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POISSON-LIE STRUCTURE ON THE TANGENT BUNDLE OF A POISSON-LIE GROUP, AND POISSON ACTION LIFTING

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Abstract. We show in this paper that the tangent bundle TG, of a Poisson-Lie group G has a Poisson-Lie group structure given by the canonical lifting of that of G. We determine the dual group of TG, its Lie bialgebra and its double Lie algebra.

We also show that any Poisson action of G on a Poisson manifold P is lifted on a Poisson action of TG on the tanget bundle TP.

1. Introduction

Poisson-Lie group theory was first introduced by Drinfel'd [1] [2] and Semenov-Tian-Shansky [11]. Semenov and Kosmann-Schwarzbach [4] used Poisson-Lie groups to understand the Hamiltonian structure of the group of dressing transformations of certain integrable systems. These Poisson-Lie groups play the role of symmetry groups. Theory of Poisson-Lie groups was remarkably developed by Weinstein [9] [13], Drinfel'd [3] and Jiang-Hua Lu [6] [7].

Let (G, ω) be a Poisson-Lie group with Lie algebra $\mathcal{G}$ and multiplication $m : G \times G \longrightarrow G$.

We assume that the tangent bundle TG is equipped with the Poisson structure Ω_{TG} introduced by Sanchez de Alvarez in [10]. In this case, TG has a Poisson-Lie group structure with dual Poisson-Lie group (TG^*, Ω_{TG^*}) and Lie bialgebra $(\mathcal{G} \dashv \mathcal{G}, \mathcal{G}^* \vdash \mathcal{G}^*)$, where G^* is the dual of G, $\mathcal{G} \dashv \mathcal{G}$ is the semi-direct product Lie algebra with bracket

$$[(x,y),(x',y')] = ([x,x'],[x,y'] + [y,x']), \text{ where } (x,y),\ (x',y') \in \mathcal{G} \times \mathcal{G}$$

and $\mathcal{G}^* \vdash \mathcal{G}^*$ is the semi-direct product Lie algebra with bracket

$$[(\alpha,\beta),(\alpha',\beta')] = ([\alpha,\beta'] + [\beta,\alpha'],[\beta,\beta']), \text{ where } (\alpha,\beta),\ (\alpha',\beta') \in \mathcal{G}^* \times \mathcal{G}^*.$$

The double Lie algebra $\check{\mathcal{D}} = (\mathcal{G} \dashv \mathcal{G}) \oplus (\mathcal{G}^* \vdash \mathcal{G}^*)$, of the Poisson-Lie group (TG, Ω_{TG}) is isomorphic to the semi-direct product Lie algebra $\mathcal{D} \dashv \mathcal{D}$, where $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ is the double Lie algebra of (G, ω).

Let H be a Poisson-Lie subgroup of (G, ω). Then the tangent bundle TH is also a Poisson-Lie subgroup of (TG, Ω_{TG}).

Let P be a Poisson manifold and $\phi : \; G \times P \longrightarrow P$, be a Poisson action of G on P. Then ϕ has a lifted Poisson action of the Poisson Lie group TG on the Poisson manifold TP. As example of Poisson action we consider the dressing action [7]. In this case we show that the lifted action of the left dressing action of G^* on G is also the left dressing action of TG^* on TG.

2. Poisson-Lie Structure on the Tangent Bundle of a Poisson-Lie Group

The notion of a Poisson-Lie group is due to Drinfiel'd [1]. Let us recall its definition and some properties.

Definition 1. *A Poisson-Lie group is a Lie group G, equipped with a Poisson structure ω such that the product*

$$m : G \times G \longrightarrow G : (g, h) \longmapsto m(g, h) = gh$$

is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

The Poisson tensor ω of a Poisson-Lie group G vanishes at the unit element e of G. Its derivative, $\mathrm{d}_e\omega \in \mathcal{G} \wedge \mathcal{G}$, at that point is a 1-cocycle of $\mathcal{G}$ relative to the adjoint representation of $\mathcal{G}$ on $\mathcal{G} \wedge \mathcal{G}$. Then, there exists a Lie bracket on $\mathcal{G}^*$ given by

$$\langle [\alpha, \beta]_\omega, x \rangle = \mathrm{d}_e\omega(x)(\alpha, \beta)$$

where $x \in T_eG = \mathcal{G}$, α and $\beta \in T_e^*G = \mathcal{G}^*$.

The connected and simply connected Lie group G^* with Lie algebra $\mathcal{G}^*$ is called the dual group of the Poisson-Lie group G. It has, too, a structure of Poisson-Lie group.

This dual group G^* acted on G by the (dressing action), whose orbits determine the symplectic leaves of G.

In this section, we give for every Poisson-Lie group G a structure of Poisson Lie group on the tangent bundle TG. The Poisson structure on TG is that given by Sanchez de Alvarez [10]. Let us recall it in the case of a Poisson manifold P.

Theorem 2. [10]. *Let P be a Poisson manifold with Poisson bracket $\{\ ,\ \}_P$. We denote by τ the canonical projection from TP on P. For all $\varphi \in C^\infty(P)$, we denote $\hat{\varphi} = \varphi \circ \tau$ and $\dot{\varphi}$ the tangent map of φ.*
Then TP has a unique Poisson structure, denoted by $\{\ ,\ \}_{TP}$ such that:

i) $\{\hat{\varphi}, \hat{\psi}\}_{TP} = 0$

ii) $\{\hat{\varphi}, \dot{\psi}\}_{TP} = \{\dot{\varphi}, \hat{\psi}\}_{TP} = \widehat{\{\varphi, \psi\}}_P$

iii) $\{\dot{\varphi}, \dot{\psi}\}_{TP} = \{\varphi, \psi\}^{\cdot}_P$, *for all* $\varphi,\ \psi \in C^\infty(P)$.

Remark 3. [10]. *Let $(x_i), i = 1, ..., n$ are local coordinates of P, such that the bracket of P is given by $\{x_i, x_j\} = \omega_{ij}(x)$. In the local coordinates $(x_i, \dot{x}_i)$ of TP, the bracket $\{\ ,\ \}_{TP}$ is given by:*

i) $\{x_i, x_j\}_{TP} = 0$

ii) $\{\dot{x}_i, x_j\}_{TP} = \{x_i, \dot{x}_j\}_{TP} = \{x_i, x_j\} = \omega_{ij}(x)$

iii) $\{\dot{x}_i, \dot{x}_j\}_{TP} = \dot{\overline{\{x_i, x_j\}}}_P = \dot{\omega}_{ij}(\dot{x}) = \sum_k \frac{\partial \omega_{ij}}{\partial x_k}(x)\dot{x}_k.$

Proposition 4. *Let G be a Lie group with Lie algebra $\mathcal{G}$. We assume that TG is equipped with the map*

$$\check{m} : TG \times TG \longrightarrow TG : (X_g, Y_h) \longmapsto L_{g*}Y_h + R_{h*}X_g.$$

Then TG is a Lie group with Lie algebra the semi-direct product of Lie algebras $\mathcal{G} \dashv \mathcal{G}$, where the bracket is given by

$$[(x, y), (x', y')] = ([x, x'], [x, y'] + [y, x']).$$

It is clear that TG is isomorphic to the semi-direct product Lie groups $G \dashv \mathcal{G}$, associated to the adjoint action of G on the abelian Lie group $\mathcal{G}$. Then the Lie algebra of TG is the semi-direct product Lie algebra $\mathcal{G} \dashv \mathcal{G}$.

With this preparation, we can give the main result of this section.

Theorem 5. *Let (G, ω) be a Poisson-Lie group. We assume that TG is equipped with the multiplication*

$$X_g.Y_h = L_{g*}Y_h + R_{h*}X_g$$

and with the Poisson structure $\{\ ,\ \}_{TG}$. Then $(TG,\ \{\ ,\ \}_{TG})$ is a Poisson-Lie group.

Proof: According to Definition 1, we have to show that

$$\{F_1, F_2\}(X_g.Y_h) = \{F_{1X_g}, F_{2X_g}\}(Y_h) + \{F_{1Y_h}, F_{2Y_h}\}(X_g)$$

for all $F_1,\ F_2 \in C^\infty(TG)$, $X_g \in T_gG$ and $Y_h \in T_hG$.

By Theorem 2, it is sufficient to consider the functions of type $\dot{\varphi}$ and $\hat{\varphi}$, where $\varphi \in C^\infty(P)$.

Let $\varphi,\ \psi \in C^\infty(P)$. We have

$$\{\hat{\varphi}, \hat{\psi}\}(X_g.Y_h) = \{\hat{\varphi}_{X_g}, \hat{\psi}_{X_g}\}(Y_h) + \{\hat{\varphi}_{Y_h}, \hat{\psi}_{Y_h}\}(X_g) = 0.$$

By a simple calculation, we get

$$\begin{aligned}\{\hat{\varphi}_{X_g}, \dot{\psi}_{X_g}\}(Y_h) + \{\hat{\varphi}_{Y_h}, \dot{\psi}_{Y_h}\}(X_g) =&\{(\varphi \circ L_g\hat{)}, (\psi \circ L_g\dot{)} + \hat{\alpha}\}(Y_h)\\ &+ \{(\varphi \circ R_h\hat{)}, (\psi \circ R_h\dot{)} + \hat{\beta}\}(X_g)\\ =&\{\hat{\varphi}, \dot{\psi}\}(X_g.Y_h)\end{aligned}$$

where $\alpha(h) = (\varphi \circ R_h\dot{)}(X_g)$ and $\beta(h) = (\psi \circ R_h\dot{)}(X_g)$.

For the last bracket we have

$$\begin{aligned}\{\dot{\varphi}, \dot{\psi}\}(X_g.Y_h) &= \{\varphi, \psi\dot{\}}(L_{g*}Y_h + R_{h*}X_g)\\ &= (\{\varphi, \psi\} \circ L_g\dot{)}(Y_h) + (\{\varphi, \psi\} \circ R_h\dot{)}(X_g).\end{aligned}$$

If we take $X_g = \sigma_x(g) = R_{g*}x$ and $Y_h = \sigma_y(h)$, where $x, y \in \mathcal{G}$ and σ_x is the fundamental vector field associated to the left translation of G, we get

$$\begin{aligned}(\{\varphi, \psi\} \circ L_g\dot{)}(\sigma_y(h)) =& \{\varphi \circ L_g, \psi \circ L_g\dot{\}}(\sigma_y(h))\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\{\varphi \circ R_{\exp ty.h}, \psi \circ R_{\exp ty.h}\}(g)_{t=0}\\ =& \{\varphi \circ L_g, \psi \circ L_g\dot{\}}(\sigma_y(h)) + \{y^l(\varphi \circ R_h), \psi \circ R_h\}(g)\\ &+ \{\varphi \circ R_h, y^l(\psi \circ R_h)\}(g)\end{aligned}$$

where y^l is the left invariant vector field whose value at e is y.

On the other hand, we have

$$\begin{aligned}(\{\varphi, \psi\} \circ R_h\dot{)}(\sigma_x(g)) =& \{\varphi \circ R_h, \psi \circ R_h\dot{\}}(\sigma_x(g))\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\{\varphi \circ L_{\exp tx.g}, \psi \circ L_{\exp tx.g}\}_{t=o}(h)\\ =& \{\varphi \circ R_h, \psi \circ R_h\dot{\}}(\sigma_x(g)) + \{\dot{\varphi}(\sigma_x \circ L_g), \psi \circ L_g\}(h)\\ &+ \{\varphi \circ L_g, \dot{\psi}(\sigma_x \circ L_g)\}(h).\end{aligned}$$

Furthermore, it is easy to verify that

$$\dot{\varphi}_{\sigma_x(g)}(\sigma_y(h)) = (\varphi \circ L_g)^{\cdot}(\sigma_y(h) + \hat{\alpha}(\sigma_y(h))$$
$$\dot{\varphi}_{\sigma_y(h)}(\sigma_x(g)) = (\varphi \circ R_h)^{\cdot}(\sigma_x(g)) + \hat{\alpha}'(\sigma_x(g))$$

where $\alpha'(g) = (\varphi \circ L_g)^{\cdot}(\sigma_y(h))$.
Then

$$\begin{aligned}
&\{\dot{\varphi}_{\sigma_x(g)}, \dot{\psi}_{\sigma_x(g)}\}(\sigma_y(h)) + \{\dot{\varphi}_{\sigma_y(h)}, \dot{\psi}_{\sigma_y(h)}\}(\sigma_x(g)) \\
=&\{(\varphi \circ L_g)^{\cdot} + \hat{\alpha}, (\psi \circ L_g)^{\cdot} + \hat{\beta}\}(\sigma_y(h)) + \{(\varphi \circ R_h)^{\cdot} + \hat{\alpha}', (\psi \circ R_h)^{\cdot} + \hat{\beta}'\}(\sigma_x(g)) \\
=&\{\varphi \circ L_g, \psi \circ L_g\}^{\cdot}(\sigma_y(h)) + \{(\varphi \circ L_g)^{\cdot}, \hat{\beta}\}(\sigma_y(h)) + \{\hat{\alpha}, (\psi \circ L_g)^{\cdot}\}(\sigma_y(h)) \\
&+ \{\varphi \circ R_h, \psi \circ R_h\}^{\cdot}(\sigma_x(g)) + \{\hat{\alpha}', (\psi \circ R_h)^{\cdot}\}(\sigma_x(g)) + \{(\varphi \circ R_h)^{\cdot}, \hat{\beta}'\}(\sigma_x(g)) \\
=&\{\varphi \circ L_g, \psi \circ L_g\}^{\cdot}(\sigma_y(h)) + \{\varphi \circ L_g, \beta\}(h) + \{\alpha, \psi \circ L_g\}(h) \\
&+ \{\varphi \circ R_h, \psi \circ R_h\}^{\cdot}(\sigma_x(g)) + \{\alpha', \psi \circ R_h\}(g) + \{\varphi \circ R_h, \beta'\}(g).
\end{aligned}$$

It suffices to verify the following expressions

$$\begin{aligned}
\alpha(h) &= \varphi^{\cdot}(\sigma_x \circ L_g)(h) \\
\beta(h) &= \dot{\psi}(\sigma_x \circ L_g)(h) \\
\alpha'(g) &= y^l(\varphi \circ R_h)(g) \\
\beta'(g) &= y^l(\psi \circ R_h)(g).
\end{aligned}$$

We replace $\alpha,\ \alpha',\ \beta\, and\, \beta'$ by these expressions we get

$$\{\dot{\varphi}, \dot{\psi}\}(\sigma_x(g).\sigma_y(h)) = \{\dot{\varphi}_{\sigma_x(g)}, \dot{\psi}_{\sigma_x(g)}\}(\sigma_y(h)) + \{\dot{\varphi}_{\sigma_y(h)}, \dot{\psi}_{\sigma_y(h)}\}(\sigma_x(g)).$$

This concludes the proof. □

Example 6. *Let $\mathcal{G}$ be a Lie algebra. We assume that $\mathcal{G}^*$ is equipped with its linear Poisson-Lie structure given, for all $\varphi,\ \psi \in C^\infty(\mathcal{G}^*)$, by*

$$\{\varphi, \psi\}(x) = \langle x, [\mathrm{d}\varphi(x), \mathrm{d}\psi(x)]\rangle.$$

In local coordinates (x_i) of $\mathcal{G}^*$, this structure is expressed by

$$\{x_i, x_j\} = \sum_k c_{ij}^k x_k$$

where c_{ij}^k are the structure constants of $\mathcal{G}$.

The linear Poisson structure of $\mathcal{G}^* \times \mathcal{G}^*$ associated to the semi-direct product $\mathcal{G} \dashv \mathcal{G}$ is given by

$$\{F, G\}(x, y) = \langle x, [\frac{\mathrm{d}F}{\mathrm{d}x}, \frac{\mathrm{d}G}{\mathrm{d}x}]\rangle + \langle y, [\frac{\mathrm{d}F}{\mathrm{d}x}, \frac{\mathrm{d}G}{\mathrm{d}y}]\rangle + \langle y, [\frac{\mathrm{d}F}{\mathrm{d}y}, \frac{\mathrm{d}G}{\mathrm{d}x}]\rangle$$

for all F, $G \in C^\infty(\mathcal{G}^* \times \mathcal{G}^*)$, $x, y \in \mathcal{G}^*$.

The local coordinates (x_i) induce local coordinates (x_i, y_j) on $\mathcal{G}^* \times \mathcal{G}^*$, such that

$$\begin{aligned}
\{y_i, y_j\}(x, y) &= 0 \\
\{y_i, x_j\}(x, y) &= \sum_k C_{ij}^k y_k = \omega_{ij}(y) \\
\{x_i, x_j\}(x, y) &= \sum_k C_{ij}^k x_k = \dot{\omega}_{ij}(x).
\end{aligned}$$

According to Remark 3, for the local coordinates $(\dot{x}_i, x_j)$ of $T\mathcal{G}^*$, this bracket coincides with that of $T\mathcal{G}^*$.

Hence, the Poisson-Lie group $T\mathcal{G}^*$ is isomorphic to the Abelien Poisson-Lie group $(\mathcal{G} \dashv \mathcal{G})^*$ associated to the semi-direct product Lie algebra $\mathcal{G} \dashv \mathcal{G}$.

3. Bialgebra and Dual of the Poisson-Lie Group TG

In this section, we study the infinitesimal version of the Poisson-Lie group TG, namely that of Lie bialgebra and double Lie algebra of TG.

Definition 7. [12] *Let $\mathcal{G}$ be a Lie algebra with dual space $\mathcal{G}^*$. We say that $(\mathcal{G}, \mathcal{G}^*)$ form a Lie bialgebra if there is given a Lie bracket on $\mathcal{G}^*$ such that*

$$\langle [\alpha, \beta], [x, y]\rangle \ = -[\mathrm{ad}_x^*\alpha, \beta](y) - [\alpha, \mathrm{ad}_x^*\beta](y) + [\mathrm{ad}_y^*\alpha, \beta](x) + [\alpha, \mathrm{ad}_y^*\beta](x).$$

By Drinfel'd [1], if (G, ω) is a Poisson-Lie group, then the derivative of ω at e defines a Lie algebra structure on $\mathcal{G}^*$, such that $(\mathcal{G}, \mathcal{G}^*)$ form a Lie bialgebra. Conversely if G is connected and simply connected, then every structure of Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$ defines a unique Poisson-Lie structure on G.

On the vector space $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$, there is a natural Lie algebra structure such that $\mathcal{G}$ and $\mathcal{G}^*$ are Lie subalgebras, whose bracket is

$$[x, \alpha] = \mathrm{ad}_x^*\alpha - \mathrm{ad}_\alpha^* x$$

where $x \in \mathcal{G}$ and $\alpha \in \mathcal{G}^*$. With that structure, $\mathcal{D}$ is called the double Lie algebra of (G, ω).

For example, let $\mathcal{G}$ be a Lie algebra. Its dual space $\mathcal{G}^*$ is an Abelian Poisson-Lie group, where the Poisson bracket is

$$\{\varphi, \psi\}(x) = \langle x, [\mathrm{d}\varphi(x), \mathrm{d}\psi(x)]\rangle$$

for all $\varphi\,, \psi\, \in C^\infty(\mathcal{G}^*)$. The Lie bialgebra of the Poisson-Lie group $\mathcal{G}^*$ is $(\mathcal{G}^*, \mathcal{G})$, where the bracket of $\mathcal{G}^*$ is zero.

Proposition 8. *Let (G, ω) be a Poisson-Lie group with Lie bialgebra $(\mathcal{G}, \mathcal{G}^*)$. Let $\mathcal{G} \dashv \mathcal{G}$ and $\mathcal{G}^* \vdash \mathcal{G}^*$ are the semi-direct products Lie algebras given above. Then $(\mathcal{G} \dashv \mathcal{G}, \mathcal{G}^* \vdash \mathcal{G}^*)$ has the structure of a Lie bialgebra.*

Proof: By a simple calculation, we get

$$\mathrm{ad}^*_{(x,y)}(\alpha, \beta) = (\mathrm{ad}^*_x\alpha + \mathrm{ad}^*_y\beta, \mathrm{ad}^*_x\beta).$$

We need only to prove the relation of definition 7. Let $(x, y), (x', y') \in \mathcal{G} \times \mathcal{G}$ and $(\alpha, \beta) \in \mathcal{G}^* \times \mathcal{G}^*$. Since $(\mathcal{G}, \mathcal{G}^*)$ is a Lie bialgebra we have

$$\begin{aligned}
&\langle[(x, y),(x', y')], [(\alpha, \beta), (\alpha', \beta')]\rangle \\
&\quad=\langle([x, x'], [x, y'] + [y, x']), ([\alpha, \beta'] + [\beta, \alpha'], [\beta, \beta'])\rangle \\
&\quad=\langle[x, x'], [\alpha, \beta']\rangle + \langle[x, x'], [\beta, \alpha']\rangle + \langle[x, y'], [\beta, \beta']\rangle + \langle[y, x'], [\beta, \beta']\rangle \\
&\quad=- [\mathrm{ad}^*_x\alpha, \beta'](x') - [\alpha, \mathrm{ad}^*_x\beta'](x') + [\mathrm{ad}^*_{x'}\alpha, \beta'](x) + [\alpha, \mathrm{ad}^*_{x'}\beta'](x) \\
&\qquad - [\mathrm{ad}^*_x\beta, \alpha']x') - [\beta, \mathrm{ad}^*_x\alpha'](x') + [\mathrm{ad}^*_{x'}\beta, \alpha'](x) + [\beta, \mathrm{ad}^*_{x'}\alpha'](x) \\
&\qquad - [\mathrm{ad}^*_x\beta, \beta'](y') - [\beta, \mathrm{ad}^*_x\beta'](y') + [\mathrm{ad}^*_{y'}\beta, \beta'](x) + [\beta, \mathrm{ad}^*_{y'}\beta'](x) \\
&\qquad - [\mathrm{ad}^*_y\beta, \beta'](x') - [\beta, \mathrm{ad}^*_y\beta'](x') + [\mathrm{ad}^*_{x'}\beta, \beta'](y) + [\beta, \mathrm{ad}^*_{x'}\beta'](y) \\
&\quad=- [\mathrm{ad}^*_{(x,y)}(\alpha, \beta), (\alpha', \beta')](x', y') - [(\alpha, \beta), \mathrm{ad}^*_{(x,y)}(\alpha', \beta')](x', y') \\
&\qquad + [\mathrm{ad}^*_{(x',y')}(\alpha, \beta), (\alpha', \beta')](x, y) + [(\alpha, \beta), [\mathrm{ad}^*_{(x,y)}(\alpha, \beta), (\alpha', \beta')](x, y).
\end{aligned}$$

Then $(\mathcal{G} \dashv \mathcal{G}, \mathcal{G}^* \vdash \mathcal{G}^*)$ is a Lie bialgebra. □

Definition 9. . *Let (G_1, ω_1) and (G_2, ω_2), be two Poisson-Lie groups with Lie bialgebras $(\mathcal{G}_1, \mathcal{G}_1^*)$ and $(\mathcal{G}_2, \mathcal{G}_2^*)$. A Lie group morphism $\varphi : G_1 \longrightarrow G_2$ is called a Poisson-Lie group morphism if it is also a Poisson map.*

A Lie algebra morphism $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ defines a Lie bialgebra morphism from $(\mathcal{G}_1, \mathcal{G}_1^*)$ to $(\mathcal{G}_2, \mathcal{G}_2^*)$, if its transposed map is also a Lie algebra morphism.

Let $\varphi : (G_1, \omega_1) \longrightarrow (G_2, \omega_2)$ be a Poisson-Lie group morphism. Then the tangent map $\varphi_*(e) : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ induces a Lie bialgebra morphism from $(\mathcal{G}_1, \mathcal{G}_1^*)$ to $(\mathcal{G}_2, \mathcal{G}_2^*)$.

Proposition 10. *Let (G,ω) be a Poisson-Lie group with Lie bialgebra $(\mathcal{G},\mathcal{G}^*)$. Then $(\mathcal{G}\dashv\mathcal{G},\mathcal{G}^*\vdash\mathcal{G}^*)$ is the Lie bialgebra of the Poisson-Lie group (TG,Ω_{TG}).*

Proof: The projection

$$\tau_G : TG \longrightarrow G : X_g \longmapsto g$$

is a Poisson-Lie group morphism, where the Poisson structure of G is zero. Then τ_G induces a Lie bialgebra morphism

$$\tau_{G*}(e) : \mathcal{G}\dashv\mathcal{G} \longrightarrow \mathcal{G} : (x,y) \longmapsto x$$

where the bracket of $\mathcal{G}^*$ is also zero.
Hence, we have

$$[(\alpha,0),(\beta,0)] = (0,0)$$

for all $\alpha,\ \beta\in\mathcal{G}^*$.
Let

$$\iota\ : \mathcal{G} \longrightarrow TG : x \longmapsto (e,x)$$

where $(e,x)\in T_eG$ is regarded as element of $\mathcal{G}$. It is clear that ι is a Poisson-Lie group morphism from $\mathcal{G}$ to TG, where $\mathcal{G}$ is the linear Poisson-Lie group associated to the Lie algebra $\mathcal{G}^*$. Then

$$\iota_*(0) : \mathcal{G} \longrightarrow \mathcal{G}\times\mathcal{G} :\ x \longmapsto (0,x)$$

induces a Lie bialgebra morphism from $(\mathcal{G},\mathcal{G}^*)$ to $(\mathcal{G}\dashv\mathcal{G},\mathcal{G}^*\vdash\mathcal{G}^*)$.
Then, for all $\alpha,\beta\in\mathcal{G}^*$, we have

$$[(0,\alpha),(0,\beta)] = (0,[\alpha,\beta]).$$

For the last bracket $[(\alpha,0),(0,\beta)]$, we need the following lemma.

Lemma 11. [12]. *Let (G,ω) be a Poisson-Lie group and (x_i) are local coordinates of G in a neighborhood of e. For all $\alpha\,,\beta\in\mathcal{G}^*$ and $x\in\mathcal{G}$ we have*

$$[\alpha,\beta]_\omega(x) = \sum\frac{\partial\omega_{ij}}{\partial x_k}(e)\alpha_i\beta_j x_k$$

where $\alpha=\Sigma\alpha_i\mathrm{d}x_i$ and $\beta=\Sigma\beta_i\mathrm{d}x_i$.

We turn to the proof of the lemma. Let (x_i) are local coordinates of G in a neighborhood of e and $(x_i, y_j) = (x_i, \dot{x}_j)$, the correspondent local coordinates of TG, in a neighberhood of (e, o). By Remark 3, the Poisson bivector of TG is expressed by

$$\Omega(g, x) = \sum_{ij} \omega_{ij}(g) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \dot{\omega}_{ij}(x) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} \ .$$

Let $\alpha = (\alpha_i)$ and $\beta \ = (\beta_j)$ be elements of $\mathcal{G}^*$. We write

$$(\alpha, 0) = \sum_i \alpha_i \mathrm{d}x_i \quad \text{ and } \quad (0, \beta) = \sum_j \beta_j \mathrm{d}y_j.$$

It follows from the lemma that

$$[(\alpha, 0), (0, \beta)](x, y) = \sum_{i,j,k} \frac{\partial \omega_{ij}(e)}{\partial x_k} \alpha_i \beta_j x_k = [\alpha, \beta](x)$$

for all $(x, y) \in \mathcal{G} \times \mathcal{G}$.
Hence

$$[(\alpha, 0), (0, \beta)] = ([\alpha, \beta], 0).$$

This concludes the proof of the proposition. □

Corollary 12. *Let (G, ω) be a Poisson-Lie group with dual group G^*. Then TG^* is the dual group of the Poisson-Lie group (TG, Ω_{TG}), i.e., $(TG)^* = T(G^*)$.*

Proof: It is clear that the map

$$\rho \ : \ \mathcal{G}^* \times \mathcal{G}^* \longrightarrow \mathcal{G}^* {\times} \mathcal{G}^*, \qquad \rho(\alpha, \beta) \longmapsto (\beta, \alpha)$$

is a Lie bialgebra isomorphism from $(\mathcal{G}^* \dashv \mathcal{G}^*, \mathcal{G} \vdash \mathcal{G})$ to $(\mathcal{G}^* \vdash \mathcal{G}^*, \mathcal{G} \dashv \mathcal{G})$. According to Proposition 10, $(\mathcal{G}^* \dashv \mathcal{G}^*, \mathcal{G} \vdash \mathcal{G})$ is the Lie bialgebra of TG^*. Since TG^* is connected and simply connected, ρ can be integrated to an isomorphism of Poisson Lie groups from TG^* to the dual of (TG, Ω_{TG}). □

Proposition 13. *Let $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ be the double Lie algebra associated to (G, ω) and $\tilde{\mathcal{D}} = (\mathcal{G} \dashv \mathcal{G}) \oplus (\mathcal{G}^* \vdash \mathcal{G}^*)$ be the double Lie algebra associated to the Poisson-Lie group (TG, Ω_{TG}). Then $\tilde{\mathcal{D}}$ is isomorphic to the semi-direct product Lie algebra $\mathcal{D} \dashv \mathcal{D}$.*

Proof: We consider the map

$$\begin{aligned} f:(\mathcal{G}\dashv\mathcal{G})\oplus(\mathcal{G}^*\vdash\mathcal{G}^*) &\longrightarrow (\mathcal{G}\oplus\mathcal{G}^*)\times(\mathcal{G}\oplus\mathcal{G}^*)\\ (x,y)+(\alpha,\beta) &\longmapsto (x+\beta,y+\alpha). \end{aligned}$$

It suffices to show that f is an isomorphism of Lie algebras from $\tilde{\mathcal{D}}$ to $\mathcal{D}\dashv\mathcal{D}$.
In fact, we have

$$\begin{aligned} f([(x,y),(\alpha,\beta)]_{\tilde{\mathcal{D}}}) &= f(\mathrm{ad}^*_{(x,y)}(\alpha,\beta)-\mathrm{ad}^*_{(\alpha,\beta)}(x,y))\\ &= f((-\mathrm{ad}^*_\beta x,-\mathrm{ad}^*_\alpha x-\mathrm{ad}^*_\beta(y))+(\mathrm{ad}^*_x\alpha+\mathrm{ad}^*_y\beta,\mathrm{ad}^*_x\beta))\\ &= (\mathrm{ad}^*_x\beta-\mathrm{ad}^*_\beta x,\mathrm{ad}^*_x\alpha-\mathrm{ad}^*_\alpha x+\mathrm{ad}^*_y\beta-\mathrm{ad}^*_\beta y)\\ &= ([x,\beta],[x,\alpha]+[y,\beta])=[(x,y),(\beta,\alpha)]_{\mathcal{D}\dashv\mathcal{D}}\\ &= [f(x,y),f(\alpha,\beta)]_{\mathcal{D}\dashv\mathcal{D}} \end{aligned}$$

for all $x\ and\ y\in\mathcal{G}$, $\alpha\ and\ \beta\in\mathcal{G}^*$.
Similarly, we get

$$\begin{aligned} f([(x,y),(x',y')]_{\tilde{\mathcal{D}}}) &= [f(x,y),f(x',y')]_{\mathcal{D}\dashv\mathcal{D}},\\ f([(\alpha,\beta),(\alpha',\beta')]_{\tilde{\mathcal{D}}}) &= [f(\alpha,\beta),f(\alpha',\beta')]_{\mathcal{D}\dashv\mathcal{D}}. \end{aligned}$$

□

Proposition 14. *Let H be a Poisson-Lie subgroup of (G,ω). Then TH is also a Poisson-Lie subgroup of TG.*

Proof: By definition, a Poisson-Lie subgroup of G is a Lie subgroup H of G, such that the injection map $\iota : H\longrightarrow G$, is a Poisson morphism.
It is clear that the tangent map $T\iota$ is a Lie group morphism from TH to TG. Furthemore, by theorem 2, the injection map $T\iota$ is also a Poisson map. Hence TH is a Poisson-Lie subgroup of TG. □

4. The Exact Case

Now, we shall discuss an important example of Poisson Lie groups, which is the exact case. Throughout this section, we suppose that G is connected.
A Poisson-Lie group (G,ω) is said to be exact, if the cocycle $\mathrm{d}_e\omega$ is a coboundary; i.e: there exists $r\in\mathcal{G}\wedge\mathcal{G}$ such that $\mathrm{d}_e\omega(x)=\mathrm{ad}_x(r)$, for all $x\in\mathcal{G}$.

Let $r \in \mathcal{G} \wedge \mathcal{G}$, we define a bivector field on G by

$$\omega(g) = L_{g*}r - R_{g*}r, \quad \text{for all} \quad g \in G.$$

By Drinfel'd [1] [2], (G, ω) is a Poisson-Lie group if and only if the algebraic Schouten bracket [r,r] is invariant under the adjoint action of G on $\mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$.

Proposition 15. *Let (G, ω) be an exact Poisson-Lie group with coboundary $\mathrm{d}_e\omega(x) = \mathrm{ad}_x(r)$, where*

$$r = \sum_{ij} r_{ij} r_i \wedge r_j \in \mathcal{G} \wedge \mathcal{G}.$$

Then (TG, Ω_{TG}) is also an exact Poisson-Lie group with coboundary

$$\mathrm{d}_{(e,0)}\Omega(x, y) = \mathrm{ad}_{(x,y)}(\check{r})$$

where

$$\check{r} = \sum_{ij} r_{ij}((r_i, 0) \wedge (0, r_j) + (0, r_i) \wedge (r_j, 0) \in \mathcal{G} \times \mathcal{G} \wedge \mathcal{G} \times \mathcal{G}.$$

Proof: We set $\varepsilon(x) = \mathrm{d}_e\omega(x)$, so that

$$\varepsilon(x)(\alpha, \beta) = [\alpha, \beta](x) = \mathrm{ad}_x(r)(\alpha, \beta) = r(\mathrm{ad}_x^*\alpha, \beta) + r(\alpha, \mathrm{ad}_x^*\beta).$$

We also set $\check{\varepsilon}(x, y) = \mathrm{d}_{(e,0)}\Omega(x, y)$.
Let $r = r_1 \wedge r_2$, for all $\alpha,\ \beta,\ \alpha',\ \beta' \in \mathcal{G}^*$ and $x,\ y \in \mathcal{G}$, we have

$$\begin{aligned}
&\check{\varepsilon}(x, y)((\alpha, \beta), (\alpha', \beta')) = [(\alpha, \beta), (\alpha', \beta')](x, y) \\
=& [\alpha, \beta](x) + [\beta, \alpha'](x) + [\beta, \beta'](y) = r_1 \wedge r_2((\mathrm{ad}_x^*\alpha, \beta') + (\alpha, \mathrm{ad}_x^*\beta')) \\
&+ r_1 \wedge r_2((\mathrm{ad}_x^*\beta, \alpha') + (\beta, (\mathrm{ad}_x^*\alpha')) + r_1 \wedge r_2(((\mathrm{ad}_y^*\beta, \beta') + (\beta, (\mathrm{ad}_y^*\beta')) \\
=& r_1(\mathrm{ad}_x^*\alpha)r_2(\beta') - r_2(\mathrm{ad}_x^*\alpha)r_1(\beta') + r_1(\alpha)r_2(\mathrm{ad}_x^*\beta') - r_2(\alpha)r_1(\mathrm{ad}_x^*\beta') \\
&+ r_1(\mathrm{ad}_x^*\beta)r_2(\alpha') - r_2(\mathrm{ad}_x^*\beta)r_1(\alpha') + r_1(\beta)r_2(\mathrm{ad}_x^*\alpha') - r_2(\beta)r_1(\mathrm{ad}_x^*\alpha') \\
&+ r_1(\mathrm{ad}_y^*\beta)r_2(\beta') - r_1(\beta')r_2(\mathrm{ad}_y^*\beta) + r_1(\beta)r_2(\mathrm{ad}_y^*\beta') \\
&- r_2(\beta)r_1(\mathrm{ad}_y^*\beta') = r_1(\mathrm{ad}_x^*\alpha + \mathrm{ad}_y^*\beta)r_2(\beta') - r_1(\alpha')r_2(\mathrm{ad}_x^*\beta) \\
&+ r_1(\alpha)r_2(\mathrm{ad}_x^*\beta') - r_2(\beta)r_1(\mathrm{ad}_x^*\alpha' + \mathrm{ad}_y^*\beta') + r_1(\mathrm{ad}_x^*\beta)r_2(\alpha') \\
&- r_1(\beta')r_2(\mathrm{ad}_x^*\alpha + \mathrm{ad}_y^*\beta) + r_2(\mathrm{ad}_x^*\alpha' + \mathrm{ad}_y^*\beta')r_1(\beta) - r_2(\alpha)r_1(\mathrm{ad}_x^*\beta') \\
=& (r_1, 0) \wedge (0, r_2)((\mathrm{ad}_x^*\alpha + \mathrm{ad}_y^*\beta, \mathrm{ad}_x^*\beta), (\alpha', \beta'))
\end{aligned}$$

$$
\begin{aligned}
&+ (r_1,0)\wedge(0,r_2)((\alpha,\beta),(\mathrm{ad}_x^*\alpha' + \mathrm{ad}_y^*\beta', \mathrm{ad}_x^*\beta'))\\
&+ (0,r_1)\wedge(r_2,0)((\mathrm{ad}_x^*\alpha + \mathrm{ad}_y^*\beta, \mathrm{ad}_x^*\beta),(\alpha',\beta'))\\
&+ (0,r_1)\wedge(r_2,0)((\alpha,\beta),(\mathrm{ad}_x^*\alpha' + \mathrm{ad}_y^*\beta', \mathrm{ad}_x^*\beta'))\\
=&((r_1,0)\wedge(0,r_2)+(0,r_1)\wedge(r_2,0))((\mathrm{ad}_{(x,y)}^*(\alpha,\beta),(\alpha',\beta'))\\
&+((\alpha,\beta),\mathrm{ad}_{(x,y)}^*(\alpha',\beta')) = \mathrm{ad}_{(x,y)}(\check{r})((\alpha,\beta),(\alpha',\beta'))
\end{aligned}
$$

where

$$\check{r} = (r_1,0)\wedge(0,r_2)+(0,r_1)\wedge(r_2,0)\in(\mathcal{G}\times\mathcal{G})\wedge(\mathcal{G}\times\mathcal{G}).$$

For the general case: $r=\sum_{ij} r_{ij} r_i\wedge r_j$, we get

$$\check{r} = \sum_{ij} r_{ij}((r_i,0)\wedge(0,r_j)+(0,r_i)\wedge(r_j,0)).$$

□

Remark 16. *If G is connected and simply connected, the bivector ω is of the form*

$$\omega(g) = L_{g*}r - R_{g*}r$$

where $[r,r]$ is Ad_G-invariant. Since TG is also connected and simply connected, and as $\check{\varepsilon}$ is exact, the bivector Ω_{TG} is given by

$$\Omega(X_g) = L_{X_g*}\check{r} - R_{X_g*}\check{r}.$$

Furthermore $[\check{r},\check{r}]$ is Ad_{TG}-invariant.

5. Poisson Action Lifting

One of the fundamental notions related to Poisson-Lie groups is that of Poisson action. The famous example of dressing action [7], plays an important role for the description of the Poisson structure of G.

In this section, we will be interested in the lifting of Poisson actions.

Definition 17. *A left action $\phi : G\times P \longrightarrow P$ of a Poisson-Lie group (G,ω) on a Poisson manifold P is called a Poisson action, if it is a Poisson map with respect to the product Poisson structure on $G\times P$.*

Let $\phi : G \times P \longrightarrow P$ be a Poisson action of G on P. Naturally, we have to regard the lifted action of G on TP given by

$$\check{\phi} : \ G \times TP \longrightarrow TP : (g, u_p) \longmapsto \phi_{g*}(u_p).$$

In the particular case, when G is equipped with the trivial Poisson structure, ϕ is just an action of G on P by Poisson morphisms. Then $\check{\phi}$ is also an action of G on TP by Poisson morphisms. Since G is trivial, $\check{\phi}$ is a Poisson action.

In the general case, this is not true. In fact, if ϕ is the left translation of G, for all $\varphi, \psi \in C^\infty(G)$, $g, h \in G$ and $X_h \in T_hG$ we have

$$\{\hat{\varphi}_g, \hat{\psi}_g\}(X_h) + \{\hat{\phi}_{X_h}, \hat{\psi}_{X_h}\}(g) = \{(\varphi \circ L_g\hat{)}, (\psi \circ L_g\hat{)}\}(X_h) + \{\varphi \circ R_h, \psi \circ R_h\}(g).$$

Since $\{\hat{\varphi}, \hat{\psi}\}(L_{g*}X_h) = 0$, $\check{\phi}$ is a Poisson action if and only if

$$\{\varphi \circ R_h, \psi \circ R_h\} = 0$$

for all $\varphi, \psi \in C^\infty(G)$, i.e. G is trivial.

For this reason, we will be interested in an other lifted action, that of TG on TP.

Theorem 18. *Let $\phi : G \times P \longrightarrow P$ be a Poisson action of the Poisson-Lie group (G, ω) on a Poisson manifold P. We assume that TP is equipped with the Poisson structure given in Theorem 2. Let*

$$\Phi \ : \ TG \times TP \longrightarrow TP \ : \ (X_g, u_p) \longmapsto T_{g,p}\phi(X_g, u_p) = \phi_{g*}u_p + \phi_{p*}X_g.$$

Then, Φ is a Poisson action of the Poisson-Lie group (TG, Ω_{TG}) on the Poisson manifold (TP, Ω_{TP}).

Proof: We know that the tangent map $T\phi \ : \ T(G \times P) \longrightarrow TP$ is a Poisson morphism. It suffices to show that the canonical bundle isomorphism

$$\begin{aligned} \rho \ : \ T(G \times P) &\longrightarrow TG \times TP \\ X &\longmapsto (\pi_{1*}X, \pi_{2*}X) \end{aligned}$$

is a Poisson morphism, where π_1 and π_2 are respectively the canonical projections from $G \times P$ on G and P.

Let (x_i) be local coordinates on G and (y_j) be local coordinates on P. Then ρ sends the local coordinates $(x_i, y_j, \dot{x}_i, \dot{y}_j)$ of $T(G \times P)$ to the local coordinates $((x_i, \dot{x}_i), (y_j, \dot{y}_j))$ of $TG \times TP$.

According to Remark 2 and Remark 3 and using the definition of direct Poisson structure, we have the following equalities:

$$
\begin{aligned}
&\{x_i, x_j\}_{T(G\times P)} = \{y_i, y_j\}_{T(G\times P)} = \{x_i, y_j\}_{T(G\times P)} = 0\\
&\{x_i, \dot{x}_j\}_{T(G\times P)} = \{x_i, x_j\}_{G\times P} = \{x_i, x_j\}_G = \omega_{ij}(x)\\
&\{y_i, \dot{y}_j\}_{T(G\times P)} = \{y_i, y_j\}_{G\times P} = \{y_i, y_j\}_P = t_{ij}(y)\\
&\{x_i, \dot{y}_j\}_{T(G\times P)} = \{x_i, \dot{y}_j\}_{G\times P} = 0\\
&\{\dot{x}_i, \dot{x}_j\}_{T(G\times P)} = \dot{\{x_i, x_j\}}_{G\times P} = \dot{\omega}_{ij}(\dot{x})\\
&\{\dot{y}_i, \dot{y}_j\}_{T(G\times P)} = \dot{\{y_i, y_j\}}_{G\times P} = \dot{t}_{ij}(\dot{y})\\
&\{\dot{x}_i, \dot{y}_j\}_{T(G\times P)} = \dot{\{x_i, y_j\}}_{G\times P} = 0.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\{x_i, x_j\}_{TG\times TP} = \{y_i, y_j\}_{TG\times TP} = \{x_i, y_j\}_{TG\times TP} = 0\\
&\{x_i, \dot{x}_j\}_{TG\times TP} = \{x_i, \dot{x}_j\}_{TG} = \{x_i, x_j\}_G = \omega_{ij}(x)\\
&\{y_i, \dot{y}_j\}_{TG\times TP} = \{y_i, \dot{y}_j\}_{TP} = \{y_i, y_j\}_P = t_{ij}(y)\\
&\{x_i, \dot{y}_j\}_{TG\times TP} = 0\\
&\{\dot{x}_i, \dot{x}_j\}_{TG\times TP} = \{\dot{x}_i, \dot{x}_j\}_{TG} = \dot{\{x_i, x_j\}}_G = \dot{\omega}_{ij}(\dot{x})\\
&\{\dot{y}_i, \dot{y}_j\}_{TG\times TP} = \{\dot{y}_i, \dot{y}_j\}_{TP} = \dot{\{y_i, y_j\}}_P = \dot{t}_{ij}(\dot{y})\\
&\{\dot{x}_i, \dot{y}_j\}_{TG\times TP} = 0.
\end{aligned}
$$

The proof is completed. □

Remark 19. *If we consider the case of the left action of G on itself we can deduce Theorem* 5*.*

Example 20. *Let*

$$\phi \;:\; G \times \mathcal{G}^* \longrightarrow \mathcal{G}^* \;: (g, \xi) \longmapsto \mathrm{Ad}_g^* \xi$$

be the coadjoint action of G on g^*. It is a Poisson action, when G is equipped with the trivial Poisson structure and $\mathcal{G}^*$ with the linear Poisson structure.

We have

$$
\begin{aligned}
\Phi \;:\; TG \times T\mathcal{G}^* &\longrightarrow T\mathcal{G}^*\\
(X_g, (\xi, \eta)) &\longmapsto (\phi_g(\xi), \phi_{g*}(\eta) + \phi_{\xi *} X_g).
\end{aligned}
$$

Then

$$\Phi \; : \; (G \times \mathcal{G}) \times (\mathcal{G}^* \times \mathcal{G}^*) \longrightarrow \mathcal{G}^* \times \mathcal{G}^*$$
$$((g,x),(\xi,\eta)) \longmapsto (\mathrm{Ad}^*_g \xi, \mathrm{Ad}^*_g \eta + \phi_{\xi *}(R_{g*}x)).$$

On the other hand, we have:

$$(\phi_\xi \circ R_g)(h) = \mathrm{Ad}^*_{gh}(\xi) = \mathrm{Ad}^*_h(\mathrm{Ad}^*_g \xi)$$
$$(\phi_\xi \circ R_g)_*(x) = -\mathrm{ad}^*_x(\mathrm{Ad}^*_g \xi).$$

Consider the semi-direct product $G \dashv \mathcal{G}$. By duality and transposition, we obtain the following formula for the coadjoint action, which is valid for all $g \in G,\ x \in \mathcal{G},\ \xi \in \mathcal{G}$ and $\eta \in \mathcal{G}^*$

$$\mathrm{Ad}_{(g,x)(\xi,\eta)} = (\mathrm{Ad}^*_g \xi - \mathrm{ad}^*_x(\mathrm{Ad}^*_g), \mathrm{Ad}^*_g \eta).$$

Corresponding to Example 6, *$T\mathcal{G}^*$ is identified with $(\mathcal{G} \dashv \mathcal{G})^*$ by*

$$T\mathcal{G}^* \longrightarrow \mathcal{G}^* \times \mathcal{G}^* \; : \; (\xi,\eta) \longmapsto (\eta,\xi).$$

Since $G \dashv \mathcal{G}$ is also equipped with the null Poisson structure and since Φ is the coadjoint action associated to the semi product $G \dashv \mathcal{G}$, the map Φ is a Poisson action.

6. Dressing Actions

Example 20 is a particular case of dressing actions [5,7]. Let us recall this notion. In the following, we assume that (G,ω) is a simply connected Poisson-Lie group, with dual group G^*. Let D be the simply connected Lie group, with Lie algebra $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$. By [7], the map

$$\psi \; : \; G \times G^* \longrightarrow D \; : \; (g,u) \longmapsto gu$$

is a local diffeomorphism. When it is a global diffeomorphism, D is called a double Lie group. In this case, let $g \in G$ and $u \in G^*$, the product ug can be uniquely written as : $ug = g^u u^g$, where $g^u \in G$ and $u^g \in G^*$. This define a left action of G^* on G by

$$\phi \; : G^* \times G \longrightarrow G \; : \; (g,u) \longmapsto g^u$$

and a right action of G on G^* by

$$\phi' \; : \; G^* \times G \longrightarrow G \; : \; (g,u) \longmapsto u^g.$$

These actions are called dressing actions, they are Poisson actions. The orbits of ϕ and ϕ' are respectively the symplectic leaves of G and G^*.

Proposition 21.

i) *Assume that D is a double Lie group. Then TD is a double Lie group.*

ii) *Let*

$$\phi \ : G^* \times G \longrightarrow G \ : \ (g,u) \longmapsto g^u$$

be the left dressing action of G^* on G. Then the lifted action

$$\Phi \ : \ TG^* \times TG \longrightarrow TG \ : \ (X_u, Y_g) \longrightarrow \phi_{u*}Y_g + \phi_{g*}X_u$$

is exactly the left dressing action of TG^* on TG.

Proof:

i) Since D is a double Lie group, then

$$T\psi \ : \ TG \times TG^* \longrightarrow TD \ : \ (X_g, Y_u) \longmapsto L_{g*}Y_u + R_{u*}X_g$$

is a vector bundle isomorphism. Furthermore for all $X_g,\ Y_u \in TD$ we have

$$X_g Y_u = L_{g*}Y_u + R_{u*}X_g.$$

Hence TD is a double Lie group associated to the Poisson-Lie group TG.

ii) By definition

$$\Phi \ : \ TG^* \times TG \longrightarrow TG \ : \ (X_u, Y_g) \longrightarrow (g^u, \phi_{u*}Y_g + \phi_{g*}X_u).$$

On the other hand,

$$ug = g^u u^g = \phi_u(g).\phi'_u(g).$$

Then, we have

$$L_{u*}Y_g = L_{g^u*}\phi'_{u*}Y_g + R_{u^g*}\phi_{u*}Y_g.$$

Similarly, we have

$$R_{g*}X_u = L_{g^u*}\phi'_{g*}(X_u) + R_{u^g*}\phi_{g*}(X_u).$$

Hence

$$\begin{aligned} X_u Y_g &= L_{u*}Y_g + R_{g*}X_u \\ &= L_{g^u*}(\phi'_{u*}Y_g + \phi'_{g*}X_u) + R_{u^g*}(\phi_{u*}Y_g + \phi_{g*}X_u) \\ &= (g^u, \phi_{u*}Y_g + \phi_{g*}X_u)(u^g, \phi'_{u*}Y_g + \phi'_{g*}X_u). \end{aligned}$$

Then the left dressing action of TG^* on TG is given by

$$TG^* \times TG \longrightarrow TG \ : \ (X_u, Y_g) \longrightarrow (g^u, \phi_{u*}Y_g + \phi_{g*}X_u).$$

This conclued the proof. □

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