JGSP **4** (2005) 45–58

JOURNAL OF
Geometry and Symmetry in Physics

ON THE SUPERSYMMETRY GROUP OF THE CLASSICAL BOSE-FERMI OSCILLATOR

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Communicated by Gregory L. Naber

Abstract. Applying the concept of a momentum map for supersymplectic supervectorspaces to the one-dimensional Bose-Fermi oscillator, we show that the largest symmetry group that admits a momentum map is the identity component of the intersection of the orthosymplectic group $\mathrm{OSp}(2|2)$ and the group of supersymplectic transformations. This gives a systematic characterization of a certain class of odd supersymmetry transformations that were originally introduced in an ad hoc way.

1. Introduction

Supermechanics is the classical counterpart of quantum field theories involving Bose and Fermi fields. The most prominent use of supermechanics from a mathematical perspective is the role of the classical free particle Lagrangian in the supersymmetric proofs of various index theorems [2]. There has also been some interest in making the geometric description of supermechanics mathematically rigorous, both from a Lagrangian and Hamiltonian point of view [3, 9, 10].

In this note, we are concerned with the classical one-dimensional supersymmetric harmonic oscillator, or Bose-Fermi oscillator. By "classical", we mean that we treat it as a supermechanical system, defined on a supersymplectic flat manifold [10, 13]. It is a simple but a really nontrivial example of a system with supersymmetries, that is, symmetries that mix the fermionic and the bosonic degrees of freedom. It first appeared as one example of a supersymmetric quantum mechanical system in Witten's ground-breaking 1981 paper [15] and was further investigated in the 1980's and 1990's.

The infinitesimal supersymmetry transformations of the harmonic oscillator were initially introduced in an ad hoc way [5]. It was later realized that the stabilizer algebra of the dynamics is the orthosymplectic superalgebra $\mathfrak{osp}(2|2)$ [4]. In this note, we use the concept of a momentum map to investigate the supersymmetries of the harmonic oscillator.

We show how the (ad hoc) supersymmetry transformations [5] can be derived in a systematic way. Namely, we construct a Lie superalgebra $\mathfrak{bf}(2|2)$ whose odd part consists of these transformations and show that $\mathfrak{bf}(2|2)$ is the intersection of $\mathfrak{osp}(2|2)$ and the Lie superalgebra of linear supersymplectic transformations. Alternatively, $\mathfrak{bf}(2|2)$ is the Lie superalgebra of the largest connected subgroup of $\mathrm{OSp}(2|2)$ that admits a momentum map. This is essentially parallel to the ungraded case, since the group of canonical transformations admits a momentum map in any dimension.

The paper is organized as follows: In Sections 2 and 3, we review some facts about the theory of supermanifolds, in particular super Lie groups and the super version of Lie's first theorem. In Section 4, we give a short description of the Bose-Fermi oscillator and write down a class of supersymmetries, taken from DeWitt's book [5]. We construct the supersymmetry Lie algebra of these supersymmetries and name it $\mathfrak{bf}(2|2)$. The corresponding group is $\mathrm{BF}(2|2)$. We then give a general definition of the concept of a momentum map in supermechanics, and show that it is preserved under the flow of Hamiltonians that are invariant under the group action. This is a generalization of the variant of Noether's theorem in ungraded classical Hamiltonian mechanics [1]. In the last section, we show that $\mathrm{BF}(2|2)$ has the properties stated above. We also verify that the momentum map is equivariant and read off the conserved quantities.

2. Supergeometry over $\mathcal{A}$

There are two approaches to the definition of supermanifolds: An algebraic approach which stresses the role of superfunctions [7, 8] and an analytic approach which stresses the points [5, 11, 13]. The analytic approach, which is implicitly used in the majority of the physics literature, has sometimes been criticized by mathematicians for its perceived lack of rigor. However, Tuynman has meticulously worked out the mathematical details in a recent textbook [13] and his definition of supergeometry over a graded commutative algebra $\mathcal{A}$ now provides a rigorous and solid foundation for the analytic approach. In the following, we give a very brief overview – ample details and careful proofs can be found in Tuynman's book [13].

The idea of the analytic approach to supergeometry is to do geometry with the real numbers replaced by some algebra of supernumbers $\mathcal{A}$. More precisely, $\mathcal{A} =$

$\mathcal{A}_0 \oplus \mathcal{A}_1$ is a $\mathbb{Z}_2$-graded commutative real algebra with a natural isomorphism $\mathcal{A} \simeq \mathbb{R} \oplus \mathcal{N}$ where $\mathcal{N}$ denotes the nilpotent elements. The projection onto the real part is called the body map and denoted by B. One also assumes that $\mathcal{A}$ has the property that for any nonzero a in $\mathcal{A}$ there exists an odd b such that $a \cdot b \neq 0$. The standard case for such an $\mathcal{A}$ is the Grassmann algebra $\bigwedge \mathbb{R}^\infty$ of an infinite dimensional real vector space [5, 11]. It is best to think of $\mathcal{A}$ as a parameter of this analytic formulation, much like the structure of the sheaf of superfunctions is a parameter of the algebraic formulation.

A free grade $\mathcal{A}$-module E of dimension $m|n$ (i.e. m even, n odd dimensions) is called an $\mathcal{A}$-vector space provided that there is a natural equivalence class of "real" bases, i.e. bases whose transition matrices have only real entries. If their $\mathbb{R}$-span is denoted by $\mathcal{R}_E$, this is equivalent of having a natural isomorphism

$$E = \mathcal{R}_E \oplus \mathcal{N}_E,$$

where $\mathcal{N}_E = \mathcal{N} \otimes E$. This gives an extension of the body map $B : E \to \mathcal{R}_E$. The DeWitt topology of E is the coarsest topology which makes B continuous.

Tuynman [**?**, 13] has given a nice rigorous construction of smooth functions that circumvents the problem that limits are not unique and hence useless in the DeWitt topology. The upshot is as follows:

Let E_0 denote the even part of an $\mathcal{A}$-vector space. Then a smooth $f : E_0 \to \mathcal{A}$ function has an expansion

$$f(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) = \sum_{i_1, \ldots, i_n = 0}^{1} \xi_1^{i_1} \ldots \xi_n^{i_n} \cdot (G f_{i_1 \ldots i_n})(x_1, \ldots, x_m) \quad (1)$$

where $f_{i_1 \ldots i_n} \in C^\infty(BU, \mathbb{R})$ and G denotes the Taylor-expansion in the nilpotent part, that is if $\mathbf{x} = B\mathbf{x} + n$, with $\mathbf{n} \in \mathcal{N}_E$, then

$$(G f_{i_1 \ldots i_n})(\mathbf{x}) = \sum_{k=0}^{n} \frac{1}{k!} \left(D^k f_{i_1 \ldots i_n} \right)_{B\mathbf{x}} (\mathbf{n}, \ldots, n).$$

In particular, a smooth function must map BE_0 to $B\mathcal{A} = \mathbb{R}$. So a constant function whose value is a nilpotent supernumber is not smooth.

The form (1) gives the characterization

$$C^\infty(E_0, \mathcal{A}) \simeq C^\infty(BE_0, \mathbb{R}) \otimes \bigwedge \mathbb{R}^n.$$

Using this as the *definition* for the sheaf of smooth functions is of course the starting point for the algebraic approach to supermanifolds.

The partial derivatives of f are

$$\frac{\partial}{\partial x_j} f = \sum_{i_1,\dots,i_n=0}^{1} \xi_1^{i_1} \dots \xi_n^{i_n} \cdot \left(G \frac{\partial f_{i_1 \dots i_n}}{\partial x_j} \right) (x_1, \dots, x_m)$$

$$\frac{\partial}{\partial \xi_k} f = \sum_{i_1,\dots,i_n=0}^{1} (-1)^{i_1 + \dots + i_{k-1}} \delta_{1,i_k} \xi_1^{i_1} \dots \widehat{\xi_k^{i_k}} \dots \xi_n^{i_n} \cdot (G f_{i_1 \dots i_n}) (x_1, \dots, x_m)$$

for $1 \leq j \leq m$ and $1 \leq j \leq n$, where $\widehat{}$ denotes omission. By definition, the chain rule takes the form

$$\frac{\partial}{\partial y_i} f(\mathbf{x}(\mathbf{y}, \boldsymbol{\eta}), \boldsymbol{\xi}(\mathbf{y}, \boldsymbol{\eta})) = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial f}{\partial x_j} + \sum_k \frac{\partial \xi_k}{\partial y_i} \frac{\partial f}{\partial \xi_k}$$

and likewise

$$\frac{\partial}{\partial \eta_l} f(\mathbf{x}(\mathbf{y}, \boldsymbol{\eta}), \boldsymbol{\xi}(\mathbf{y}, \boldsymbol{\eta})) = \sum_j \frac{\partial x_j}{\partial \eta_l} \frac{\partial f}{\partial x_j} + \sum_k \frac{\partial \xi_k}{\partial \eta_l} \frac{\partial f}{\partial \xi_k}.$$

With the concept of smoothness, one can now develop the theory of $\mathcal{A}$-manifolds, vector bundles, Lie groups, etc. almost exactly as in the ungraded case. We will need the concept of flows of vector fields [9]. Since in a way only even vector fields are really infinitesimal directions, this is usually done for even vector fields only. The flow of a vector field $X \in \mathfrak{X}_0(M)$ is a smooth map $\Phi : U \to M$, where U is an open subset of $\mathcal{A}_0 \times M$, such that if ∂_t is the pullback of the canonical vector field on $\mathcal{A}_0$ to U. Then

$$T\Phi \circ \partial_t = X \circ \Phi.$$

With the initial condition $\Phi(\mathbf{x}, 0) = \mathbf{x}$ we have existence and uniqueness of the flow in the usual sense [9]. In a way, the functions $t \mapsto \Phi(t, \mathbf{x}_0)$ (for fixed $\mathbf{x}_0$) are the integral curves of X, only they need not be smooth in the above sense.

3. Super Linear Algebra

In this section, we fix further notations and give our conventions for supervectors and supermatrices. We essentially follow Leĭtes [8].

For $m, n \in \mathbb{N}$, we consider the $\mathcal{A}$-vector space $\mathcal{A}^{m \oplus n} = \mathcal{A}^m \times \mathcal{A}^n$, with the following grading: an element $(\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{A}^{m \oplus n}$ is even iff $\mathbf{x} \in \mathcal{A}_0^m$ and $\boldsymbol{\xi} \in \mathcal{A}_1^m$ and it is odd iff $\mathbf{x} \in \mathcal{A}_1^m$ and $\boldsymbol{\xi} \in \mathcal{A}_0^m$. There are two natural superscalar multiplications: the right one and the left one. Since it will be more convenient when dealing with matrices, we fix the action of $\mathcal{A}$ on $\mathcal{A}^{m\oplus n}$ to be the right multiplication, i.e.

$$(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n) \cdot a = (x_1 a, \ldots, x_m a, \xi_1 a, \ldots, \xi_n a) .$$

The generalization of the Euclidean space is the superspace $\mathcal{A}^{m|n} := (\mathcal{A}^{m\oplus n})_0 = \mathcal{A}_0^m \times \mathcal{A}_1^n$.

Matrices with entries from $\mathcal{A}$ will be called supermatrices, and will be denoted by $X = (x_{ij})_{ij}$. We can identify the set of $(m+n)\times(m+n)$ supermatrices, $\mathrm{Mat}(m|n)$, with $\mathrm{End}_R\left(\mathcal{A}^{m\oplus n}\right)$, the space of right linear $\mathcal{A}$ endomorphisms

$$\mathrm{End}_R\left(\mathcal{A}^{m\oplus n}\right) \cong \mathrm{Mat}(m|n), \quad \phi \leftrightarrow (x_{ij}) = \Big([\phi(e_i)]_j\Big)_{\substack{i=1,\ldots,m+n\\ j=1,\ldots,m+n}} . \tag{2}$$

Then $\phi(x) = Xx$, where the right hand side is the usual matrix multiplication. Likewise, if $\phi \leftrightarrow X$ and $\psi \leftrightarrow Y$, then $\phi \circ \psi \leftrightarrow XY$.

The following grading on $\mathrm{Mat}(m|n)$ is natural: call a supermatrix X even if it preserves $(\mathcal{A}^{m\oplus n})_0$, and odd if it maps $(\mathcal{A}^{m\oplus n})_0$ to $(\mathcal{A}^{m\oplus n})_1$. Equivalently, if

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

where A is an $m\times m$, B an $m \times n$, C an $n\times m$, and D an $n \times n$ matrix with entries from $\mathcal{A}$, then X is even iff all entries of A and D are even, and all entries of B and C are odd, and X is odd iff all entries of A and D are odd, and all entries of B and C are even. Now, the pull back of the $\mathcal{A}$ graded bimodule structure on $\mathrm{End}_R\left(\mathcal{A}^{m\oplus n}\right)$ to $\mathrm{Mat}(m|n)$ via (2) yields the following rules for multiplications by a superscalar:

$$a \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA & aB \\ (-1)^{p(a)}\, aC & (-1)^{p(a)}\, aD \end{pmatrix}$$

for a homogeneous element $a \in \mathcal{A}$ with parity $p(a)$, where the multiplications on the right hand side are just the entry-wise multiplications in $\mathcal{A}$.

For an even supermatrix, the supertranspose is defined to be

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$$

where ^{T} denotes the usual transposition. This way, for $X \in \mathrm{Mat}(m|n)_0$ and $\mathbf{x} \in \mathcal{A}^{m|n}$, we have that $(X\mathbf{x})^{ST} = \mathbf{x}^{ST} X^{ST}$, so that the matrix B of a bilinear form on $\mathcal{A}^{m|n}$ transforms under a linear coordinate change with matrix X as $B \to X^{ST} BX$.

If we define the body $B\,(X)$ of a supermatrix to be the real matrix whose entries are the bodies of the entries of X, it holds that an even supermatrix X of the form (3) is invertible iff $B\,(A)$ and $B\,(D)$ are invertible as real square matrices [8]. The group of even invertible $(m+n) \times (m+n)$ supermatrices is super Lie group of dimension $\left(m^2+n^2|2mn\right)$. It is denoted by $\mathrm{GL}(m|n)$. Its Lie superalgebra is $\mathrm{Mat}(m|n)$, and the super Lie bracket is the usual supercommutator

$$[X,Y] = XY - (-1)^{p(X)\cdot p(Y)}\, YX.$$

The concept of the exponential of matrices carries over to the super case, namely

$$\mathrm{Exp}\ (X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

is smooth and maps $\mathrm{Mat}(m|n)_0$ to $\mathrm{GL}(m|n)$. It is locally invertible.

Finally, let us mention that we have the usual relation between connected sub Lie supergroups and sub Lie superalgebras, that is, to any sub Lie superalgebra of $\mathrm{Mat}(m|n)$ corresponds one and only one connected super Lie subgroup of $\mathrm{GL}(m|n)$ [13]. This correspondence is given by exponentiation and taking the tangent space at the identity, respectively. It allows us to move freely from Lie supermatrix algebras to Lie supermatrix groups. We will make frequent use of it. A detailed exposition and proofs may be found in Tuynman's book [13].

4. The Bose-Fermi Oscillator

In the Lagrangian formulation, the configuration space of the Bose-Fermi oscillator is $\mathcal{A}^{1|2}$ equipped with the Lagrangian

$$\mathcal{L}(q,\xi_1,\xi_2,\dot{q},\dot{\xi_1},\dot{\xi_2}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}(\dot{\xi}_1\xi_1 + \dot{\xi}_2\xi_2) - \frac{1}{2}\omega^2 q^2 + \omega\xi_1\xi_2$$

where q is an even, ξ_1 and ξ_2 two odd variables, and ω is a positive parameter. The Lagrangian consists of kinetic and potential energies in the bosonic and fermionic sectors, respectively.

We use the following Hamiltonian description of this system – the phase space is $\mathcal{A}^{2|2}$ with Hamiltonian

$$H\,(q,p,\xi_1,\xi_2) = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) - \omega\xi_1\xi_2.$$

If we write $\mathbb{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbb{I}_\omega = \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = (q, p, \xi_1, \xi_2)^T$, we can write H as the graded quadratic form

$$H = \frac{1}{2}\mathbf{Q}^{ST} \begin{pmatrix} \mathbb{I}_\omega & 0 \\ 0 & -\omega\mathbb{M} \end{pmatrix} \mathbf{Q} = \frac{1}{2}\mathbf{Q}^{ST}\mathbb{H}\mathbf{Q}.$$

Here we wrote $\mathbb{H}$ for the matrix of the Hamiltonian H. We equip $\mathcal{A}^{2|2}$ with the standard supersymplectic form

$$\Omega = \mathrm{d}p\,\mathrm{d}q + \frac{1}{2}\left(\mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2\right)$$

i.e., in matrix form, we have $\Omega = \begin{pmatrix} -\mathbb{M} & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}$. Then Hamilton's equations $\dot{\mathbf{Q}} = \Omega^{-1}\nabla H(\mathbf{Q})$ yield

$$\dot{q} = p, \qquad \dot{p} = -\omega^2 q, \qquad \dot{\boldsymbol{\xi}} = -\omega\mathbb{M}\boldsymbol{\xi}.$$

Here, q, p and $\boldsymbol{\xi}$ denote the components of the flow of the Hamiltonian, rather than components of a supercurve, see Section 2. One checks (formally) that these are the dynamic equations of the Lagrangian $\mathcal{L}$.

The Hamiltonian is preserved under the following infinitesimal supersymmetry transformations, taken from DeWitt [5]:

$$\delta q = \boldsymbol{\xi}^T \delta\boldsymbol{\alpha} \tag{4}$$

$$\delta p = \omega\boldsymbol{\xi}^T \mathbb{M}\delta\boldsymbol{\alpha} \tag{5}$$

$$\delta\boldsymbol{\xi} = (p\mathbb{I}_2 - \omega q\mathbb{M})\,\delta\boldsymbol{\alpha} \tag{6}$$

where $\delta\boldsymbol{\alpha} \in \mathcal{A}_1^2$ is a vector of odd supernumbers. That is

$$\delta q \cdot \frac{\partial H}{\partial q} + \delta p \cdot \frac{\partial H}{\partial p} + \delta\boldsymbol{\xi}^T \frac{\partial H}{\partial \boldsymbol{\xi}} = 0.$$

Note that these transformations mix the odd and even components, that is, the bosonic and the fermionic variables. This is really what makes it *supersymmetric*. In matrix notation $(4) - (6)$ read

$$\begin{pmatrix} \delta q \\ \delta p \\ \delta\boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -(\delta\boldsymbol{\alpha})^T \\ 0 & 0 & \omega(\delta\boldsymbol{\alpha})^T \mathbb{M} \\ -\omega\mathbb{M}\delta\boldsymbol{\alpha} & \delta\boldsymbol{\alpha} & \mathbf{0} \end{pmatrix} \begin{pmatrix} q \\ p \\ \boldsymbol{\xi} \end{pmatrix}. \tag{7}$$

We now construct a sub Lie algebra of $\mathrm{Mat}(2|2)$ that contains as a subspace the transformation of the form (4)-(4). For this, let us define the $2|2$ square supermatrices

$$A_i = \begin{pmatrix} 0 & 0 & -\mathbf{e}_i^T \\ 0 & 0 & \omega \cdot \mathbf{e}_i^T . \mathbb{M} \\ \omega \mathbb{M} \mathbf{e}_i & -\mathbf{e}_i & 0 \end{pmatrix} \qquad \text{for } i = 1, 2$$

$$C_1 = \begin{pmatrix} \begin{matrix} 0 & -1 \\ \omega^2 & \mathbf{0} \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & -\omega \cdot \mathbb{M} \end{pmatrix}$$

where $\mathbf{e}_i$ denotes the canonical basis of $\mathbb{R}^2$. Then $A_i \in \mathrm{Mat}(2|2)_1$ and $C_i \in \mathrm{Mat}(2|2)_0$, and these matrices have the following supercommutators:

$$[A_i, A_j] = 2\delta_{ij} (-C_1 + C_2) \tag{8}$$

$$[A_1, C_i] = -\omega \cdot A_2 \tag{9}$$

$$[A_2, C_i] = \omega \cdot A_1 \tag{10}$$

$$[C_i, C_j] = 0. \tag{11}$$

Thus, their $\mathcal{A}$-span defines a sub Lie superalgebra in $\mathrm{Mat}(2|2)$. In the following, we call this span the Lie superalgebra of Bose-Fermi supersymmetry and denote it by $\mathfrak{bf}(2|2) = \mathrm{span}_{\mathcal{A}} (A_1, A_2, C_1, C_2)$. The corresponding connected sub Lie supergroup of $\mathrm{GL}(2|2)$, $\mathrm{BF}\,(2|2) = \langle \mathrm{Exp}\ (\mathfrak{bf}\,(2|2)_0) \rangle$ will be referred to as the Lie supergroup of Bose-Fermi supersymmetry.

We note that $\mathfrak{bf}\,(2|2)_0 = (\mathcal{A}_0 \otimes_{\mathbb{R}} \mathrm{span}_{\mathbb{R}}\,(C_i)) \oplus (\mathcal{A}_1 \otimes_{\mathbb{R}} \mathrm{span}_{\mathbb{R}}\,(A_i))$, and that the matrix in (7) is $\delta\alpha_1 \cdot A_1 + \delta\alpha_2 \cdot A_2$. Also, one sees nicely that the matrices A_i represent transformations that mix odd and even variables whereas the matrices C_i represents transformations among the bosonic and the fermionic parts themselves.

We show that $\mathrm{BF}\,(2|2)$ leaves the Hamiltonian H invariant, as expected. H is preserved under the action by the orthosymplectic group $\mathrm{OSp}\,(2|2)$, that is, the group of all supermatrices R that satisfy $R^{ST} H R = H$. It is an embedded super Lie subgroup of $\mathrm{GL}(2|2)$ of dimension $4|4$. The even part of its Lie superalgebra $\mathfrak{osp}\,(2|2)_0$ consists of all matrices X that satisfy the infinitesimal version of the preservation of H,

$$X^{ST} \mathbb{H} + \mathbb{H} X = 0.$$

With $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this reads

$$A^T \mathbb{I}_\omega = -\mathbb{I}_\omega A, \qquad D^T \mathbb{M} = -\mathbb{M} D, \quad \text{and } \omega C = \mathbb{M} B^T \mathbb{I}_\omega. \tag{12}$$

One can check that $\mathfrak{bf}\,(2|2)_0 \subseteq \mathfrak{osp}\,(2|2)_0$ by verifying that the matrices C_i and $a \cdot A_i$ (for an odd supernumber a satisfy (12). So the Bose-Fermi supersymmetry group $\mathrm{BF}\,(2|2)$ is a super Lie subgroup of $\mathrm{OSp}\,(2|2)$ of dimension $2|2$. Thus the Bose-Fermi supersymmetry group is not the whole stabilizer of the Hamiltonian. In section 6, we give two reasons why this makes sense.

We close this section with the following useful alternative characterization of $\mathfrak{bf}(2|2)$, which is not hard to check:

An element $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{osp}(2|2)_0$ lies in $\mathfrak{bf}(2|2)_0$ if and only if

$$\mathbb{M}B = C^T, \qquad (\mathbb{M}A)^T = \mathbb{M}A, \quad \text{and } \; D^T = -D. \tag{13}$$

5. Momentum Map

Before we come to the characterization of $\mathrm{BF}(2|2)$ indicated in the introduction, we have to introduce the concept of a momentum map. Since this is a general concept, we formulate it for an $\mathcal{A}$-vector space of arbitrary dimension equipped with an even supersymplectic form.

Recall that in ungraded symplectic geometry, a momentum map for the action of a Lie group G on a symplectic manifold (M, ω) is a linear map $\hat{J} : \mathfrak{g} \to C^\infty\,(M)$ such that the Hamiltonian vector field generated by $\hat{J}\,(\xi)$ is the vector field associated to $\xi \in \mathfrak{g}$ via the differential of the action Equivalently, we can consider $J : M \to \mathfrak{g}^*$ by setting $(J\,(x)\,, \xi) = \hat{J}\,(\xi)\,(x)$.

This definition can essentially be carried over to the graded case: Let E be an $\mathcal{A}$-vector space, and let Ω be an even supersymplectic form on E.

Definition 1. *A momentum map for the action of a supermatrix group G with Lie superalgebra $\mathfrak{g}$ on E is a smooth map*

$$\hat{J} : \mathfrak{g}_0 \times E_0 \to \mathcal{A}$$

such that

$$\Omega^{-1} \nabla_x \hat{J}\,(X, x) = Xx \tag{14}$$

for all $X \in \mathfrak{g}_0$, $x \in E_0$, where ∇_x denotes the spatial gradient.

Note that there is a technical problem though. For fixed X, the function $x \mapsto \hat{J}\,(X, x)$ is in general *not* smooth. Namely, if X is not real, there is nothing that guarantees that the images of real vectors will be real. We solve this technical problem by considering $\hat{J}$ to be defined on $\mathfrak{g}_0 \times E_0$. Then $\nabla_x \hat{J}$ is defined for any X and the above condition makes sense.

We have the following result analogous to the variant of Noether's theorem in ungraded mechanics [1]

Lemma 2. *Suppose $H \in C^\infty(E_0, \mathcal{A})$ is a Hamiltonian that is invariant under the action of G and that G admits a momentum map $\hat{J}$ on E_0.*
Then $\hat{J}$ is preserved under the Hamiltonian flow. That is, if $\Phi(t,x)$ denotes the flow of the Hamiltonian vector field, then

$$\frac{\partial}{\partial t}\hat{J}(X, \Phi(t,x)) \equiv 0$$

for any $X \in \mathfrak{g}$.

The verification of this fact works almost like in the ungraded case. There is one complication, and we therefore indicate the computation. Let us say that E_0 has dimension $m|n$. We introduce the matrix $\mathbb{P}_{m|n} = \begin{pmatrix} +\mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$. Then $(x^{ST})^{ST} = \mathbb{P}_{m|n} x$ for any supervector in E_0. Also, the matrix of the supersymplectic form Ω satisfies $\Omega^{ST} = -\Omega \mathbb{P}_{m|n}$. With this, we have:

$$\begin{aligned} \frac{\partial}{\partial t}\hat{J}(X, \Phi(t,x)) &= (\partial_t \Phi)^{ST} \nabla_x \hat{J}(X, \Phi) = (\Omega^{-1}\nabla H)^{ST} \Omega X \Phi \\ &= (\nabla H)^{ST} (\Omega^{-1})^{ST} \Omega X \Phi = -(\nabla H)^{ST} \mathbb{P}_{m|n} X \Phi \\ &= -(\mathbb{P}_{m|n} X \Phi)^{ST} ((\nabla H)^{ST})^{ST} = (X\Phi)^{ST} \mathbb{P}^2_{m|n} \nabla H \\ &= (X\Phi)^{ST} \nabla H \end{aligned}$$

where we used $(\Omega^{-1})^{ST}\Omega = -\mathbb{P}_{m|n}$ and the dynamical equations $\dot{x} = \Omega^{-1}\nabla H\,(x)$. But this last expression is zero, as it is the derivative of H along X, which preserves H.

6. Two Characterizations of $\mathrm{BF}(2|2)$

We now come to the main result:

Theorem 3. 1. $\mathrm{BF}(2|2)$ *is the identity component of the intersection of* $\mathrm{OSp}(2|2)$ *and the group of linear supersymplectic transformations.*

2. $\mathrm{BF}(2|2)$ *is the largest connected subgroup of* $\mathrm{OSp}(2|2)$ *whose action on* $\mathcal{A}^{2|2}$ *admits a momentum map.*

Proof: In order to understand the Theorem 3, let us note that the infinitesimal version of preserving the supersymplectic form is $X^{ST}\Omega + \Omega X = 0$. One checks that this is equivalent to the conditions in (13).

We now prove Theorem 3. The differential equation (14) for the momentum map $\hat{J}$ reads

$$\left[\nabla_{(\boldsymbol{q},\boldsymbol{\xi})}\hat{J}\right](X,\boldsymbol{q},\boldsymbol{\xi}) = \Omega X \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{\xi} \end{pmatrix}.$$

Here we have made use of the abbreviation $\boldsymbol{q} = (q,p)^T$. So with $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\nabla_{\mathbf{q}}\hat{J} = -\mathbb{M}A\mathbf{q} - \mathbb{M}B\boldsymbol{\xi} \tag{15}$$

$$\nabla_{\boldsymbol{\xi}}\hat{J} = C\mathbf{q} + D\boldsymbol{\xi}. \tag{16}$$

We determine for which X this has a solution for all $(\boldsymbol{q},\boldsymbol{\xi})$. Assume first that there exists a solution. Then

$$\partial_{\xi_i}\partial_{q_j}\hat{J} = -\partial_{\xi_i}\left(\mathbf{e}_j^T\mathbb{M}B\boldsymbol{\xi}\right) = \mathbf{e}_j^T\mathbb{M}B\mathbf{e}_i$$
$$\partial_{q_j}\partial_{\xi_i}\hat{J} = \partial_{q_j}\left(\mathbf{e}_i^T C\mathbf{q}\right) = \mathbf{e}_i^T C\mathbf{e}_j = \mathbf{e}_j^T C^T\mathbf{e}_i$$

so that necessarily $\mathbb{M}B = C^T$.
Therefore

$$\nabla_{\mathbf{q}}\left(\hat{J} - \boldsymbol{\xi}^T C\mathbf{q}\right) = -\mathbb{M}A\mathbf{q}$$
$$\nabla_{\boldsymbol{\xi}}\left(\hat{J} - \boldsymbol{\xi}^T C\mathbf{q}\right) = D\boldsymbol{\xi}.$$

Hence $\hat{J} - \boldsymbol{\xi}^T C\mathbf{q}$ must be of the form

$$\hat{J} - \boldsymbol{\xi}^T C\mathbf{q} = q^T A'\boldsymbol{q} + \boldsymbol{\xi}^T D'\boldsymbol{\xi}$$

with some 2×2 matrices A' and D' with entries from $\mathcal{A}_0$. Then

$$\nabla_{\mathbf{q}}\left(\hat{J} - \boldsymbol{\xi}^T C\mathbf{q}\right) = \left(A' + A'^T\right)\mathbf{q}$$
$$\nabla_{\boldsymbol{\xi}}\left(\hat{J} - \boldsymbol{\xi}^T C\mathbf{q}\right) = \left(D' - D'^T\right)\boldsymbol{\xi}$$

so that $-\mathbb{M}A = A' + A'^T$ and $D = D' - D'^T$. Thus $\mathbb{M}A$ is symmetric and D antisymmetric: $(\mathbb{M}A)^T = \mathbb{M}A$ and $D^T = -D$.

Hence a necessary condition for the existence of a solution is that X satisfies the conditions (13).

On the other hand, if these conditions are satisfied, then

$$\hat{J}\,(X) = \frac{1}{2}\mathbf{Q}^{ST}\Omega X\mathbf{Q} = \frac{1}{2}(-\mathbf{q}^T\mathbb{M}A\mathbf{q} + \boldsymbol{\xi}^T D\boldsymbol{\xi}) + \boldsymbol{\xi}^T C\mathbf{q} \tag{17}$$

solves $(15) - (16)$, so that they are also sufficient. This completes the proof of the theorem.

With the explicit formula (17), one can now verify that $\hat{J}$ is $\mathrm{BF}\,(2|2)$ equivariant, i.e. that

$$\hat{J}\,(X, G\mathbf{Q}) = \hat{J}\left(G^{-1}XG, \mathbf{Q}\right)$$

for $X \in \mathfrak{bf}\,(2|2)_0$, $G \in \mathrm{BF}\,(2|2)$ and $\mathbf{Q} \in \mathcal{A}^{2|2}$.

Indeed, this follows from the fact that $\mathrm{BF}(2|2)$ preserves the supersymplectic form Ω.

We can read off conserved quantities from (17), given the result about the preservation of the momentum map from section 5. Namely, explicitly, if we have $X = \sum_{i=1,2}(c_i \cdot C_i + a_i \cdot A_i) \in \mathfrak{bf}\,(2|2)_0$ with $c_i \in \mathcal{A}_0$ and $a_i \in \mathcal{A}_1$, then

$$\begin{aligned}
\hat{J}\,(X) &= \frac{1}{2}\left(-\mathbf{q}^T\mathbb{M}A\mathbf{q} + \boldsymbol{\xi}^T D\boldsymbol{\xi}\right) + \boldsymbol{\xi}^T C\mathbf{q} \\
&= \frac{1}{2}\left(c_1 \cdot \left(\omega^2 q^2 + p^2\right) + c_2 \cdot \omega\boldsymbol{\xi}^T\mathbb{M}\boldsymbol{\xi}\right) + p\,(\xi_1 \cdot a_1 + \xi_2 \cdot a_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \omega q\,(\xi_1 \cdot a_2 - \xi_2 \cdot a_1) \\
&= \begin{pmatrix} \frac{1}{2}\left(\omega^2 q^2 + p^2\right) \\ \frac{1}{2}\omega\boldsymbol{\xi}^T\mathbb{M}\boldsymbol{\xi} \end{pmatrix}^T \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \left[(p\mathbb{I}_2 + \omega q\mathbb{M})\,\boldsymbol{\xi}\right]^T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.
\end{aligned}$$

This recovers the first integrals of motion $\frac{1}{2}\left(\omega^2 q^2 + p^2\right)$ and $\frac{1}{2}\omega\boldsymbol{\xi}^T\mathbb{M}\boldsymbol{\xi}$, the Bose- and Fermi-energies, as well as two new supersymmetric conserved quantity, given by $(p\mathbb{I}_2 + \omega q\mathbb{M})\,\boldsymbol{\xi}$. One checks that this last quantity is in fact the preserved super Noether charge that one obtains by a formal use of Noether's theorem from the transformations $(4) - (6)$ applied to the Lagrangian $\mathcal{L}$.

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