JGSP **39** (2015) 45–53

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

RICCI FLOW ON MODIFIED RIEMANN EXTENSIONS

HALAMMANAVAR G. NAGARAJA AND HARISH DAMMU

Communicated by Bernard Kay

Abstract. We study the properties of the modified Riemann extensions evolving under the Ricci flow with examples.

MSC: 53C20, 53C44
Keywords: Evolution equations, Ricci flow, Riemann extension

1. Introduction

The Ricci flow and the evolution equations of the Riemannian curvature tensor were initially introduced by Hamilton [8] and was later studied to a large extent by Perelman [13–15], Cao and Zhu [4], Morgan and Tian [10]. Indeed, the theory of Ricci flow has been used to prove the geometrization and Poincare conjectures [1]. However not much work has been done on Ricci flows on modified Riemann extensions. The Ricci flow equation is the evolution equation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ where g_{ij} and R_{ij} are metric components respectively. As flow progresses the metric changes and hence the properties related to it.

Patterson and Walker [11] have defined Riemann extensions and showed how a Riemannian structure can be given to the $2n$ dimensional tangent bundle of an n-dimensional manifold with given non-Riemannian structure. This shows that Riemann extension provides a solution of the general problem of embedding a manifold M carrying a given structure in a manifold $\tilde{M}$ carrying another structure, the embedding being carried out in such a way that the structure on $\tilde{M}$ induces in a natural way the given structure on M. The Riemann extension of Riemannian or non-Riemannian spaces can be constructed with the help of the Christoffel coefficients Γ^i_{jk} of corresponding Riemann space or with connection coefficients Π^i_{jk} in the case of the space of affine connection [5]. The theory of Riemann extensions has been extensively studied by Afifi [1]. Though the Riemann extensions itself is rich in geometry, here in our discussions, the modified Riemann extensions fit naturally in to the frame work. Modified Riemann extensions are introduced in [2] and their properties we list briefly in the next section.

doi: 10.7546/jgsp-39-2015-45-53

In this paper we discuss some interesting properties satisfied by curvature tensors under the influence of the Ricci flow on modified Riemann extensions. We give a brief introduction to modified Riemann extensions [11] in Section 2. In Section 3 we find the rate of change of concircular, conharmonic and conformal curvature tensors under the Ricci flow. The Ricci flow on modified Riemann extensions are discussed in Section 4.

2. Preliminaries

Let (M, g) be a n-dimensional Riemannian manifold. Then Ricci flow is the evolution of the metric given by

$$\frac{\partial g_{il}}{\partial t} = -2R_{il} \tag{1}$$

where g_{il} is the metric component and R_{il} is the component of the Ricci curvature tensor.

For a time dependent metric under Ricci flow, the evolution equations for Riemann curvature tensor, Ricci tensor and scalar curvature are given by [8]

$$\begin{aligned}\frac{\partial R_{ijkl}}{\partial t} &= \triangle R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql})\end{aligned} \tag{2}$$

$$\frac{\partial R_{ij}}{\partial t} = \triangle R_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj} \tag{3}$$

and

$$\frac{\partial R}{\partial t} = \triangle R + 2g^{ij}g^{kl}R_{ik}R_{jl} \tag{4}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and R_{ijkl}, R_{ij}, R are the Riemannian curvature tensor, Ricci tensor and scalar curvature respectively.

Let ∇ be a torsion-free affine connection of M. The modified Riemann extension of (M, ∇) is the cotangent bundle T^*M equipped with a metric $\bar{g}$ whose local components given by

$$\bar{g}_{ij} = -2\omega_l\Gamma^l_{ij} + c_{ij}, \qquad \bar{g}_{ij^*} = \delta^j_i, \qquad \bar{g}_{i^*j} = \delta^j_i \qquad \text{and} \qquad \bar{g}_{i^*j^*} = 0 \tag{5}$$

where Γ^l_{ij} are the connection coefficients of M.

The contravariant components are

$$\bar{g}^{ij} = 0, \qquad \bar{g}^{ij^*} = \delta^j_i, \qquad \bar{g}^{i^*j} = \delta^j_i \qquad \text{and} \qquad \bar{g}^{i^*j^*} = 2\omega_l\Gamma^l_{ij} - c_{ij} \tag{6}$$

for i, j ranging from 1 to n and i^*, j^* ranging from $n+1$ to $2n$, where ω_l are extended coordinates and c_{ij} is a $(0,2)$ tensor on M.

We note following results for the connection coefficients on extended space

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}, \qquad \bar{\Gamma}^k_{i^*j} = 0, \qquad \bar{\Gamma}^{k^*}_{i^*j} = -\Gamma^i_{jk}, \qquad \bar{\Gamma}^k_{i^*j^*} = 0$$
$$\bar{\Gamma}^{k^*}_{i^*j^*} = 0, \qquad \bar{\Gamma}^{k^*}_{ij} = \omega_l R^l_{kji} + \frac{1}{2}(\nabla_i c_{jk} + \nabla_j c_{ik} - \nabla_h c_{ij}).$$

The components of the Riemann curvature tensor of the extended space are given by

$$\bar{R}^{i^*}_{jkl} = \frac{1}{2}(\nabla_j(\nabla_l c_{ki} - \nabla_i c_{kl}) - \nabla_k(\nabla_l c_{ji} - \nabla_i c_{jl}) - R^m_{jkl}c_{mi} - R^m_{jki}c_{lm}) + \omega_a(\nabla_j R^a_{ilk} - \nabla_k R^a_{ilj})$$
$$\bar{R}^{i^*}_{j^*kl} = R^j_{ilk}, \qquad \bar{R}^{i^*}_{jk^*l} = -R^k_{ilj}, \qquad \bar{R}^i_{jkl} = R^i_{jkl} \quad \text{and} \quad \bar{R}^{i^*}_{jkl^*} = R^l_{kji}.$$

The others are zero. Here i^*, j^*, k^*, l^* ranges from $n+1$ to $2n$.

It may be noted by simple calculation that $\bar{R}_{i^*jk^*l} = 0$ which we require later on. Further, $\bar{R}_{ij} = R_{ij} + R_{ji}$, $\bar{R}_{i^*j} = 0$ and $\bar{R}_{i^*j^*} = 0$. Here bar is used for components of extended space. However in last section we do not use bar though they are components of modified Riemann extension as the calculations involve only the extended space.

3. Evolution

Under the Ricci flow given in equation (1), the rate of change of conformal curvature tensor depends on the difference of conharmonic and Riemannian curvature tensors. The concircular and conharmonic curvature tensors are respectively given by

$$C_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik}) \tag{7}$$

and

$$L_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{jk}R_{il} + g_{il}R_{jk} - g_{ik}R_{jl} - g_{jl}R_{ik}). \tag{8}$$

Under Ricci flow, we give a relation between conformal tensor and conharmonic tensor.

Theorem 1. *For a manifold with non zero scalar curvature under Ricci flow, the rate of change of concircular tensor is related to conharmonic tensor by*

$$\frac{\partial}{\partial t}\Big(\frac{C_{ijkl} - R_{ijkl}}{R}\Big) = \frac{2(n-2)}{n(n-1)}(R_{ijkl} - L_{ijkl}). \tag{9}$$

Proof: Differentiating equation (7) we get

$$\frac{\partial C_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{1}{n(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik})\frac{\partial R}{\partial t} - \frac{R}{n(n-1)}\left(\frac{\partial g_{il}}{\partial t}g_{jk} + \frac{\partial g_{jk}}{\partial t}g_{il} - \frac{\partial g_{ik}}{\partial t}g_{jl} - \frac{\partial g_{jl}}{\partial t}g_{ik}\right). \tag{10}$$

Using (1) in (10), we get

$$\frac{\partial C_{ijkl}}{\partial t} - \frac{\partial R_{ijkl}}{\partial t} = -\frac{1}{n(n-1)}(g_{il}g_{jk} - g_{jl}g_{ik})\frac{\partial R}{\partial t} + \frac{R}{n(n-1)}\left(2R_{il}g_{jk} + 2R_{jk}g_{il} - 2R_{ik}g_{jl} - 2R_{jl}g_{ik}\right). \tag{11}$$

But from (7) and (8), we obtain

$$g_{il}g_{jk} - g_{jl}g_{ik} = \frac{n(n-1)}{R}(R_{ijkl} - C_{ijkl}) \tag{12}$$

and

$$R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} = (n-2)(R_{ijkl} - L_{ijkl}). \tag{13}$$

Substituting equations (12) and (13) in (15) we get

$$\frac{\partial C_{ijkl}}{\partial t} - \frac{\partial R_{ijkl}}{\partial t} = -\frac{1}{R}(R_{ijkl} - C_{ijkl})\frac{\partial R}{\partial t} + \frac{2(n-2)R}{n(n-1)}(R_{ijkl} - L_{ijkl}). \tag{14}$$

Hence

$$R\frac{\partial}{\partial t}(C_{ijkl} - R_{ijkl}) - (C_{ijkl} - R_{ijkl})\frac{\partial R}{\partial t} = \frac{2(n-2)R^2}{n(n-1)}(R_{ijkl} - L_{ijkl}). \tag{15}$$

Therefore

$$\frac{\partial}{\partial t}\left(\frac{C_{ijkl} - R_{ijkl}}{R}\right) = \frac{2(n-2)}{n(n-1)}(R_{ijkl} - L_{ijkl}). \tag{16}$$

■

Example 2. Let M be a Riemannian manifold with a space of constant curvature with $K = \frac{1}{1-n}$. Then evolution of the metric under Ricci flow is given by $g_{ij}(t) = g_{ij}(0)\mathrm{e}^{-2t}$ and $R_{ijkl}(t) = R_{ijkl}(0)\mathrm{e}^{-4t}$. Further, $C_{ijkl} - R_{ijkl} = -\frac{R}{n}R_{ijkl}$ and $L_{ijkl} - R_{ijkl} = \frac{2(n-1)}{n-2}R_{ijkl}$. Substituting this in equation (16) the above result is verified.

The Weyl conformal tensor is given by

$$\begin{aligned} W_{ijkl} = & R_{ijkl} - \frac{1}{n-2}(g_{jk}R_{il} - g_{ik}R_{jl} + g_{il}R_{jk} - g_{jl}R_{ik}) \\ & + \frac{R}{(n-1)(n-2)}(g_{il}g_{jk} - g_{jl}g_{ik}). \end{aligned} \tag{17}$$

Equations (7), (8) and (17) can be combined to form

$$W_{ijkl} - L_{ijkl} = -\frac{n}{n-2}(C_{ijkl} - R_{ijkl}). \tag{18}$$

Theorem 3. *For a n-manifold under the Ricci flow*

$$\frac{\partial}{\partial t}\Big(\frac{W_{ijkl} - L_{ijkl}}{R}\Big) = \frac{2}{n-1}(L_{ijkl} - R_{ijkl}). \tag{19}$$

Proof: Differentiating equation (18) with respect to 't' and using Theorem 1 the result follows. ■

4. Extensions

We note that for modified Riemann extensions, since the scalar curvature vanishes, the concircular curvature tensor is same as the Riemannian curvature tensor. Further the conharmonic curvature tensor is equal to the conformal curvature tensor.

The Ricci flow on modified Riemann extensions is the evolution of metric such that the class of metrics obtained under Ricci flow can be expressed as modified Riemann extensions of a base metric. We prove the following results for Ricci flow on modified Riemann extensions.

Lemma 4. *Laplacian of Ricci tensor is zero on modified Riemann extension.*

Proof: The Laplacian of the Ricci tensor is given by

$$\triangle R_{ij} = g^{kl}R_{ij:k:l}. \tag{20}$$

But

$$\begin{aligned} g^{kl}R_{ij:k:l} = & g^{kl}R_{ij,k,l} - g^{kl}\Gamma^{\alpha}_{jk,l}R_{\alpha i} - g^{kl}\Gamma^{\alpha}_{jk}R_{\alpha i,l} - g^{kl}\Gamma^{\alpha}_{ik,l}R_{\alpha j} \\ & - g^{kl}\Gamma^{\alpha}_{ik}R_{\alpha j,l} - g^{kl}\Gamma^{\alpha}_{il}R_{\alpha j,k} + g^{kl}\Gamma^{\alpha}_{il}\Gamma^{\beta}_{k\alpha}R_{\beta j} + g^{kl}\Gamma^{\alpha}_{il}\Gamma^{\beta}_{jk}R_{\beta\alpha} \\ & - g^{kl}\Gamma^{\alpha}_{jl}R_{i\alpha,k} + g^{kl}\Gamma^{\alpha}_{jl}\Gamma^{\beta}_{\alpha k}R_{i\beta} + g^{kl}\Gamma^{\alpha}_{jl}\Gamma^{\beta}_{ik}R_{\beta\alpha} - g^{kl}\Gamma^{\alpha}_{kl}R_{ij,\alpha} \\ & + g^{kl}\Gamma^{\alpha}_{kl}\Gamma^{\beta}_{i\alpha}R_{\beta j} + g^{kl}\Gamma^{\alpha}_{kl}\Gamma^{\beta}_{j\alpha}R_{i\beta}. \end{aligned} \tag{21}$$

From the properties of extended metric components we have, g^{kl} to be non zero at least one of k or l must be greater than n. Suppose $k > n$, then $R_{ij,k} = 0$. Also $R_{\alpha i} \neq 0$ only when $\alpha < n$ and $i < n$. But if $\alpha \leq n$ then $\Gamma^{\alpha}_{jk} = 0$, since $k > n$. Similar argument makes all the terms on the right side of the equation to vanish. If $l > n$ then again $R_{ij,k,l}$ vanishes since $R_{ij,k}$ is a function of first n coordinates. Also, since Christoffel symbols are preserved by extension, $\Gamma^{\alpha}_{jk,l}$ vanishes. Hence the result. ■

Theorem 5. *The Ricci curvature tensor is independent of time for Ricci flow on modified Riemann extensions.*

Proof: The rate of change of Ricci tensor is given by

$$\frac{\partial R_{ik}}{\partial t} = \triangle R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}. \tag{22}$$

For i or k greater than n, $R_{ik} = 0$ where n is the dimension of the manifold. It is sufficient to prove for i, k ranging from 1 to n. For g^{pr} and g^{qs} to be non zero, either $p > n$ or $r > n$ and $q > n$ or $s > n$. Suppose $p > n$ and $q > n$. Then as discussed earlier $R_{piqk} = 0$. If $s > n$ or $r > n$ then $R_{rs} = 0$. Thus $2g^{pr}g^{qs}R_{piqk}R_{rs} = 0$. Now g^{pq} is non zero for $p > n$ or $q > n$. But if $p > n$, $R_{pi} = 0$ and similarly if $q > n$, $R_{qk} = 0$. Hence the result. ■

It must be noted here that the flow is not on the base manifold but on the extended space. We have proved the necessary and sufficient conditions for modified Riemann extension under Ricci flow to stay as modified Riemann extensions.

We can restate the result in terms of metric.

Theorem 6. *The Ricci flow on modified Riemann extensions is linear.*

Proof: Under the Ricci flow on modified Riemann extensions, the Ricci tensor is time invariant. Hence on solving (1) we get

$$g_{jk}(t) = R_{jk}t + g_{jk}(0). \tag{23}$$

Thus the metric is linearly varying with time. ■

Example 7. *Modified Riemann extension of Schwarzchild metric has vanishing Ricci tensor and hence remains a trivial example.*

Example 8. Consider the hyperbolic metric $\mathrm{d}s^2 = \frac{1}{y^2}\mathrm{d}x^2 + \frac{1}{y^2}\mathrm{d}y^2$. The modified Riemannian extension of this metric is

$$\mathrm{d}s^2 = -\frac{4P}{y}\mathrm{d}x^2 - \frac{8P}{y}\mathrm{d}x\mathrm{d}y + \frac{4Q}{y}\mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}P + 2\mathrm{d}Q\mathrm{d}y \tag{24}$$

where $c_{ij} = 0$ (equation (5)).
Then $R_{11} = \frac{2}{y^2} = R_{22}$ and rest of the components equal to zero. Thus $g_{11} = \frac{2}{y^2}t - \frac{4Q}{y}$ and $g_{22} = -\frac{4Q}{y} + \frac{2}{y^2}t$ with rest of the components independent of time which are the required class of metric components.

Theorem 9. *For modified Riemann extensions under Ricci flow, the rate of change of extended components of the Weyl conformal tensor is the same as the rate of change of extended components of the Riemann curvature tensor.*

Proof: For the extended space, the Weyl conformal tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik}). \tag{25}$$

Differentiating partially with respect to 't' we get

$$\begin{aligned}\frac{\partial W_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{1}{n-2}\Bigg(&\frac{\partial g_{ik}}{\partial t}R_{jl} + g_{ik}\frac{\partial R_{jl}}{\partial t} - \frac{\partial g_{il}}{\partial t}R_{jk} - g_{il}\frac{\partial R_{jk}}{\partial t} \\ &- \frac{\partial g_{jk}}{\partial t}R_{il} - g_{jk}\frac{\partial R_{il}}{\partial t} + \frac{g_{jl}}{\partial t}R_{ik} + g_{jl}\frac{\partial R_{ik}}{\partial t}\Bigg).\end{aligned} \tag{26}$$

Using previous theorem and (1), we get

$$\frac{\partial W_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{4}{n-2}(R_{il}R_{jk} - R_{ik}R_{jl}). \tag{27}$$

Here again for any two of i, j, k, l greater than n the Ricci components are zero. In particular for all of them greater than n, we get the above result. ■

Conclusion

We have found the necessary and sufficient conditions for the modified Riemann extension under Ricci flow evolving to obtain a class of metrics which again are modified Riemann extensions. While dealing with flow on manifold with general pseudo Riemannian metric we have to prove existence and uniqueness theorems. However when the flow is restricted to modified Riemannian extensions, we get the solutions straightaway.

Acknowledgements

This work is supported by CSIR 09/039(0106)2012-EMR-I. We would also like to thank E. Calviño-Louzao for his kind suggestions.

References

[1] Afifi Z., *Riemann Extension of Affine Connected Spaces,* Quart. J. Math. Oxford **5** (1954) 312–330.

[2] Calviño-Louzao E., García-Río E., Gilkey P. and Vazquez-Lorenzo A., *The Geometry of Modified Riemannian Extensions*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **465** (2009) 2023–2040.

[3] Calviño-Louzao E., García-Río E. and Vázquez-Lorenzo R., *Riemann Extensions of Torsionfree Connections with Degenerate Ricci Tensor,* Can. J. Math. **62** (2010) 1037–1057.

[4] Cao H.-D. and Zhua X.-P., *Complete Proof of The Poincare and Geometrization conjectures - Application of the Hamilton-Perelman Theory of The Ricci Flow,* Asian J. Math. **10** (2006) 165-492.

[5] Dryuma V., *Four Dimensional Einstein Spaces on Six Dimensional Ricci Flat Base Spaces,* arXiv:gr-qc/0601051.

[6] Eisenhart L., *Fields Of Parallel Vectors in Riemannian Space,* Ann. Math. **39** (1938) 316–321.

[7] Gezer A., Bilen L. and Cakmak A., *Properties of Modified Riemannian Extensions,* arXiv:1305.4478v2 [math.DG].

[8] Hamilton R., *Three-Manifolds with Positive Ricci Curvature,* J. Diff. Geom. **17** (1982) 255–306.

[9] Kowalski O. and Sekizawa M., *The Riemann Extensions With Cyclic Parallel Ricci Tensor,* Math.Nachr. **287** (2014) 955–961.

[10] Morgan J. and Tian G., *Ricci Flow and the Poincare Conjecture,* arXiv:math.DG/0607607 v1.

[11] Paterson E. and Walker A., *Riemann Extensions,* Quart. J. Math. Oxford **3** (1952) 19–28.

[12] Walker A., *Canonical Form for a Riemannian Space with Parallel Field of Null Planes,* Quart. J. Math. Oxford, **1** (1950) 67–69.

[13] Perelman G., *The Entropy Formula for the Ricci Flow and Its Geometric Applications,* arXiv:math/0211159v1 [math.DG].

[14] Perelman G., *Ricci Flow with Surgery on Three-Manifolds,* arXiv:math/0303109v1 [math.DG].

[15] Perelman G., *Finite Extinction Time For the Solutions to the Ricci Flow on Certain Three-Manifolds,* arXiv:math/0307245 v1 [math.DG].

Received 30 June 2015

Halammanavar G. Nagaraja
Department of Mathematics
Bangalore University
Bengaluru-560001
INDIA
E-mail address: hgnraj@yahoo.com

Harish Dammu
Department of Mathematics
Bangalore University
Bengaluru-560001
INDIA
E-mail address: itsme.harishd@gmail.com