



ON THE PRE-METRIC FORMULATION AND NONLINEARIZATION OF CHARGE-FREE ELECTRODYNAMICS

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Abstract. The purpose of this paper is to give a coordinate free pre-metric formulation of charge free electrodynamics, appropriate, in our view, for non-linearization of Maxwell equations in order to obtain pre-metric description of spatially finite electromagnetic field objects of photon-like nature. First we introduce some formal relations from multilinear algebra and differential geometry to be used further. Then we recall and appropriately modify the existing pre-metric formulation of linear charge free electrodynamics in pre-relativistic and relativistic forms as preparation to turn to corresponding pre-metric non-linearization. Then after some preliminary examples and notes on non-linearization, we motivate our view for existence and explicit formulation of time stable subsystems of the physical objects considered. Section 5 presents the formal results of our approach on the pre-metric nonlinear formulations in static case, in time-dependent case, and in space-time formulation. In the Conclusion we give our general view on “why and how to non-linearize”.

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1. Introduction

We start with some general remarks concerning our view on physical systems and the necessary physical motivations when a theoretical model is meant to be appropriately constructed. Appropriate here means that, the description of structure and behavior must be based on the natural assumption that physical world demonstrates itself through relatively stable objects, being able to survive, or to be destroyed, during mutual interactions. Our knowledge about physical world comes namely from this mutual interaction, no matter whether this interaction is direct, or indirect. We should not equalize the concepts of *explanation* and *description*, our view is that theoretical physics must *explain* any description of natural objects and processes. So, physical motivations should lead to explanation of the description model built. Every built model must give: first, corresponding mathematical structure of the physical system studied, which includes *constituents* + *subsystems*, second, mathematical images of the physical characteristics by means of which the system considered demonstrates *existence* and *participation* among the other systems. For example, it is one thing to model natural flows by vector fields, and another thing to calculate the exchanged energy-momentum between these flows. We may conclude that mathematical models should rather give equations of balance of corresponding quantities, than equations that equalize the mathematical images of the interacting systems. For example, Poynting's equation $\text{div}(\mathbf{E} \times \mathbf{B}) = -\frac{1}{2c} \frac{\partial}{\partial t} (\mathbf{E}^2 + \mathbf{B}^2)$ is a balance relation, while Maxwell equation $\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \text{curl } \mathbf{B}$ equalizes differentiated mathematical images of initial constituents. Any physical system has spatial structure and shows definite stability properties, so, it can support its existence and compensate in definite degree the external disturbances through appropriate shape changes and kinematical behavior without losing identity. Shortly speaking, its time existence is a dynamical process being strongly connected with various and continuous internal and external stress-energy-momentum exchange processes. All these processes are real phenomena and any attempt for their description should be based on appropriate mathematical structures. Of course, all real changes during existence of a physical system also must be formally identified with appropriate mathematical objects. Clearly, if we are going to describe formally a real physical system together with all admissible

changes, the corresponding mathematical images of the system, including its recognizable subsystems and all real changes, must have *tensor* nature in order not be possible to be removed by coordinate transformations.

All these moments suggest to pay respect due to modern differential geometry as appropriate mathematical language. In fact, the more than a century intensive interaction between differential geometry and theoretical and mathematical physics proved to be exclusively useful, suggestive and creative process.

The plan of the paper is the following.

First we introduce some formal relations from multilinear algebra and differential geometry to be used further. Then in Section 3 we start the pre-metric approach in the linear charge free electrodynamics in pre-relativistic and relativistic forms as preparation to turn to corresponding pre-metric non-linearization. Section 4 gives some preliminary examples and notes on non-linearization, and motivates our view for existence and explicit formulation of time stable subsystems of the physical objects considered. Section 5 considers *nonlinearization*: pre-metric formulation in the static case, in time-dependent case, and in space-time formulation. In the Conclusions we give our general view on “why and how to non-linearize”.

In the next section, to facilitate the reader, we are going to introduce the objects and relations to be used further in the paper. Our basic mathematical object will be an oriented manifold (M, ω) with appropriate dimension. The orientation defined by ω will allow to compute integral characteristics of the objects considered on one hand, and to make use of the Poincare isomorphisms between p -vector fields and $(\dim M - p)$ -exterior forms. The two basic operators to use will be, of course, the interior product $i_T \alpha$, where T is a p -vector, α is a q -form, $p \leq q$, and the exterior derivative $d: \Lambda^p(M) \rightarrow \Lambda^{p+1}(M)$.

2. Some Formal Relations

Let E and E^* be two dual real finite dimensional vector spaces. The duality between E and E^* allows to distinguish the following well known (anti)derivation. Let $h \in E$, then we obtain the derivation $i(h)$, or i_h , in $\Lambda(E^*)$ of degree (-1) (sometimes called substitution/contraction/insertion operator, interior product, algebraic flow) according to [13]

$$i(h)(x^{*1} \wedge \cdots \wedge x^{*p}) = \sum_{i=1}^p (-1)^{(i-1)} \langle x^{*i}, h \rangle x^{*1} \wedge \cdots \wedge \hat{x^{*i}} \wedge \cdots \wedge x^{*p}.$$

Clearly, if $u^* \in \Lambda^p(E^*)$ and $v^* \in \Lambda(E^*)$ then

$$i(h)(u^* \wedge v^*) = (i(h)u^*) \wedge v^* + (-1)^p u^* \wedge i(h)v^*.$$

Also, we get

$$\begin{aligned} i(h)u^*(x_1, \dots, x_{p-1}) &= u^*(h, x_1, \dots, x_{p-1}) \\ i(x) \circ i(y) &= -i(y) \circ i(x). \end{aligned}$$

This antiderivation can be extended immediately to a mapping $i(h_1 \wedge \dots \wedge h_p): \Lambda^m(E^*) \rightarrow \Lambda^{(m-p)}(E^*)$, $m \geq p$, according to the rule

$$i(h_1 \wedge h_2 \wedge \dots \wedge h_p)u^* = i(h_p) \circ \dots \circ i(h_1)u^*.$$

Note that this extended mapping is not an antiderivation, except for $p = 1$.

Clearly, this interior product may be used in “opposite direction”, i.e., if u^* is one-form, then

$$\begin{aligned} i(u^*)(h_1 \wedge h_2 \wedge \dots \wedge h_p) &= \langle u^*, h_1 \rangle (h_2 \wedge \dots \wedge h_p) - \langle u^*, h_2 \rangle (h_1 \wedge h_3 \wedge \dots \wedge h_p) \\ &\quad + \dots + (-1)^{(p-1)} \langle u^*, h_p \rangle (h_1 \wedge h_2 \wedge \dots \wedge h_{p-1}). \end{aligned}$$

This mapping is extended to multivectors and exterior forms which are linear combinations: if $\Psi = \Psi_1 + \Psi_2 + \dots$ is an arbitrary multivector on E and $\Phi = \Phi^1 + \Phi^2 + \dots$ is an arbitrary exterior form on E^* then $i_\Psi \Phi$ is defined as extension by linearity, e.g.

$$i(\Psi_1 + \Psi_2)(\Phi^1 + \Phi^2) = i(\Psi_1)\Phi^1 + i(\Psi_1)\Phi^2 + i(\Psi_2)\Phi^1 + i(\Psi_2)\Phi^2.$$

The above extension of the interior product allows to extend the Lie derivative of a differential form Φ along a vector field X to a derivative of Φ along a multivector field T [22], according to the formula

$$\mathcal{L}_T(\Phi) = d \circ i_T \Phi - (-1)^{\deg(T)} i_T \circ d\Phi. \quad (1)$$

If $\mathcal{L}_T(\Phi) = 0$ this extension allows to consider T as a local symmetry of Φ .

The specialization of these interior products for the cases $i(\Phi)\omega$ and $i(\beta)\omega^*$, where Φ is a multivector, $\omega^* = \varepsilon^1 \wedge \varepsilon^2 \wedge \dots \wedge \varepsilon^n$ is a volume form in E^* , β is a p -form, and $\omega = e_1 \wedge e_2 \wedge \dots \wedge e_n$ is the volume form in E , dual to ω^* : $\langle \omega^*, \omega \rangle = 1$, is well known [13] and used in multilinear algebra and differential topology under the name of Poincare isomorphisms, or Poincare dualities. In fact, we note that the two spaces $\Lambda^p(E) \otimes \Lambda^n(E^*)$ and $\Lambda^{n-p}(E^*)$ have the same dimension, so, every nonzero $\omega \in \Lambda^n(E^*)$ generates isomorphism D^p , between these two spaces according to $(u, \omega) \rightarrow i(u)\omega$, where $u \in \Lambda^p(E)$ is a p -vector over E . In particular, if $\{e_i\}$ and $\{\varepsilon^j\}$ are dual bases, the corresponding basis elements

$$e_{\nu_1} \wedge \dots \wedge e_{\nu_p}, \quad \nu_1 < \nu_2 < \dots < \nu_p$$

and

$$\varepsilon^{\nu_{p+1}} \wedge \cdots \wedge \varepsilon^{\nu_n}, \quad \nu_{p+1} < \nu_{p+2} < \cdots < \nu_n$$

are connected according to the rule

$$D^p(e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}) = (-1)^\sigma \varepsilon^{\nu_{p+1}} \wedge \cdots \wedge \varepsilon^{\nu_n}, \quad \sigma = \sum_{i=1}^p (\nu_i - i).$$

Also

$$\begin{aligned} D_p(\varepsilon^{\nu_1} \wedge \cdots \wedge \varepsilon^{\nu_p}) &= (-1)^\sigma e_{\nu_{p+1}} \wedge \cdots \wedge e_{\nu_n} \\ D_p(\varepsilon^{\nu_{p+1}} \wedge \cdots \wedge \varepsilon^{\nu_n}) &= (-1)^{p(n-p)+\sigma} e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}. \end{aligned}$$

Clearly, we have

$$i(e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}) D^p(e_{\nu_1} \wedge \cdots \wedge e_{\nu_p}) = 0.$$

Also, we note that in this way every subspace $V^p \subset E$ leads to defining three other spaces

$$(V^p)^* \subset E^*, \quad D^p(V^p) \subset E^*, \quad D_p((V^p)^*) \subset E$$

where

$$E = V^p \oplus (D^p(V^p))^*, \quad E^* = (V^p)^* \oplus D^p(V^p).$$

We remind that these isomorphisms depend on the chosen element $\omega \in \Lambda^n(E^*)$, but further in this section we shall omit writing ω for clarity.

These last two formulas allow to make use of any isomorphism between E and E^* for defining isomorphisms $\Lambda^p(E) \cong \Lambda^{n-p}(E)$, and $\Lambda^p(E^*) \cong \Lambda^{n-p}(E^*)$. For these isomorphisms and their duals

$$\begin{aligned} D^p : \Lambda^p(E) &\rightarrow \Lambda^{n-p}(E^*), & (D^p)^* : \Lambda^{n-p}(E) &\rightarrow \Lambda^p(E^*) \\ D_p : \Lambda^p(E^*) &\rightarrow \Lambda^{n-p}(E), & (D_p)^* : \Lambda^{n-p}(E^*) &\rightarrow \Lambda^p(E) \end{aligned}$$

the following relations also hold

$$\begin{aligned} (D_p)^* &= (D^p)^{-1} = (-1)^{p(n-p)} D_{n-p}, & (D^p)^* &= (D_p)^{-1} = (-1)^{p(n-p)} D^{n-p} \\ D_{n-p} \circ D^p &= (-1)^{p(n-p)} \text{id}, & D^{n-p} \circ D_p &= (-1)^{p(n-p)} \text{id} \end{aligned}$$

where id denotes the corresponding identity map. So, up to a sign factor, D^p and D_{n-p} are inverse linear isomorphisms. When $T \in \Lambda(E)$ and $\Psi \in \Lambda(E^*)$ are represented by exterior products, for example: $T = u \wedge v$, $\Psi = \alpha \wedge \beta$, and their degrees are not given, then we denote $D^p \rightarrow \mathfrak{D}$ and $D_p \rightarrow \mathbb{D}$, and the following relations hold

$$\mathfrak{D}(T) = i_v \circ i_u \omega, \quad \mathbb{D}(\Psi) = i_\beta \circ i_\alpha \omega^*.$$

We extend now these insertion operators in different direction. Let E_1 and E_2 be two real vector spaces with corresponding bases $e_i, i = 1, 2, \dots, \dim E_1$ and $k_j, j = 1, 2, \dots, \dim E_2$, $T = t^i \otimes e_i$ be a E_1 -valued q -vector, $\Phi = \alpha^j \otimes k_j$ be a E_2 -valued p -form with $q \leq p$, and $\varphi : E_1 \times E_2 \rightarrow F$ be a bilinear map into the vector space F . Now we define $i_T^\varphi \Phi \in \Lambda^{p-q}(M, F)$ [9]

$$i_T^\varphi \Phi = i_{t^i} \alpha^j \otimes \varphi(e_i, k_j), \quad i = 1, 2, \dots, \dim(E_1), \quad j = 1, 2, \dots, \dim(E_2). \quad (2)$$

Also, if T_1, T_2 are two multivectors and α, β are two forms then $(i \otimes i)_{T_1 \otimes T_2}(\alpha \otimes \beta)$ is defined by

$$(i \otimes i)_{T_1 \otimes T_2}(\alpha \otimes \beta) = i_{T_1} \alpha \otimes i_{T_2} \beta.$$

We can define now the φ -extended Lie derivative. Let M be a n -dimensional manifold, Φ be a E_1 -valued differential p -form on M , $T \in \mathfrak{X}^q(M, E_2)$ be a E_2 -valued q -multivector field on M , with $q \leq p$ and $\varphi : E_1 \times E_2 \rightarrow F$ be a bilinear map. The φ -extended Lie derivative

$$\mathcal{L}_T^\varphi : \Lambda^p(M, E_1) \times \mathfrak{X}^q(M, E_2) \rightarrow \Lambda^{p-q+1}(M, F)$$

is defined as follows [9, p.53]

$$\mathcal{L}_T^\varphi(\Phi) = d \circ i_T^\varphi \Phi - (-1)^{\deg(T) \cdot \deg(d)} i_T^\varphi \circ d\Phi \quad (3)$$

where d is the exterior derivative on M , so, $\deg(d) = 1$. This definition suggests to consider the tensor field T as a *local* φ -symmetry of the differential form Φ when $\mathcal{L}_T^\varphi(\Phi) = 0$.

3. Pre-Metric Formulation in the Linear Case

Since we are not going to use metric tensors in the cases considered, for more information, including history, analysis and citations, on the so-called *pre-metric* formulation of electrodynamics, not only in the charge free case, the reader may find in [5].

3.1. Pre-Relativistic Approach

Let X be a vector field on (\mathbb{R}^3, ω) , where $\omega = dx \wedge dy \wedge dz$ is the usual volume three-form on \mathbb{R}^3 . The usual interaction partners of any vector field X are the differential forms $\alpha \in \Lambda(\mathbb{R}^3)$ on the same manifold since the generated by X flow $i_X \alpha$ across α may change α , i.e., $i_X \alpha$ may be not zero: $i_X \alpha \neq 0$. In searching for such a differential form partner, our vector field X finds only one that is specially

indicated, the volume three-form ω . Now the following question arises: does this flow $i_X\omega \neq 0$ change from point to point? Formally this means whether the Lie derivative $L_X\omega$ change from point to point along X , or not. For this Lie derivative we obtain

$$L_X\omega = d i_X\omega + i_X d\omega = d i_X\omega = d(X^1 dy \wedge dz - X^2 dx \wedge dz + X^3 dx \wedge dy)$$

since $d\omega = 0$ by dimensional reasons. So, if this flow does not change from point to point, i.e., $L_X\omega = 0$, the two-form $i_X\omega$ must be closed: $d i_X\omega = 0$. As is easily verified, this means that the vector field X has zero divergence

$$d i_X\omega = \left(\frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z} \right) dx \wedge dy \wedge dz = (\operatorname{div} X) \cdot \omega = 0 \rightarrow \operatorname{div} X = 0.$$

We specially note that the two-form $i_X\omega$ is *closed*: $d i_X\omega = 0$. Therefore, there exists a class of one-forms $\alpha + df$ such that, locally, we may write $i_X\omega = d(\alpha + df)$. We conclude

Every divergence-free vector field on (\mathbb{R}^3, ω) is naturally connected with some differential one-forms.

Recalling now the two Maxwell equations $\operatorname{div} E = 0, \operatorname{div} B = 0$ we conclude also, that these two vector fields (E, B) are *intrinsically* connected with corresponding differential one-forms, and this connection does NOT require any metric tensor.

However, if the flows generated by E and B admit *time-change* although each of them does not change the volume form ω ? This question suggests to look more carefully for other possible ways for mutual influence of the two flows generated by E and B .

According to the above notations the two *linearly independent* vector fields (E, B) appear together with their corresponding two-forms $i_E\omega$ and $i_B\omega$. Moreover, the equations $\operatorname{div} E = 0, \operatorname{div} B = 0$ require these two two-forms $(i_E\omega, i_B\omega)$ to be closed, and so - locally exact. We conclude that there *must exist* two one-forms α and β satisfying the equations

$$i_E\omega = d\alpha, \quad i_B\omega = d\beta, \quad i_E d\alpha = 0, \quad i_B d\beta = 0.$$

Another suggesting moment in this direction is the heavily used “curl” operator in Maxwell equations. Making use of the mentioned Poincare isomorphisms between p -forms and $(n-p)$ -vectors on oriented manifolds, it is easily found that this “curl” operator is strongly connected to the exterior derivative operator d , acting in the graded algebra of differential forms. Explicitly, in the case of our manifold (\mathbb{R}^3, ω) , the restriction \mathbb{D} of the Poincare isomorphism to differential two-forms of the kind

$d\alpha$, where α is arbitrary one-form, maps $d\alpha$ to a unique vector field of the kind $\text{curl } V$

$$\mathbb{D}(d\alpha) = \text{curl } V \quad \text{or} \quad \mathfrak{D}(\text{curl } V) = d\alpha$$

and the components of $d\alpha$ in the basis $(dx \wedge dy, dx \wedge dz, dy \wedge dz)$ coincide with the components of $\text{curl } V$ in the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. Of course, most easily this can be achieved if the components of α and the components of V are the same in the corresponding dual bases $(dx, dy, dz) \leftrightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Also, Poincare isomorphism generates two bivectors \bar{H} and \bar{K} according to

$$\alpha = i_{\bar{H}}\omega_3, \quad \beta = i_{\bar{K}}\omega_3.$$

All this suggests to look for realizable $E \leftrightarrow B$ *time*-interaction making use of α and β .

In Maxwell theory time is introduced as *external* to the coordinates (x, y, z) parameter, so, the time differentiation of tensor fields on \mathbb{R}^3 does NOT change the tensor nature of the object. It seems that $i_E\omega$ and $i_B\omega$ may have a chance to find their corresponding “curl” partners paying appropriate respect to their own time derivatives. This suggests to answer positively to the question *do available non-zero time derivatives of (E, B) find way for realization through the one-form partners*. As is well known, classical charge free electrodynamics, as represented by Maxwell equations, gives a positive answer to this question. In our approach it looks as follows.

The closed nature of $i_E\omega$ and $i_B\omega$ and the above consideration and notations allow in the corresponding bases $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $(dx \wedge dy, dx \wedge dz, dy \wedge dz)$ to write down

$$\mathfrak{D}(\text{curl } E) = d\alpha \quad \text{or} \quad \mathbb{D}(d\alpha) = \text{curl } E$$

and

$$\mathfrak{D}(\text{curl } B) = d\beta \quad \text{or} \quad \mathbb{D}(d\beta) = \text{curl } B.$$

Now, in order to model mutual and *time-realizable* interaction between E and B making help of α and β , *Maxwell consistency condition* requires

$$\frac{1}{c} \frac{\partial}{\partial t} i_E\omega = d\beta, \quad \frac{1}{c} \frac{\partial}{\partial t} i_B\omega = -d\alpha \quad (4)$$

where c denotes an invariant constant velocity, characterizing the translational propagation of the electromagnetic field object considered. Hence, equations (4) formally introduce a kind of *cross-connection* between the *divergence-free nature* of (E, B) and the *external* nature of their possible time dependence.

So, the content of this positive answer physically means that time derivatives of (E, B) *interchange* their one-form partners: a time change of E requires appropriate local spatial change of the one-form partner β of B , and a time change of B requires appropriate local spatial change of the one-form partner α of E . We note that, from formal viewpoint, this partnership makes use of the *invariant* operator *exterior derivative*, so the partnership is *real*, and the introduced cross-connection of this reality with the corresponding time derivatives allows to consider the time dependent electromagnetic field objects as *real* ones.

Recall now the concept of local *helicity* generated by any one-form η on \mathbb{R}^3 , it is defined as the three-form $\eta \wedge d\eta$ and carries information about the rotational properties of the flow generated by a divergence-free vector field X that is Euclidean metric image of η : $\eta_i = g_{ij}X^j$, or $X^i = g^{ij}\eta_j$, where g denotes the Euclidean metric tensor on \mathbb{R}^3 . In fact, $\eta \wedge d\eta = g(X, \text{curl } X)\omega$.

The above equations (4) determine time-partnership respectively between (E, β) and between (B, α) . This suggests to show appropriate respect to the corresponding *cross helicities*, given by $\alpha \wedge d\beta$ and $\beta \wedge d\alpha$. From classical vector analysis we have the relation

$$L_{E \times B}\omega = (B.\text{curl } E - E.\text{curl } B)\omega = \text{div}(E \times B)\omega.$$

The corresponding Poynting relation in Maxwell theory suggests now the following. Since our field object will *propagate* in the three-space, then at the points where the field functions are not zero at the moment t , generally speaking, there must be expected time changes of the *important local characteristics* of the object at these points. Trying to present these changes without making use of a metric, we can make use of the above introduced cross helicities as follows

$$\beta \wedge d\alpha - \alpha \wedge d\beta = - \left(\frac{\partial}{\partial \xi} \frac{1}{2} (\langle \alpha, E \rangle + \langle \beta, B \rangle) \right) \omega.$$

We form now the following $(1, 1)$ tensor

$$\mathbb{T} = \alpha \otimes E + \beta \otimes B - \frac{1}{2} (\langle \alpha, E \rangle + \langle \beta, B \rangle) \text{id}_{T\mathbb{R}^3}.$$

Obviously, this tensor represents a *metric-free* analogue of Maxwell stress tensor.

A more careful look at \mathbb{T} naturally sets the question “weather the two vector fields E and B , which are *constituents* of \mathbb{T} , define eigen directions of \mathbb{T} at every point”. The corresponding to this question equations read

$$i_E \mathbb{T} = \lambda_1 . E, \quad i_B \mathbb{T} = \lambda_2 . B.$$

Since E and B are linearly independent, it is elementary to see that these two relations hold only if

$$\langle \alpha, B \rangle = 0, \quad \langle \beta, E \rangle = 0.$$

So, we have two more algebraic equations connecting (E, B) with (α, β) .

Finally, the mutually dependent vector fields (E, B) define the bi-vector $E \wedge B$, and a natural algebraic characteristic of the assumed dependence between (E, B) and (α, β) appears to be the flow of $E \wedge B$ across the two-form $\alpha \wedge \beta$. In view of the above last relations for this flow we obtain

$$i_{E \wedge B}(\alpha \wedge \beta) = i_B \circ i_E(\alpha \wedge \beta) = \langle \alpha, E \rangle \langle \beta, B \rangle - \langle \alpha, B \rangle \langle \beta, E \rangle = \langle \alpha, E \rangle \langle \beta, B \rangle.$$

We may assume now the following way to relate this nonlinear flow with the linear flow of E across α , given by $\langle \alpha, E \rangle$, and of B across β , given by $\langle \beta, B \rangle$

$$i_{E \wedge B}(\alpha \wedge \beta) = \left(\frac{\langle \alpha, E \rangle + \langle \beta, B \rangle}{2} \right)^2.$$

From these last two algebraic relations it will follow

$$\langle \alpha, E \rangle^2 - 2\langle \alpha, E \rangle \cdot \langle \beta, B \rangle + \langle \beta, B \rangle^2 = (\langle \alpha, E \rangle - \langle \beta, B \rangle)^2 = 0$$

i.e., $\langle \alpha, E \rangle = \langle \beta, B \rangle$.

Remark 1. *Considering the flow of $E \wedge B$ across ω we get the one-form*

$$\theta = i_{E \wedge B} \omega.$$

Since $i_{E \wedge B} = i_B \circ i_E$, we get $i_E \theta = i_B \theta = 0$, therefore this one-form θ may carry information about the integrability of the two-dimensional distribution Γ on \mathbb{R}^3 defined by E and B . It seems reasonable to admit that in some cases Γ may be integrable which formally means that θ must satisfy the equation

$$d\theta \wedge \theta = 0$$

for example, when time stable and spatially finite solutions are under consideration. In the considered linear case it is hard to believe that such solutions exist at all, but let it be in sight.

In this way we have 11 equations for the 12 components of E, B, α, β . They read

$$L_E \omega = 0, \quad L_B \omega = 0, \quad \frac{\partial}{\partial \xi} i_E \omega = d\beta, \quad \frac{\partial}{\partial \xi} i_B \omega = -d\alpha \quad (5)$$

$$i_E \mathbb{T} = \lambda_1 \cdot E, \quad i_B \mathbb{T} = \lambda_2 \cdot B, \quad \langle \alpha, E \rangle = \langle \beta, B \rangle, \quad \xi = ct. \quad (6)$$

Note that equations (5) reduce to Maxwell equations if $E^i = \alpha_i$ and $B^i = \beta_i$, $i = 1, 2, 3$, which is a particular case of our equations.

The above equations (5), (6) do *not* use any metric, so, this approach may represent the *nonrelativistic* view on so called *pre-metric* formulation of charge free Maxwell equations.

3.2. Relativistic Approach

The relativistic approach to electrodynamic phenomena is based on the assumption that electromagnetic field objects propagate in free space with constant speed c , where “constant” means “*the same with respect to any reference frame*”, i.e., with respect to any free mass body. This brought the necessity to introduce the notion that time-measuring depends on the reference frame with respect to which the time-measuring system does not move. In this way the fourth coordinate $\xi = ct$, t is the measured time interval, was introduced, the three-space \mathbb{R}^3 was extended to \mathbb{R}^4 , and the three-space volume form $\omega_3 = dx \wedge dy \wedge dz$ was extended to four-spacetime volume form ω

$$\omega = \omega_3 \wedge d\xi = dx \wedge dy \wedge dz \wedge d\xi, \quad d\xi = cdt.$$

The above mentioned *Maxwell consistency condition* suggests the following *extension* of the two two-forms $i_E \omega_3$ and $i_B \omega_3$ to two spacetime two-forms

$$\begin{aligned} F &= i_B \omega_3 + \alpha \wedge d\xi = B^3 dx \wedge dy - B^2 dx \wedge dz + B^1 dy \wedge dz \\ &\quad + \alpha_1 dx \wedge d\xi + \alpha_2 dy \wedge d\xi + \alpha_3 dz \wedge d\xi \\ G &= i_E \omega_3 - \beta \wedge d\xi = E^3 dx \wedge dy - E^2 dx \wedge dz + E^1 dy \wedge dz \\ &\quad - \beta_1 dx \wedge d\xi - \beta_2 dy \wedge d\xi - \beta_3 dz \wedge d\xi. \end{aligned}$$

Making use now of the new volume four-form ω , the corresponding bi-vector fields \bar{F} and \bar{G} are defined by

$$F = i_{\bar{G}} \omega, \quad G = -i_{\bar{F}} \omega.$$

Explicitly

$$\begin{aligned} \bar{F} &= \beta_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \beta_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \beta_1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - E^1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \xi} - E^2 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi} - E^3 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi} \\ \bar{G} &= \alpha_3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \alpha_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \alpha_1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + B^1 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \xi} + B^2 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \xi} + B^3 \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}. \end{aligned}$$

The following relations hold

$$\begin{aligned}\langle F, \bar{F} \rangle &= \langle \beta, B \rangle - \langle \alpha, E \rangle, & \langle G, \bar{G} \rangle &= \langle \alpha, E \rangle - \langle \beta, B \rangle \\ \langle F, \bar{G} \rangle &= 2\langle \alpha, B \rangle, & \langle G, \bar{F} \rangle &= 2\langle \beta, E \rangle\end{aligned}$$

so, $\langle F, \bar{F} \rangle = -\langle G, \bar{G} \rangle$.

Now, since $d\omega = 0$, the mentioned extension of the Lie derivative (1) gives the equations

$$\mathcal{L}_{\bar{F}}\omega = di_{\bar{F}}\omega = -dG = 0, \quad \mathcal{L}_{\bar{G}}\omega = di_{\bar{G}}\omega = dF = 0 \quad (7)$$

which reproduce equations (5) in a relativistic *metric-free* way, i.e., the two two-forms F, G are closed: $dF = 0$, $dG = 0$. In other words, the flows of the bi-vectors \bar{F} and \bar{G} do NOT change the volume four-form ω .

But let us not forget that in our approach $E^i \neq \alpha_i$ and $B^i \neq \beta_i$ in general, so, equations (7) give maximum eight scalar equations for twelve scalar functions $(E^i, B^i, \alpha_i, \beta_i)$, while in the Maxwell approach this system is overdetermined - eight scalar equations for the six scalar functions (E^i, B^i) .

Let X be an arbitrary vector field on \mathbb{R}^4 and θ be an arbitrary one-form on \mathbb{R}^4 . We can form $i_X F = X^\sigma F_{\sigma\nu} dx^\nu$ and $i_\theta \bar{F} = \theta_\sigma \bar{F}^{\sigma\nu} \frac{\partial}{\partial x^\nu}$. The (1,1)-stress-energy-momentum tensor $\mathbb{T}_{(F,G)}$ now is defined as bilinear map in the following way

$$\mathbb{T}(X, \theta) = -\frac{1}{2} (\langle i_X F, i_\theta \bar{F} \rangle + \langle i_X G, i_\theta \bar{G} \rangle). \quad (8)$$

Equivalently, if C_2^2 is the contraction operator with respect to the second member, then we have

$$\mathbb{T} = -\frac{1}{2} (C_2^2 (F \otimes \bar{F}) + C_2^2 (G \otimes \bar{G})).$$

In this way we get a relativistic *pre-metric* view on Maxwell charge-free equations, as well as, on more general equations, containing as a particular class of solutions all Maxwell solutions, and this is done without Minkowski pseudometric, or Hodge “*” (detailed comments see in [5]).

4. Preliminary Notes on Non-Linearization

Nonlinearizations of Maxwell charge free electrodynamics the reader may find in [1-5, 6-9, 11, 12, 14-21]. Passing to nonlinear field equations we begin with two appropriate examples [9, p.370-371, 10].

The examples consider the nonlinear equations defining the so called *autoparallel time-like* and *null* vector fields with respect to a given connection, restricting to

the simple case of Levi-Civita connection $\Gamma = (\Gamma_{\mu\nu}^\sigma)$ on Minkowski space-time (\mathbb{R}^4, η) .

Recalling the definition of autoparallel vector field: $\nabla_X X = 0$, or $i_X \nabla X = 0$, it is interesting to note that this nonlinear system of equations admits (3+1)-soliton-like, even *spatially finite*, solutions on Minkowski space-time.

In fact, in canonical coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ we have $\Gamma_{\mu\nu}^\sigma = 0$, and let $u^\mu = (0, 0, \pm \frac{v}{c} f, f)$ be the components of the time-like vector field u , $\eta(u, u) > 0$, where $0 < v = \text{const} < c$, and c is the velocity of light, so $\frac{v}{c} < 1$ and $\eta(u, u) = \left(1 - \frac{v^2}{c^2}\right) f^2 > 0$. Then every function f of the kind

$$f(x, y, z, \xi) = f\left(x, y, h \cdot \left(z \mp \frac{v}{c} \xi\right)\right), \quad h = \text{const}, \quad \text{e.g.} \quad h = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

defines a solution.

If $\eta(u, u) = 0$ then the equations are equivalent to $u^\mu (d\tilde{\eta}(u))_{\mu\nu} = 0$, where d is the exterior derivative. In fact, since the connection used is Riemannian, we have $0 = \nabla_\mu \frac{1}{2}(u^\nu u_\nu) = u^\nu \nabla_\mu u_\nu$, so the relation $u^\nu \nabla_\nu u_\mu - u^\nu \nabla_\mu u_\nu = 0$ holds and is obviously equal to $u^\mu (du)_{\mu\nu} = 0$. The soliton-like solution is defined by $u^\mu = (0, 0, \pm f, f)$ where the function f is of the form

$$f(x, y, z, \xi) = f(x, y, z \mp \xi).$$

Clearly, for every autoparallel vector field u (or one-form u) there exists a canonical coordinate system on the Minkowski space-time, in which u takes such a simple form: $u^\mu = (0, 0, kf, f)$, $k = \text{const}$. The dependence of f on the three spatial coordinates (x, y, z) is *arbitrary*, so it is allowed to be chosen *soliton-like* and, even, *spatially finite*.

So, although the trajectories of these autoparallel vector fields are straight lines and in this sense are naturally considered as appropriate description of free point-like objects, the field nature of these vector fields deserves corresponding attention and respect as a generator of (3+1)-spatially finite solutions. Moreover, these equations suggest one possible way to find appropriate nonlinearization in other more complex cases: having the mathematical image Φ of the physical object of interest and its change $\nabla\Phi$, compute the flow \mathfrak{F} of Φ across $\nabla\Phi$, and put this flow equal to zero. This means that the change $\nabla\Phi$ is *admissible* for Φ , i.e., the physical object considered appropriately changes under the action of external fields of different nature in order to *survive*, and to stay *recognizable* as the same. Of course, if the object considered consists of several interacting subsystems, then the external and the mutual among the subsystems interaction has to be correspondingly identified

and formally represented in order to get the right entire and complete picture of the object's appearance and behaviour.

In the frame of such approach we are going to non-linearize the charge free equations (7) in a *pre-metric* way, trying to build spatially finite image of a photon-like object, consisting of two interacting subsystems, and the constituents of these two subsystems to be formally represented by already introduced four fields $(E, B; \alpha, \beta)$.

Our first step to non-linearization considers the case, of course, of *non-zero divergences* of E and B , i.e., $d_{iE}\omega_3 \neq 0$, $d_{iB}\omega_3 \neq 0$, in general, and *available local interaction* between the flows generated by the two vector fields (E, B) , not ignoring, of course, the participation of their one-form companions (α, β) . Note that, in the linear case, the basic constituents of the situation were these four objects, and no definite local time-stable subsystems created by them were defined and used. Our view on available intrinsic interaction, i.e., *local stress-energy-momentum exchange*, inside an electromagnetic field object is based on the following presumption.

Existence of recognizable time-stable subsystems of the object considered, which subsystems are able to carry and exchange stress-energy-momentum, is required.

Therefore, our first step should be to formally identify and further recognize such subsystems. Our approach is based on the hypothesis that the subsystems are just *two*. As *formal constituents* of these two subsystems we choose the mathematical objects $(E, B; \alpha, \beta)$, where, as we mentioned, (E, B) are allowed now to have NON-zero divergences, so, in general, $i_E d\alpha \neq 0$, $i_B d\beta \neq 0$, $i_E d\beta \neq 0$ and $i_B d\alpha \neq 0$.

5. Pre-Metric Formulation in the Non-Linear Case

5.1. Static Case

In this case in order to follow the above given view, we consider defined on \mathbb{R}^3 *vector-valued objects*, in terms of which the mathematical images of *two subsystems* will be created. Since we assume the subsystems to be two, our objects will take values in a two-dimensional real vector space denoted further by V , and equipped with a basis (e_1, e_2) .

The first subsystem $(\Omega, \bar{\Omega})$ consists of the following two V -valued objects

$$\Omega = \alpha \otimes e_1 + \beta \otimes e_2, \quad \bar{\Omega} = E \otimes e_1 + B \otimes e_2. \quad (9)$$

Recalling now that the two one-forms (α, β) have their bi-vector ω_3 -images \bar{H} and \bar{K} according to $\alpha = i_{\bar{H}}\omega_3$, $\beta = i_{\bar{K}}\omega_3$, we define the second subsystem $(\Sigma, \bar{\Sigma})$ as

follows

$$\Sigma = -i_B \omega_3 \otimes e_1 + i_E \omega_3 \otimes e_2, \quad \bar{\Sigma} = -\bar{K} \otimes e_1 + \bar{H} \otimes e_2. \quad (10)$$

Note that while Ω is V -valued one-form, Σ is V -valued two-form. Correspondingly, $\bar{\Omega}$ is V -valued vector field, and $\bar{\Sigma}$ is V -valued bi-vector field.

In this *static* situation, i.e., there are no running processes, any admissible stress-energy exchange between the two subsystems should be realized as *dynamical stress equilibrium*, i.e., it must have the following special property: any exchange must be *simultaneous* and *in equal quantities*, otherwise the required static situation will be disturbed. Formally this means that any local stress-energy *loss/gain* of $(\Omega, \bar{\Omega})$ is simultaneously compensated by corresponding local stress-energy *gain/loss* of $(\Sigma, \bar{\Sigma})$, and vice versa. We could say that our two subsystems demonstrate *stable local stress equilibrium*. These preliminary remarks suggest: such *stress-equilibrium situations need NOT available nonzero interaction stress-energy*, which corresponds to the Maxwell stress-energy tensor which does not contain interaction stress-energy between the two constituents E and B: the full stress-energy is the sum of the stress-energy carried by the two constituents

$$\begin{aligned} M_j^i(E, B) &= M_j^i(E) + M_j^i(B) = \left(E^i E_j - \frac{1}{2} E^2 \delta_j^i \right) + \left(B^i B_j - \frac{1}{2} B^2 \delta_j^i \right) \\ &= \frac{1}{2} \left(E^i E_j + (i_E \omega_3)^{ik} (i_E \omega_3)_{kj} + B^i B_j + (i_B \omega_3)^{ik} (i_B \omega_3)_{kj} \right). \end{aligned}$$

The corresponding equations, defining local stress equilibrium, must represent the following picture: each subsystem keeps locally its stress-energy and the possible changes are mutually compensated. Assuming that these changes are *real*, their corresponding formal expressions *should have tensor nature*, so, the most appropriate formal expressions seem to be given by $d\Omega$ and $d\Sigma$. The corresponding flows should also pay respect to the vector-valued nature of the formal images of the object's subsystems. These specificities of the stress-energy exchange suggest: the stress-energy balance between the two subsystems to make use of the “ \vee ”-extension (2) of the interior product as follows

$$i_{\bar{\Omega}}^{\vee} d\Omega = -i_{\bar{\Sigma}}^{\vee} d\Sigma. \quad (11)$$

Our view is that equation (11) adequately corresponds to the fact that there is NO interaction stress in $M_j^i(E, B)$: the whole stress should be carried by $(\Omega, \bar{\Omega})$ and $(\Sigma, \bar{\Sigma})$, and the hidden “dynamical” aspect of this equilibrium is adequately represented by the “ \vee ”-extension of the interior product. Recalling that $di_X \omega_3 = (\operatorname{div} X) \omega_3$ and the above notations: $\alpha = i_{\bar{H}} \omega_3$ and $\beta = i_{\bar{K}} \omega_3$ from (11) we obtain

$$d\Omega = d\alpha \otimes e_1 + d\beta \otimes e_2, \quad d\Sigma = -di_B \omega_3 \otimes e_1 + di_E \omega_3 \otimes e_2$$

$$i_{\Omega}^{\vee} d\Omega = i_E d\alpha \otimes e_1 \vee e_1 + i_B d\beta \otimes e_2 \vee e_2 + (i_E d\beta + i_B d\alpha) \otimes e_1 \vee e_2$$

$$i_{\Sigma}^{\vee} d\Sigma = \beta \operatorname{div}(B) \otimes e_1 \vee e_1 + \alpha \operatorname{div}(E) \otimes e_2 \vee e_2 - (\beta \operatorname{div} E + \alpha \operatorname{div} B) \otimes e_1 \vee e_2.$$

Hence, the balance relation (11) leads to the following three-vector, or, maximum, nine-scalar equations

$$i_E d\alpha + \operatorname{div}(B)\beta = 0, \quad i_B d\beta + \operatorname{div}(E)\alpha = 0$$

$$i_E d\beta + i_B d\alpha - \operatorname{div}(E)\beta - \operatorname{div}(B)\alpha = 0.$$

The nonzero values of $\operatorname{div} E$, $\operatorname{div} B$, $i_E d\alpha$, $i_B d\beta$, $i_E d\beta$ and $i_B d\alpha$ provide possible interaction between the two subsystems formally represented by $(\Omega, \bar{\Omega})$ and $(\Sigma, \bar{\Sigma})$, so, the two subsystems acquire status of *interacting* subsystems of a larger *stress-balanced* system, i.e., being in the state of *dynamical equilibrium*.

5.2. Time Dependent Case

First we note that introducing *time* is considered here as a quantitative comparing of the courses of two physically independent processes, the one of which we call *referent*, e.g., the progress of appropriate watch, then the other one attains significance of *parameterized* process.

Hence, we have to specially note that the *time* parameter t used in this subsection we continue to consider as *external* to the spatial coordinates (x, y, z) parameter, and the corresponding referent process must NOT influence the parameterized process. The main formal consequence of this consideration is, as we mentioned earlier, that time-derivatives do *not* change the tensor nature of the t -differentiated object.

Naturally, from physical viewpoint, any observed time change of the above discussed stress balance in the *static* case should presume corresponding *influence*, leading to its violation, and, of course, to violation of its formal representation - relation (11). Physically it may be expected the electromagnetic field object described to survive through some kind of *time "pulsating"* at the space points, or through a *propagation as a whole* in the three-space, or, both. So: *the local static balance should be replaced by an appropriate intrinsically compatible local dynamical and time dependent balance*. Hence, in order to survive, our object must be able to generate *appropriate* spatial changes inside any spatial area that it occupies at any moment of its existence.

To this time-dependence of the behaviour of our electromagnetic field object we are going to give formal description by means of finding appropriate change of the static balance equation (11).

Equation (11) formally postulates equivalence between two vector valued one-forms, so, any introduced influence object, representing how the new time-dependent balance would look like, is expected, formally, also to be vector valued one-form, containing appropriately first order ($\xi = ct$) - derivative(s) and valued in the same vector space. This allows a natural return to the static balance equation through setting this new object equal to zero.

Also, since the available spatial differential operators in (11) are just of *first order*, it seems natural the corresponding formal influence object to contain time derivatives of *not higher* than first order. Clearly, in view of the flow nature of the objects across their own spatially differentiated objects in the static relation (11), the influence object is expected to express formally also a flow, but a flow across *time differentiated* object. Moreover, it should be expected also this time dependence to generate direct mutual influence between the two now time-dependent subsystems. Finally, since time derivation must not change the tensor nature of the differentiated object, and since Ω is one-form, then the two-form Σ is the natural candidate to be ξ -differentiated, and the “ ∇ ”-flow of $\bar{\Omega}$ across the ξ -differentiated Σ will give vector valued one-form, which naturally appears as appropriate formal measure of the local stress-energy time exchange. So, we may write

$$i_{\bar{\Omega}}^{\nabla} d\Omega + i_{\Sigma}^{\nabla} d\Sigma = i_{\bar{\Omega}}^{\nabla} \frac{\partial}{\partial \xi} \Sigma. \quad (12)$$

This equation (12) gives the following three vector equations

$$\begin{aligned} i_E d\alpha + (\operatorname{div} B)\beta &= -i\left(\frac{\partial B}{\partial \xi} \wedge E\right)\omega_3 \\ i_B d\beta + (\operatorname{div} E)\alpha &= i\left(\frac{\partial E}{\partial \xi} \wedge B\right)\omega_3 \\ i_E d\beta + i_B d\alpha - (\operatorname{div} E)\beta - (\operatorname{div} B)\alpha &= i\left(\frac{\partial E}{\partial \xi} \wedge E\right)\omega_3 - i\left(\frac{\partial B}{\partial \xi} \wedge B\right)\omega_3. \end{aligned}$$

5.3. Space-Time Representation

In the frame of the space-time view on physical processes the introduced variable $\xi = ct$ is no more independent on the choice of physical frames with respect to which we introduce spatial coordinates and write down time-dependent formal relations. Now ξ is considered as appropriate coordinate, it generates local coordinate base vector $\frac{\partial}{\partial \xi}$ and corresponding co-vector (or one-form) $d\xi$, $\langle d\xi, \frac{\partial}{\partial \xi} \rangle = 1$. So, the three-volume $\omega_3 = dx \wedge dy \wedge dz$ naturally extends to the four-volume $\omega = dx \wedge dy \wedge dz \wedge d\xi$ on \mathbb{R}^4 . Our purpose now is to find appropriate four-dimensional balance law, suggested by the previous balance laws formally given by equations (11) and (12).

Recall our two basic objects: the vector valued differential one-form $\Omega = \alpha \otimes e_1 + \beta \otimes e_2$ and the vector valued differential two-form $\Sigma = -i_B \omega_3 \otimes e_1 + i_E \omega_3 \otimes e_2$, been defined entirely in terms of objects previously introduced on \mathbb{R}^3 . We want now these objects to depend on ξ as they depend on the spatial coordinates, so to be appropriately extended to objects on \mathbb{R}^4 .

Now, the fourth dimension ξ generates the coordinate one-form $d\xi$, so, the vector valued one-form Ω turns to $d\xi$ for help to extend to a two-form on \mathbb{R}^4 , which is done in the simplest way: $\Omega \rightarrow \Omega \wedge d\xi$. We are in position now to consider the difference $\Omega \wedge d\xi - \Sigma$

$$\begin{aligned} \Omega \wedge d\xi - \Sigma &= (\alpha \wedge d\xi) \otimes e_1 + (\beta \wedge d\xi) \otimes e_2 + i_B \omega_3 \otimes e_1 - i_E \omega_3 \otimes e_2 \\ &= (i_B \omega_3 + \alpha \wedge d\xi) \otimes e_1 - (i_E \omega_3 - \beta \wedge d\xi) \otimes e_2. \end{aligned}$$

In this way we get two V -valued differential two-forms on \mathbb{R}^4 naturally recognized by the basis vectors (e_1, e_2) of the external vector space V

$$F = i_B \omega_3 + \alpha \wedge d\xi \quad \text{and} \quad G = i_E \omega_3 - \beta \wedge d\xi.$$

These two two-forms we consider further as vector components of *one* V -valued two-form \mathfrak{J}

$$\mathfrak{J} = F \otimes e_1 + G \otimes e_2.$$

In order to define corresponding flow, as we did it in previous subsections, we have to construct $\bar{\mathfrak{J}}$. The corresponding two-vectors \bar{F} and \bar{G} are easily introduced making use of the isomorphism between two-forms and two-vectors defined by the volume four-form $\omega = dx \wedge dy \wedge dz \wedge d\xi$ according to

$$G = -i_{\bar{F}} \omega, \quad F = i_{\bar{G}} \omega : \quad \rightarrow \quad \bar{\mathfrak{J}} = \bar{F} \otimes e_1 + \bar{G} \otimes e_2.$$

Making use of the mentioned φ -extension of the interior product (Section 1), we turn now to the corresponding balance law, it reads

$$i_{\bar{\mathfrak{J}}}^\vee d\mathfrak{J} = 0 \tag{13}$$

i.e., the “ \vee ”-flow of $\bar{\mathfrak{J}}$ across the change $d\mathfrak{J}$ of \mathfrak{J} does NOT lead to losses. It has to be noted, that this balance law is written down without making use of (pseudo)metric, the volume form ω serves sufficiently well. We obtain

$$\begin{aligned} i_{\bar{\mathfrak{J}}}^\vee d\mathfrak{J} &= i_{(\bar{F} \otimes e_1 + \bar{G} \otimes e_2)}^\vee (dF \otimes e_1 + dG \otimes e_2) \\ &= i_{\bar{F}} dF \otimes e_1 \vee e_1 + i_{\bar{G}} dG \otimes e_2 \vee e_2 + (i_{\bar{F}} dG + i_{\bar{G}} dF) \otimes e_1 \vee e_2 = 0. \end{aligned}$$

So, equation (13) is equivalent to the following three equations

$$i_{\bar{F}} dF = 0, \quad i_{\bar{G}} dG = 0, \quad i_{\bar{F}} dG + i_{\bar{G}} dF = 0. \tag{14}$$

We have here maximum 12 equations for the 12 components of (F, G) .

We give now another form to equations (14) making use of the Poincare isomorphisms \mathfrak{D} and \mathbb{D} , recalling: $\mathbb{D}_{n-p} \circ \mathfrak{D}^p = (-1)^{p(n-p)} \text{id}$ and $\mathfrak{D}^{n-p} \circ \mathbb{D}_p = (-1)^{p(n-p)} \text{id}$, also, the divergence operator $\delta^p = (-1)^p \mathbb{D}_{n-p+1} \circ \text{d} \circ \mathfrak{D}^p$. Clearly, δ^p maps p -vector fields to $(p-1)$ -vector fields. Since in our case

$$G = -i_{\bar{F}}\omega = -\mathfrak{D}\bar{F}, \quad F = i_{\bar{G}}\omega = \mathfrak{D}\bar{G}$$

where \bar{F} and \bar{G} are bi-vector fields, then $\delta\bar{F}$ and $\delta\bar{G}$ will be just vector fields. In this way we obtain

$$\begin{aligned} \mathfrak{D}(\bar{F} \wedge \delta\bar{G}) &= i_{\delta\bar{G}}\mathfrak{D}(\bar{F}) = i_{\delta\bar{G}}(i_{\bar{F}}\omega) = i_{\delta\bar{G}}(-G) = \mathfrak{D}(\delta\bar{G} \wedge \bar{F}) = i_{\bar{F}}\mathfrak{D}\delta\bar{G} \\ &= i_{\bar{F}}\mathfrak{D}^1(-1)^2\mathbb{D}_3 \text{d} \mathfrak{D}^2\bar{G} = i_{\bar{F}}(-\text{id})\text{d} \mathfrak{D}^2\bar{G} = -i_{\bar{F}}\text{d}(i_{\bar{G}}\omega) = -i_{\bar{F}}\text{d}F. \end{aligned}$$

So, $i_{\delta\bar{G}}G = i_{\bar{F}}\text{d}F$. In the same way following the same line of transformations, from $\mathfrak{D}(\bar{F} \wedge \delta\bar{F})$, $\mathfrak{D}(\bar{G} \wedge \delta\bar{F})$, $\mathfrak{D}(\bar{G} \wedge \delta\bar{G})$ we obtain consecutively

$$i_{\delta\bar{F}}F = i_{\bar{G}}\text{d}G, \quad i_{\delta\bar{F}}G = -i_{\bar{F}}\text{d}G, \quad i_{\delta\bar{G}}F = -i_{\bar{G}}\text{d}F.$$

Hence, equations (14) are equivalent to

$$i_{\delta\bar{G}}G = 0, \quad i_{\delta\bar{F}}F = 0, \quad i_{\delta\bar{F}}G + i_{\delta\bar{G}}F = 0. \quad (15)$$

Note that, from *algebraic* viewpoint, each of the two equations $i_{\delta\bar{G}}G = 0$, $i_{\delta\bar{F}}F = 0$, represents a homogeneous algebraic system of equations for the components of $\delta\bar{G}$ and $\delta\bar{F}$, which components in the both cases are just *four*. Since the nonlinear solutions would require $\delta\bar{G} \neq 0$ and $\delta\bar{F} \neq 0$, a nonzero nonlinear solution would be possible only if $\det|(F_{\mu\nu})| = 0$ and $\det|(G_{\mu\nu})| = 0$, i.e., when $\langle\alpha, B\rangle = \langle\beta, E\rangle = 0$, or, $F \wedge F = 0$, $G \wedge G = 0$. In the Minkowski metric case this corresponds to $F_{\mu\nu}(*F)^{\mu\nu} = 0$, which means Euclidean orthogonality of E and B , $E \cdot B = 0$.

Summing up the above three equations (15) we obtain

$$i_{(\delta\bar{F} + \delta\bar{G})}(F + G) = 0$$

and since in the nonlinear case $(\delta\bar{F} + \delta\bar{G}) \neq 0$ in general, we end up with equation $\det|(F + G)_{\mu\nu}| = 0$, i.e.,

$$(F + G) \wedge (F + G) = F \wedge F + 2F \wedge G + G \wedge G = 2F \wedge G = 2(\langle\alpha, E\rangle - \langle\beta, B\rangle) = 0.$$

The two cases $\delta\bar{F} = \delta\bar{G} \neq 0$ and $\delta\bar{F} = -\delta\bar{G} \neq 0$ also require $\langle\alpha, E\rangle = \langle\beta, B\rangle$. In the Minkowski metric case this is equivalent to $F_{\mu\nu}F^{\mu\nu} = 0$, which in the Euclidean metric terms means $E^2 = B^2$. Hence, in terms of Minkowski spacetime

we can say: all nonlinear solutions require *null* nature of the bi-vector fields \bar{F} and \bar{G} , as well as *null* nature of the two-forms $F = \mathfrak{D}\bar{G}$ and $G = -\mathfrak{D}\bar{F}$.

Finally, we may say that all nonlinear solutions of (14) require $i_{\mathfrak{J}}^{\vee}\mathfrak{J} = 0$, which leads to $\mathcal{L}_{\mathfrak{J}}^{\vee}\mathfrak{J} = 0$.

The energy-momentum tensor looks the same as in the linear relativistic case (8). Here we may approach the conservation aspects as follows. It should be clear that the object considered has *dynamical structure* and propagates as a whole, so, the local conservation aspects must be considered with respect to corresponding vector fields X , determining the corresponding internal processes and propagation as a whole. Therefore, in order to find the corresponding to such a vector field X conserved quantity, the stress-energy tensor (8) must be appropriately “projected” along this vector field X . Hence, the quantity under consideration should locally look like $\mathfrak{P} = X^{\sigma}\mathbb{T}_{\sigma}^{\mu}\frac{\partial}{\partial x^{\mu}}$. Since this vector field \mathfrak{P} must locally express *corresponding dynamical equilibrium* with the media volume where our object exists and propagates, the coordinate free mathematical image of such a property should require the corresponding Lie derivative of the four-volume form ω with respect \mathfrak{P} to be zero: $L_{\mathfrak{P}}\omega = 0$. Now, since $d\omega = 0$, for this Lie derivative we obtain

$$L_{\mathfrak{P}}\omega = di_{\mathfrak{P}}\omega + i_{\mathfrak{P}}d\omega = di_{\mathfrak{P}}\omega = d\mathfrak{D}(\mathfrak{P}) = 0. \quad (16)$$

The so obtained three-form $\mathfrak{D}(\mathfrak{P})$ is closed, and the corresponding to X locally conserved quantity is explicitly represented, by restricting this three-form to the three-space, we can integrate over the occupied by the object three-volume in order to compute the integral value of this quantity, carried by the electromagnetic object considered. Of course, this final equation (16) may be considered also as equation for the vector field X when the solution entering the tensor \mathbb{T} is explicitly used, so, different solutions are not forbidden, in general, to have different local symmetries.

It deserves now noting the following. In this relativistic formulation, where classical *absolute* time becomes *relative*, the speed of light becomes *absolute* constant, and general reference frames and arbitrary coordinates are allowed and used, we may not stick rigidly to the traditional interpretation of the components of F and G , i.e., each of the new constituents F and G has six components and that is all, getting back to $(E, B; \alpha, \beta)$ will require special frames and coordinates.

The equations obtained suggest some connection with the concepts of *absolute* and *relative* integral invariants of a vector field X on a manifold M introduced and used by Cartan [4]. These are differential forms $\alpha \in \Lambda(M)$ satisfying respectively the relations $i(X)\alpha = 0$, $i(X)d\alpha = 0$, leading to $L_X\alpha = 0$, and just $i(X)d\alpha = 0$. Our relations may be considered as corresponding extensions

a vector field \rightarrow vector valued multivector field

a differential form \rightarrow vector valued differential form

and these extensions allow the mentioned in Section 2 extension of the Lie derivative of a differential form along *multivector* fields. The new moment in our extension is that we consider vector valued multivectors along which vector valued forms to be differentiated *with respect to some bilinear map* $\varphi: V \times V \rightarrow W$, where W is appropriately determined vector space, and $\varphi = \vee$ in our case, which corresponds to the specific kind of interaction between the two subsystems: absence of non-zero interaction stress-energy.

In general, we note that, the triple $(V, W; \varphi)$ determines appropriate interaction between the subsystems of the field object considered, which subsystems in our considerations are formally represented by (F, \bar{F}) and (G, \bar{G}) , without excluding lower level interaction between their constituents (F, \bar{G}, G, \bar{F}) .

6. Conclusions

From the general point of view, getting knowledge about the internal compatibility and external stability of a physical object is being done by measuring the corresponding to these physical appearances appropriate physical quantities. Such physical quantities may vary in admissible, or not admissible degree: in the first case we talk about admissible changes, and in the second case we talk about changes leading to destruction of the object. Formally, this is usually verified by calculating the flow of the formal image of the (sub)system considered through its appropriately modelled change, as it is seen, e.g., in equations (11), (12), (13), i.e., by means of finding corresponding *differential self flows of the subsystems*, e.g., $i_{\bar{F}}dF$, and *differential mutual flows*, e.g., $i_{\bar{F}}dG$. Since every measuring process requires stress-energy-momentum transferring between the object studied and the measuring system, the role of finding corresponding *tensor* representatives of these change-objects and of the corresponding flows is of serious importance. Therefore, having adequate stress-energy-momentum tensor \mathbb{T} for the considered case, the clearly individualized nonlinear tensor members of its appropriate projections on the corresponding vector fields X should represent qualitatively and quantitatively important aspects of the *intrinsic* and *extrinsic* dynamical nature of the object considered. As for the volume form ω , it can always be *locally* represented as exterior product of the coordinate one-forms (dx^1, \dots, dx^n) which seems in good correspondence with theoretical viewpoint for spatially finite nature of the majority of physical objects.

The existing knowledge about the structure and internal dynamics of free electromagnetic field objects made us assume the notion for *two partner-fields internal structure*. Each of these two partner-fields (F, \bar{F}) and (G, \bar{G}) has by construction, at most, six independent scalar components, and each partner-field is able to carry local stress-energy-momentum, allowing appropriate local energy-“intercommunication”. The two subsystems realize local mutual energy exchange *without available interaction energy*. Moreover, they strictly respect each other: the exchange is *simultaneous* and in *equal quantities*, so, each of the two partner-fields keeps its identity and recognizability. The corresponding internal dynamical structure is quite appropriate for the translational-rotational aspects of the object’s space-time propagation with the fundamental velocity. All Maxwell solutions are duly respected. As it was shown [9], in the corresponding Minkowski space-time consideration the new nonlinear solutions, i.e., those satisfying $dF \neq 0, d*F \neq 0$, are *time-stable*, they admit *FINITE SPATIAL SUPPORT*, and demonstrate *compatible translational-rotational dynamical structure*.

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