymbol{x}\big)}{\sqrt{V(r)}}, \qquad e^i = \sqrt{V(r)}\,\mathrm{d}x^i.$$

Thus, remembering that $g = \delta_{ij}\, e^i \otimes e^j$ the hyper-Kähler forms (73) are

$$\omega^i = \omega^i_{jk}\mathrm{d}x^j \wedge \mathrm{d}x^k = J^i_{jk}e^j \wedge e^k = \varepsilon^i{}_{jk}\, V(r)\,\mathrm{d}x^j \wedge \mathrm{d}x^k - e^0 \wedge e^i$$

$$\omega^i = \varepsilon^i{}_{jk}V(r)\mathrm{d}x^j \wedge \mathrm{d}x^k \pm \big(\mathrm{d}\tau \wedge \mathrm{d}x^i + \sigma_n\,\mathrm{d}x^n \wedge \mathrm{d}x^i\big). \tag{74}$$

For the Taub-NUT, choosing only anti-self-dual components for $V(r) = l + \dfrac{1}{r}$ and restricting $\boldsymbol{\sigma}$ to lie on a plane ($\boldsymbol{\sigma} = (0, \sigma_2, \sigma_3)$), the reduced symplectic forms are

$$\begin{aligned}
\omega^1 &= \mathrm{d}x^1 \wedge \mathrm{d}\tau + \sigma_2\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \sigma_3\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \left(l + \frac{1}{r}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\omega^2 &= \mathrm{d}x^2 \wedge \mathrm{d}\tau + \sigma_3\mathrm{d}x^2 \wedge \mathrm{d}x^3 - \left(l + \frac{1}{r}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\
\omega^3 &= \mathrm{d}x^3 \wedge \mathrm{d}\tau - \sigma_2\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(l + \frac{1}{r}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned} \tag{75}$$

Table 2. Comparison between Killing-Yano tensors and hyperkähler structures, where $\mathrm{d}\varsigma = \big(\mathrm{d}\psi + A_n \mathrm{d}x^n\big), \quad \gamma^{ij} = \mathrm{d}x^i \wedge \mathrm{d}x^j$.

i	**Killing-Yano tensor f^i**	**Hyperkähler structure ω^i**
i	$\pm 4\mathrm{d}\varsigma \wedge \mathrm{d}x^i - \varepsilon^i{}_{jk} f(r)\gamma^{jk}$	$\pm\big(\mathrm{d}\tau + \sigma_n.\mathrm{d}x^n\big) \wedge \mathrm{d}x^i + \varepsilon^i{}_{jk} V(r)\gamma^{jk}$
1	$\mp 4\mathrm{d}x^1 \wedge \mathrm{d}\varsigma + \left(1 + \frac{4}{r}\right)\gamma^{23}$	$\mathrm{d}x^1 \wedge \big(\mathrm{d}\tau + \sigma_2 \mathrm{d}x^2 + \sigma_3 \mathrm{d}x^3\big) + \left(l + \frac{1}{r}\right)\gamma^{23}$
2	$\mp 4\mathrm{d}x^2 \wedge \mathrm{d}\varsigma - \left(1 + \frac{4}{r}\right)\gamma^{13}$	$\mathrm{d}x^2 \wedge \big(\mathrm{d}\tau + \sigma_3 \mathrm{d}x^3\big) - \left(l + \frac{1}{r}\right)\gamma^{13}$
3	$\mp 4\mathrm{d}x^3 \wedge \mathrm{d}\varsigma + \left(1 + \frac{4}{r}\right)\gamma^{12}$	$\mathrm{d}x^3 \wedge \big(\mathrm{d}\tau - \sigma_2 \mathrm{d}x^2\big) + \left(l + \frac{1}{r}\right)\gamma^{12}$

This construction of hyperkähler structures is similar to deduction of spatial Killing-Yano tensors, proving that Killing-Yano tensors are the hyperkähler structures of the Taub-NUT. Few points are worth mentioning here. By studying the G_2 holonomy equation for biaxial anti-self dual Bianchi IX base Gibbons et.al [11] found associated first order equations satisfied by the metric coefficients to yield self-dual Ricci flat Taub-NUT metrics where $\mathrm{SO}(3) \subset \mathrm{U}(2)$ rotates the three hyperkähler forms as a triplet.

7. Discussion

In this article we see that Taub-NUT is comparable to Euclideanized Bertrand spacetime with magnetic fields due to the shared geometry, conserved quantities, and dual configuration as oscillator or Kepler systems. Identical conserved quantities imply identical symmetry and Killing tensors. These are the Killing-Stäckel and Killing-Yano tensors embedded as co-efficients within the Laplace-Runge- Lenz and angular momentum vectors respectively. The Killing-Yano tensors exhibit quaternionic algebra, implying that Killing-Yano tensors and hyperkähler structures are the same for Taub-NUT. Since euclideanization does not effect symmetry, we can expect Bertrand spacetimes with magnetic fields to exhibit the same properties.

Taub-NUT is a special case of anti-self-dual Bianchi-IX spaces [5], derived from the equations that arise upon applying the corresponding settings to the classical Darboux-Halphen system. Shared features, such as Ricci flow, integrability aspects and integrable reductions to Painleve systems can be explored to some extent. In special cases, self-dual Einstein Bianchi-IX metrics reduce to Taub-NUT de Sitter metric with two parameters of the biaxial solutions identified as the NUT parameter and the cosmological constant. Its anti-self dual curvature confirms that Taub-NUT is a gravitational instanton, and thus, Ricci-flat with topological invariants comparable with other possible diffeomorphically equivalent Ricci-flat manifolds. According to Kronheimer classifications [30,31] all hyperkähler metrics like Taub-NUT in four dimensions are always anti-self dual, so the hyperkähler quotient construction due to Hitchin, Karlhede, Lindstrom and Rocek [23] carries an anti-self dual conformal structure, allowing Penrose's twistor theory [40] techniques to be applied here.

Recent works in emergent gravity [32] aim at constructing a Riemannian geometry from $\mathrm{U}(1)$ gauge fields on noncommutative spacetime. This construction is invertible to find corresponding $\mathrm{U}(1)$ gauge fields on a (generalized) Poisson manifold for a metric (M, g). There exist detailed tests [33] of the emergent gravity picture with explicit solutions in both gravity and gauge theory. Symplectic $\mathrm{U}(1)$ gauge fields have been derived starting from the Eguchi-Hanson metric in four- dimensional Euclidean gravity. The result precisely reproduces $\mathrm{U}(1)$ gauge fields of the Nekrasov-Schwarz instanton derived from the top-down approach. To clarify the role of noncommutative spacetime, the prescription was inverted and Braden-Nekrasov $\mathrm{U}(1)$ instanton defined in commutative spacetime was used to derive a gravitational metric just to show that the Kähler manifold determined by the Braden-Nekrasov instanton exhibits a spacetime singularity while the Nekrasov-Schwarz instanton gives rise to a regular geometry in the form of Eguchi-Hanson space. This implies the importance noncommutativity of spacetime plays in resolving spacetime singularities [34] in general relativity. Some relevant studies related to emergent nature of Schwarzschild spacetime was also performed in [6].

One may wonder if we can similarly get $\mathrm{U}(1)$ gauge fields from the Taub-NUT metric. A critical difference from the Eguchi-Hanson metric [13] is that the Taub-NUT (10) is locally asymptotic at infinity to $\mathbb{R}^3 \times \mathbb{S}^1$, so it belongs to the class of Asymptotically Locally Flat (ALF) spaces. Thus, Hopf coordinates cannot represent the Taub-NUT metric, and it is difficult to naively generalize the same construction to ALF spaces. From gauge theory perspective, it may be related to ALF spaces arising from NC monopoles [36] whose underlying equation is defined by an $\mathbb{S}^1$-compactification of the self(anti)-dual-instanton equation, the so-called Nahm equation. We will discuss in [35] a possible generalization to include Taub-NUT in the bottom-up approach of emergent gravity.

For a special choice of the NUT parameter we get a regular metric, but generally, one encounters singularities at either end of the four-dim radial coordinate. In the most generic case, for a specific choice of azimuthal angle period, one can get away with the bolt-singularity. The NUT singularity (co-dimension 4 orbifold singularity) stays, possibly admitting an M theory interpretation associated with the corresponding non-abelian gauge symmetries [2].

Recently, Ricci flat metrics of ultrahyperbolic signature were constructed [8] with l-conformal Galilei symmetry, involving an AdS_2 part reminiscent of the near horizon geoemtry of extremal black holes. Similarly, it should be interesting to see if Taub-NUT spaces are associable with geodesics that can describe second order dynamical systems. Perhaps the most interesting issue will be to explore whether something like "`Taub-NUT/CFT`" correspondence can be conjectured.

8. Appendix

This section contains some important computations and derivations used in the article.

8.1. Basic Killing Tensors from Holten's Algorithm

Angular Momentum

If we choose to set $C^{(n)}_{\{i\}} = 0,\ n \geq 2$, we get the Killing equations

$$\nabla_{(i} C^{(1)}_{j)} = 0. \tag{76}$$

There are two parts of this solution to study in detail. We can write (76) as

$$\nabla_i C^{(1)}_j + \nabla_j C^{(1)}_i = 0 \qquad \Rightarrow \qquad \nabla_i C^{(1)}_j = -\nabla_j C^{(1)}_i .$$

This is an anti-symmetric matrix, written as $\theta_{ij} = -\theta_{ji}$. Further elaboration gives

$$\theta_{ij}(\vec{x}) = \varepsilon_{ijk}(\vec{x})\theta^k = g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k$$
$$\therefore \quad -\nabla_j C^{(1)}_i = g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k \quad \Rightarrow \quad C^{(1)}_i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j .$$

Thus, we have the rotation operator as the first order co-efficient

$$C^{(1)}_i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j . \tag{77}$$

Applying this (77) into the first term of the power series, we get

$$Q^{(1)} = C_i^{(1)}\Pi^i = -g_{im}(\vec{x})\varepsilon^m{}_{jk}\theta^k x^j \Pi^i$$
$$\Rightarrow \qquad \boldsymbol{L}.\boldsymbol{\theta} = -\big(\varepsilon_{ijk}\Pi^i x^j\big)\theta^k = \big(\boldsymbol{x}\times\boldsymbol{\Pi}\big).\boldsymbol{\theta}$$

$$\therefore \qquad \boldsymbol{L} = \boldsymbol{x}\times\boldsymbol{\Pi}. \tag{78}$$

This eventually becomes the conserved quantity known as the angular momentum.

Laplace-Runge-Lenz Vector

Now when we choose to set $C^{(n)}_{\{i\}} = 0,\ n \geq 3$, we get the Killing equations

$$\nabla_i C^{(2)}_{jk} + \nabla_j C^{(2)}_{ki} + \nabla_k C^{(2)}_{ij} = 0. \tag{79}$$

Clearly, (79) perfectly matches the property of the Killing-Yano and Stäckel tensors. The Runge-Lenz like quantity is given by a symmetric sum as shown below

$$\Big[\boldsymbol{A}\times\big(\boldsymbol{B}\times\boldsymbol{C}\big)\Big]_i = \varepsilon_{ilm}\varepsilon^m{}_{jk}A^l B^j C^k, \qquad \varepsilon_{ilm}\varepsilon^m{}_{jk} = \delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}$$

$$\begin{aligned}\nabla_k C^{(2)}_{ij} &= \varepsilon_{ilm}(\vec{x})\varepsilon^m{}_{jk}(\vec{x})n^l + (i \leftrightarrow j)\\ &= \big(2g_{ij}(\vec{x})g_{kl}(\vec{x}) - g_{ik}(\vec{x})g_{jl}(\vec{x}) - g_{il}(\vec{x})g_{kj}(\vec{x})\big)n^l x^k\end{aligned}$$

$$C^{(2)}_{ij} = \big(2g_{ij}(\vec{x})n_k - g_{ik}(\vec{x})n_j - g_{kj}(\vec{x})n_i\big)x^k. \tag{80}$$

As before, applying (80) to the second order term in the power series gives

$$\begin{aligned}Q^{(2)} &= \frac{1}{2}C^{(2)}_{ij}\Pi^i\Pi^j = \big\{\big|\boldsymbol{\Pi}\big|^2\big(\boldsymbol{n}.\boldsymbol{x}\big) - \big(\boldsymbol{\Pi}.\boldsymbol{x}\big)\big(\boldsymbol{\Pi}.\boldsymbol{n}\big)\big\}\\ &= \boldsymbol{N}.\boldsymbol{n} = \big\{\big|\boldsymbol{\Pi}\big|^2\boldsymbol{x} - \big(\boldsymbol{\Pi}.\boldsymbol{x}\big)\boldsymbol{\Pi}\big\}.\boldsymbol{n} = \big\{\boldsymbol{\Pi}\times\big(\boldsymbol{x}\times\boldsymbol{\Pi}\big)\big\}.\boldsymbol{n}.\end{aligned}$$

$$\therefore \qquad \boldsymbol{N} = \boldsymbol{\Pi}\times\big(\boldsymbol{x}\times\boldsymbol{\Pi}\big). \tag{81}$$

This quantity is a term that is present in another conserved quantity known as the Laplace-Runge-Lenz vector. Having found the two familiar types of conserved quantities, we can now proceed to see what it looks like for the Taub-NUT metric.

8.2. The Bohlin Transformation of Harmonic Oscillator Dynamics

The Bohlin transformation in a 2D plane is $f : z \to \xi = z^2 = R\,\mathrm{e}^{\mathrm{i}\phi};\ z, \xi \in \mathbb{C}$. Another invariant, the angular momentum, will change form under this transformation. To preserve its form, we re-parametrize accordingly

$$l = r^2\dot{\theta} = |z|^2\dot{\theta} = |\xi|^2\phi' \quad \Rightarrow \quad |\xi|\frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau}\theta' = |\xi|^2\theta'$$

$$\therefore \quad \tau \longrightarrow \tilde{\tau} : \frac{\mathrm{d}\widetilde{\tau}}{\mathrm{d}\tau} = |\xi|. \tag{82}$$

Using (82), the velocity and acceleration can be given as

$$\dot{z} = \frac{1}{2}\big(\bar{\xi}\big)^{\frac{1}{2}}\xi', \qquad \ddot{z} = \frac{1}{2}\frac{|\xi|^2}{\big(\xi\big)^{\frac{1}{2}}}\xi'' + \frac{1}{4}\big(\xi\big)^{\frac{1}{2}}|\xi'|^2. \tag{83}$$

Applying (83), the Harmonic Oscillator equation $\ddot{z} = -\frac{k}{m}z$ becomes

$$\xi'' = -\left(\frac{1}{2}|\xi'|^2 + \frac{2k}{m}\right)\frac{\xi}{|\xi|^2}.$$

The oscillator Hamiltonian $\mathcal{H}$ can be re-written to complete the transformation

$$\mathcal{H} = \frac{m}{4}\left(\frac{1}{2}|\xi'|^2 + \frac{2k}{m}\right)|\xi| \quad \Rightarrow \quad \left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right) = \frac{4\mathcal{H}}{m}\frac{1}{|\xi|} = \kappa\frac{1}{|\xi|}$$

$$\therefore \qquad \xi'' = -\left(\frac{|\xi'|^2}{2} + \frac{2k}{m}\right)\frac{\xi}{|\xi|^2} \equiv -\kappa\frac{\xi}{|\xi|^3}. \tag{84}$$

Showing that it restores the central force nature of the system, giving us the equation of motion for inverse square law forces.

8.3. Double Derivative of Killing-Yano Tensors

Similar to Killing vectors, rank n Killing-Yano tensors exhibit a curvature equation

$$\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)K_{c_1\ldots c_n} = \sum_{i=1}^{n} R_{abc_i}{}^{d}K_{c_1\ldots d\ldots c_n}. \tag{85}$$

For the LHS of (85), by permuting the indices according to the rules, we will get

$$\begin{aligned}\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)K_{c_1\ldots c_n} &= -\nabla_a\nabla_{c_1}K_{bc_2\ldots c_n} + \nabla_b\nabla_{c_1}K_{ac_2\ldots c_n}\\ &= 2\nabla_{c_1}\nabla_b K_{ac_2\ldots c_n} - R_{abc_1}{}^{d}K_{dc_2\ldots c_n}\\ &\quad + \sum_{i=2}^{n}\big(R_{bc_1c_i}{}^{d}K_{ac_2\ldots d\ldots c_n} - R_{ac_1c_i}{}^{d}K_{bc_2\ldots d\ldots c_n}\big)\\ &= R_{abc_1}{}^{d}K_{dc_2\ldots c_n} + \sum_{i=2}^{n} R_{abc_i}{}^{d}K_{c_1\ldots d\ldots c_n}\end{aligned}$$

$$\begin{aligned} 2\nabla_{c_1}\nabla_b K_{ac_2...c_n} = 2R_{abc_1}{}^d K_{dc_2...c_n} + \big(\underbrace{R_{abc_i}{}^d K_{c_1...d...c_n}}_{\mathrm{I}} \\ + \sum_{i=2}^{n} \underbrace{R_{ac_1c_i}{}^d K_{bc_2...d...c_n} - R_{bc_1c_i}{}^d K_{ac_2...d...c_n}}_{\mathrm{II}}\big) \end{aligned} \tag{86}$$

$$\because \qquad \nabla_{c_1}\nabla_b K_{ac_2...c_n} = \nabla_{c_1}\nabla_{[b} K_{ac_2...c_n]},$$
$$W_{c_1} := \nabla_{c_1}\nabla_{[b} K_{ac_2...c_n]} e^a \wedge e^b \wedge e^{c_2} ... \wedge e^{c_n}.$$

On writing (86) as a three-form, we can say that for I and II

$$\begin{aligned} \mathrm{I}: \quad & R_{abc_i}{}^d e^a \wedge e^b \wedge e^{c_i} = \frac{1}{3}\big(R_{abc_i}{}^d + R_{bc_ia}{}^d + R_{c_iab}{}^d\big) e^a \wedge e^b \wedge e^{c_i} = 0 \\ \mathrm{II}: \quad & R_{ac_1c_i}{}^d K_{bc_2...d...c_n} e^a \wedge e^b \wedge e^{c_i} = -R_{c_1c_ib}{}^d K_{ac_2...d...c_n} e^a \wedge e^b \wedge e^{c_i}. \end{aligned}$$

Thus, on using Bianchi identity for curvature, II of (86) will become

$$\begin{aligned} & -\sum_{i=2}^{n} \big(R_{c_1c_ib}{}^d + R_{bc_1c_i}{}^d\big) K_{ac_2...d...c_n} e^a \wedge e^b \wedge e^{c_i} \\ & = \sum_{i=2}^{n} R_{c_ibc_1}{}^d K_{ac_2...d...c_n} e^a \wedge e^b \wedge e^{c_i} \\ & = \sum_{i=2}^{n} R_{abc_1}{}^d K_{dc_2...c_i...c_n} e^a \wedge e^b \wedge e^{c_i} = (n-1) R_{abc_1}{}^d K_{dc_2...c_i...c_n} e^a \wedge e^b \wedge e^{c_i}. \end{aligned}$$

Applying this result back in the main equation (86), we get

$$2\nabla_{c_1}\nabla_b K_{ac_2...c_n} e^a \wedge e^b \wedge e^{c_i} = \big[2R_{abc_1}{}^d + (n-1)R_{abc_1}{}^d\big] K_{dc_2...c_n} e^a \wedge e^b \wedge e^{c_i}$$

$$2\nabla_{c_1}\nabla_b K_{ac_2...c_n} e^a \wedge e^b \wedge e^{c_i} = (n+1) R_{abc_1}{}^d K_{dc_2...c_n} e^a \wedge e^b \wedge e^{c_i}.$$

Finally, we get the double-derivative of Killing-Yano tensors as

$$\therefore \qquad \nabla_a \nabla_b K_{c_1c_2...c_n} = (-1)^{n+1}\frac{n+1}{2} R_{[bc_1|a|}{}^d K_{c_2c_3...c_n]d}. \tag{87}$$

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References

[1] Arnold V. and Vassiliev V., *Newton's Principia Read 300 Years Later*, Notices Am. Math. Soc. **36** (1989) 1148 -1154; **37**, (1990) 144 (addendum).

[2] Behrndt K., *Singular Seven Manifolds with G(2) Holonomy and Intersecting Six-Branes*, Nucl. Phys. B **635** (2002) 158-174 , [hep-th/0204061].

[3] Besse A., *Einstein Manifolds*, Springer, Berlin 1987.

[4] Bohlin M., *Note Sur le Probléme des Deux Corps et Sur Une Intégration Nouvelle Dans le Probléme des Trois Corps*, B.A. **28** (1911) 113-119.

[5] Chanda S., Guha P., and Roychowdhury R., *Bianchi-IX, Darboux-Halphen and Chazy-Ramanujan*, Int. J. Geom. Methods **13** (2016) 1650042-1650066, arXiv:1512.01662 [hep-th].

[6] Chanda S., Guha P., and Roychowdhury R., *Emergent Schwarzschild: Symplectic, Geometric and Topological Perspective*, arXiv:1406.6459[hep-th],1-34.

[7] Chanda S., Guha P., and Roychowdhury R., *First Integrals of Generalized Darboux-Halphen Systems and Membrane Paradigm*, arXiv:1606.02910 [hep-th], 1-5.

[8] Chernyavsky D. and Galajinsky A., *Ricci-Flat Spacetimes with l-Conformal Galilei Symmetry*, Phys. Lett. B **754** (2016) 249-253, arXiv:1512.06226[hep-th].

[9] Comtet A. and Horváthy P., *The Dirac Equation in Taub-NUT Space*, Phys. Lett. B **349** (1995) 49-56, [hep-th/0511190].

[10] Cordani B., Fehér L. and Horváthy P., *O(4,2) Dynamical Symmetry of the Kaluza-Klein Monopole*, Phys. Lett. B **201** (1988) 481-486.

[11] Cvetic M., Gibbons G., Lu H. and Pope C., *Bianchi IX Self-Dual Einstein Metrics and Singular G(2) Manifolds*, Classical Quant. Grav. **20** (2003) 4239-4268, [hep-th/0206151].

[12] Douglas M. and Moore G., *D-Branes, Quivers, and ALE Instantons*, hep-th/9603167, 1-50.

[13] Eguchi T., Gilkey P. and Hanson A., *Gravitation, Gauge Theories and Differential Geometry*, Phys. Rep. **66** (1980) 213-393.

[14] Eguchi T. and Hanson A., *Self-Dual Solutions to Euclidean Gravity.*, Ann. Phys. **120** (1979) 82-106.

[15] Fehér L., *Dynamical Symmetries of The Kaluza-Klein Monopole*, Relativity Today (1988) 215-224.

[16] Fehér L. and Horváthy P., *Dynamical Symmetry of Monopole Scattering*, Phys. Lett. B **183** (1987) 182-186, [Erratum-ibid. **188** (1987) 512].

[17] Gaeta G. and Rodriguez M., *Hyperkähler Structure of The Taub-NUT Metric*, J. Nonlinear Math. Phys. **19** (2012) 1250014-1250023.

[18] Gibbons G. and Manton N., *Classical and Quantum Dynamics of BPS Monopoles*, Nucl. Phys. B **274** (1986) 183-224.

[19] Gibbons G. and Perry M., *New Gravitational Instantons and Their Interactions*, Phys. Rev. D **22** (1980) 313-321.

[20] Gibbons G. and Ruback P., *The Hidden Symmetries of Multicenter Metrics*, Commun. Math. Phys. **115** (1988) 267-300.

[21] Gibbons G. and Ruback P., *The Hidden Symmetries of Taub-NUT and Monopole Scattering*, Phys. Lett. B **188** (1987) 226-230.

[22] Grandati Y., Bérard A. and Mohrbach H., *Bohlin-Arnold-Vassiliev's Duality and Conserved Quantities*, arXiv:0803.2610v2 [math-ph], 1-8.

[23] Hitchin N., Karlhede A., Lindstrom U. and Rocek M., *Hyperkähler Metrics and Supersymmetry*, Commun. Math. Phys. **108** (1987) 535-589.

[24] Holten J., *Gravity, Geometry and Physics*, NIKHEF /96-22, 1-112.

[25] Holten J., *Supersymmetry and The Geometry of Taub-NUT*, Phys. Lett. B **342** (1995) 47-52, [hep-th/9409139].

[26] Iwai T., Katayama N., *On Extended Taub-NUT Metrics*, J. Geom. Phys. **12** (1993) 55-75.

[27] Iwai T., Katayama N., *Two Kinds of Generalized Taub-NUT Metric and The Symmetry of Associated Dynamical Systems*, J. Phys. A-Math. & Gen. **27** (1994) 3179-3190.

[28] Jante R. and Schroers B., *Taub-NUT Dynamics with a Magnetic Field.* J. Geom. Phys. **104** (2016) 305-328, arXiv:1507.08165 [hep-th].

[29] Kastor D., Ray S. and Traschen J., *Do Killing-Yano Tensors Form a Lie Algebra?*, Classical Quant. Grav. **24** (2007) 3759-3768, arXiv:0705.0535 [hep-th].

[30] Kronheimer P., *A Torelli-Type Theorem for Gravitational Instantons*, J. Differ. Geom. **29** (1989) 685-697.

[31] Kronheimer P., *The Construction of ALE Spaces as Hyperkähler Quotients*, J. Differ. Geom. **29** (1989) 665-683.

[32] Lee S., Roychowdhury R. and Yang H., *Notes on Emergent Gravity*, J. High Energy Phys. **09** (2012) 1-29, arXiv:1206.0678 [hep-th].

[33] Lee S., Roychowdhury R. and Yang H., *Test of Emergent Gravity*, Phys. Rev. D. **88** (2013) 086007-086028, arXiv:1211.0207 [hep-th].

[34] Lee S., Roychowdhury R. and Yang H., *Topology Change of Spacetime and Resolution of Spacetime Singularity in Emergent Gravity*, Phys. Rev. D **87** (2013) 126002-126007, arXiv:1212.3000 [hep-th].

[35] Lee S., Roychowdhury R. and Yang H., *Emergent Gravity From Bottom-Up Approach*, unpublished.

[36] Lee K. and Yi P., *Quantum Spectrum of Instanton Solitons in Five-Dimensional Noncommutative $U(N)$ Theories*, Phys. Rev. D. **61** (2000) 125015-125028, hep-th/9911186v1.

[37] Miyake Y., *Self-Dual Generalized Taub-NUT Metrics*, Osaka J. Math. **32** (1995) 659-675.

[38] Ngome J., *Curved Manifolds with Conserved Runge-Lenz Vectors*, J. Math. Phys. **50** (2009) 122901- 122913, arXiv:0908.1204 [math-ph].

[39] Oh J., Park C. and Yang H., *Yang-Mills Instantons From Gravitational Instantons*, J. High Energy Phys. **4** (2011) 1-37, arXiv:1101.1357v3 [hep-th].

[40] Penrose R., *Nonlinear Gravitons and Curved Twistor Theory*, Gen. Relat. Gravit. **7** (1976) 31-52.

[41] Perlick V., *Bertrand Spacetimes*, Classical Quant. Grav. **9** (1992) 1009-1021.

[42] Saggio M., *Bohlin Transformation*: *The Hidden Symmetry that Connects Hooke to Newton*, Eur. J. Phys. **34** (2013) 129-138.

[43] Santillan O., *Killing-Yano Tensors and Some Applications*, J. Math. Phys. **53** (2012) 043509-043534, arXiv: 1108.0149v2 [hep-th].

[44] Visinescu M., *Generalized Taub-NUT Metrics and Killing-Yano Tensors*, J. Phys. A-Math. Gen. **33** (2000) 4383-4391, arXiv: hep-th/9911126v2.

[45] Witten E., *Branes, Instantons, and Taub-NUT Spaces*, J. High Energy Phys. **06** (2009) 067-121, arXiv:0902.0948v2 [hep-th].

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