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ON MKDV EQUATIONS RELATED TO THE AFFINE KAC-MOODY ALGEBRA $A_5^{(2)}$

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Abstract. We have derived a new system of mKdV-type equations which can be related to the affine Lie algebra $A_5^{(2)}$. This system of partial differential equations is integrable via the inverse scattering method. It admits a Hamiltonian formulation and the corresponding Hamiltonian is also given. The Riemann-Hilbert problem for the Lax operator is formulated and its spectral properties are discussed.

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1. Introduction

The general theory of the nonlinear evolution equations (NLEE) allowing Lax representation is well developed [1, 3, 6, 9, 10, 21]. In this paper our aim is to derive a set of modified Korteveg–de Vries (mKdV) equations related to three affine Lie algebras using the procedure introduced by Mikhailov [20]. This means that the equations can be written as the commutativity condition of two ordinary differential operators of the type

$$\begin{aligned} L\psi &\equiv \mathrm{i}\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi = 0 \\ M\psi &\equiv \mathrm{i}\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi = \psi\Gamma(\lambda) \end{aligned} \tag{1}$$

where $U(x,t,\lambda)$, $V(x,t,\lambda)$ and $\Gamma(\lambda)$ are some polynomials of λ to be defined below. We request also that the Lax pair (1) possesses appropriate reduction group [20], for example if the reduction group is $\mathbb{Z}_h$ (h is a positive number) the reduction condition is

$$C\big(U(x,t,\lambda)\big) = U(x,t,\omega\lambda), \qquad C\big(V(x,t,\lambda)\big) = V(x,t,\omega\lambda). \tag{2}$$

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This work can be considered as a continuation of our recent publications [11–13]. Below we consider the three cases separately. The underlying Kac-Moody algebras are $B_2^{(1)}$, $A_4^{(2)}$, $A_5^{(2)}$ and the groups of reductions are correspondingly $\mathbb{Z}_4$, $\mathbb{Z}_5 \times \mathbb{Z}_2$, $\mathbb{Z}_5 \times \mathbb{Z}_2$. For the first two cases the Hamiltonians are well known [5]. A key motivation for choosing this particular algebras is that the derived equations will have very simple and elegant form.

Section 2 contains a derivation of the mKdV equations related to $B_2^{(1)}$. We start with the Lax representation which is a subject to $\mathbb{Z}_4$-reduction group [20], find the equations and derive the corresponding Hamiltonians. Then using the Lax representation which is a subject to $\mathbb{Z}_5 \times \mathbb{Z}_2$-reduction group [20] we derive the system of mKdV equations related to $A_4^{(2)}$ and finally derive the corresponding Hamiltonians. In the next Section 3 we make the same procedure but this time the algebra is $A_5^{(2)}$. Section 4 is devoted to the spectral properties of Lax operator for each algebra. Finally we relate this to the famous Riemann-Hilbert problem (RHP). We finish with some discussion and conclusion.

2. Preliminaries

2.1. Equations Related to $B_2^{(1)}$

We assume that the reader is familiar with the theory of semisimple Lie algebras [18] and affine Lie algebras [4]. The rank of $B_2^{(1)}$ is 2, its Coxeter number is $h = 4$ and its exponents are $1, 3$. Thus the Coxeter automorphism (see [11]) introduces a grading in $B_2^{(1)}$ as follows

$$B_2^{(1)} = \bigoplus_{k=0}^{3} \mathfrak{g}^{(k)}. \tag{3}$$

The grading condition holds

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right] \subset \mathfrak{g}^{(k+l)} \tag{4}$$

where $k + l$ is taken modulo four.

A convenient basis compatible with the grading of $B_2^{(1)}$ algebra is [11]

$$\begin{aligned} &\mathfrak{g}^{(0)} \equiv \operatorname{span}\{\mathcal{E}_{11}^{+}, \mathcal{E}_{22}^{+}\}, \qquad && \mathfrak{g}^{(1)} \equiv \operatorname{span}\{\mathcal{E}_{12}^{+}, \mathcal{E}_{23}^{+}, \mathcal{E}_{41}^{+}\} \\ &\mathfrak{g}^{(2)} \equiv \operatorname{span}\{\mathcal{E}_{13}^{+}, \mathcal{E}_{31}^{+}\}, \qquad && \mathfrak{g}^{(3)} \equiv \operatorname{span}\{\mathcal{E}_{21}^{+}, \mathcal{E}_{32}^{+}, \mathcal{E}_{14}^{+}\} \end{aligned} \tag{5}$$

where we use

$$\mathcal{E}_{ij}^{\pm} = E_{i,j} \mp S_1 E_{ij}^T S_1^{-1} = E_{i,j} \mp (-1)^{i+j} E_{6-j,6-i}. \tag{6}$$

In this Section E_{ij} is a 5×5 matrix equal to $(E_{ij})_{n,p} = \delta_{in}\delta_{jp}$ and

$$S_1 = E_{15} - E_{24} + E_{33} - E_{42} + E_{51}, \qquad S_1^2 = \mathbb{1} \tag{7}$$

provides the action of the external automorphism of $A_4 \simeq \mathfrak{sl}(5)$ related to the symmetry of its Dynkin diagram [18]. Obviously all $\mathfrak{E}_{ij}^+$ belong to the subalgebra $B_2 \simeq \mathfrak{so}(5)$ of A_4.

For deriving the equations we start with a Lax pair of the form (for details see [11])

$$\begin{aligned} L &= \mathrm{i}\partial_x + Q(x,t) - \lambda J \\ M &= \mathrm{i}\partial_t + V^{(0)}(x,t) + \lambda V^{(1)}(x,t) + \lambda^2 V^{(2)}(x,t) - \lambda^3 K \end{aligned} \tag{8}$$

with

$$\begin{aligned} Q(x,t) &= \frac{\mathrm{i}}{2}\left(u_1(x,t)\mathcal{E}_{11}^+ - u_2(x,t)\mathcal{E}_{22}^+\right), \qquad J = \mathcal{E}_{12}^+ + \mathcal{E}_{23}^+ + \mathcal{E}_{41}^+ \\ V^{(0)}(x,t) &= v_1^{(0)}\mathcal{E}_{11}^+ + v_2^{(0)}\mathcal{E}_{22}^+ \\ V^{(1)}(x,t) &= v_1^{(1)}\mathcal{E}_{12}^+ + v_2^{(1)}\mathcal{E}_{23}^+ + v_3^{(1)}\mathcal{E}_{41}^+ \\ V^{(2)}(x,t) &= v_1^{(2)}\mathcal{E}_{13}^+ + v_2^{(2)}\mathcal{E}_{31}^+, \qquad K = 2^6(\mathcal{E}_{21}^+ + 2\mathcal{E}_{32}^+ + \mathcal{E}_{14}^+). \end{aligned} \tag{9}$$

We require that $[L,M] = 0$ for any λ. The condition $[L,M] = 0$ leads to a set of recurrent relations (see [1, 8, 16]) which allow us to determine $V^{(k)}(x,t)$ in terms of the potential $Q(x,t)$ and its x-derivatives.

After the transformation $x \mapsto 2x$ and $t \to 4t'$ the equations become

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= 4\frac{\partial}{\partial x}\left(-u_1^3 + 3u_2^2 u_1 - 4\frac{\partial^2 u_1}{\partial x^2} + 6u_1\frac{\partial u_2}{\partial x}\right) \\ \frac{\partial u_2}{\partial t} &= 4\frac{\partial}{\partial x}\left(-u_2^3 + 3u_1^2 u_2 + 2\frac{\partial^2 u_2}{\partial x^2} - 6u_1\frac{\partial u_1}{\partial x}\right). \end{aligned} \tag{10}$$

They can be written as Hamiltonian equations of motion

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta q_i} \tag{11}$$

with the Hamiltonian

$$H = -\int_{-\infty}^{\infty} \mathrm{d}x\left(u_1^4 + u_2^4 - 6u_1^2 u_2^2 - 8\left(\frac{\partial u_1}{\partial x}\right)^2 + 4\left(\frac{\partial u_2}{\partial x}\right)^2 - 12u_1^2\left(\frac{\partial u_2}{\partial x}\right)\right) \tag{12}$$

which coincides with the one in [5].

2.2. Equations Related to $A_4^{(2)}$

Similarly we treat the $A_4^{(2)}$ case. The rank of this algebra is two, its Coxeter number is $h = 10$ and its exponents are $1, 3, 7, 9$. Now the Coxeter automorphism is of order 10 and introduces a grading in $A_4^{(2)}$ as follows

$$A_4^{(2)} = \mathop{\oplus}_{k=0}^{9} \mathfrak{g}^{(k)}. \tag{13}$$

The grading condition holds

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right] \subset \mathfrak{g}^{(k+l)} \tag{14}$$

where now $k + l$ is taken mod10, [4,5], see also [11].
A convenient basis compatible with the grading of $B_2^{(1)}$ algebra is [11]

$$\begin{aligned}
&\mathfrak{g}^{(0)} \equiv \mathrm{span}\{\mathcal{E}_{11}^+, \mathcal{E}_{22}^+\}, && \mathfrak{g}^{(1)} \equiv \mathrm{span}\{\mathcal{E}_{14}^-, \mathcal{E}_{31}^-, \mathcal{E}_{42}^-\} \\
&\mathfrak{g}^{(2)} \equiv \mathrm{span}\{\mathcal{E}_{12}^+, \mathcal{E}_{23}^+\}, && \mathfrak{g}^{(3)} \equiv \mathrm{span}\{\mathcal{E}_{21}^-, \mathcal{E}_{32}^-, \mathcal{E}_{15}^-\} \\
&\mathfrak{g}^{(4)} \equiv \mathrm{span}\{\mathcal{E}_{13}^+, \mathcal{E}_{41}^+\}, && \mathfrak{g}^{(5)} \equiv \mathrm{span}\{\mathcal{E}_{11}^- - \mathcal{E}_{22}^-, \mathcal{E}_{22}^- - \mathcal{E}_{33}^-\} \\
&\mathfrak{g}^{(6)} \equiv \mathrm{span}\{\mathcal{E}_{14}^+, \mathcal{E}_{31}^+\}, && \mathfrak{g}^{(7)} \equiv \mathrm{span}\{\mathcal{E}_{12}^-, \mathcal{E}_{23}^-, \mathcal{E}_{51}^-\} \\
&\mathfrak{g}^{(8)} \equiv \mathrm{span}\{\mathcal{E}_{21}^+, \mathcal{E}_{32}^+\}, && \mathfrak{g}^{(9)} \equiv \mathrm{span}\{\mathcal{E}_{13}^-, \mathcal{E}_{41}^-, \mathcal{E}_{24}^-\}
\end{aligned} \tag{15}$$

where we have used the basis (6) generated by the same matrix S_1 (7).
The relevant Lax pair is of the form (for details see [11])

$$\begin{aligned}
L &= \mathrm{i}\partial_x + Q(x,t) - \lambda J \\
M &= \mathrm{i}\partial_t + V^{(0)}(x,t) + \lambda V^{(1)}(x,t) + \lambda^2 V^{(2)}(x,t) - \lambda^3 K
\end{aligned} \tag{16}$$

with

$$\begin{aligned}
Q(x,t) &= \mathrm{i}u_2(x,t)\mathcal{E}_{11}^+ - \mathrm{i}u_1(x,t)\mathcal{E}_{22}^+, \qquad J = \mathcal{E}_{14}^- + \mathcal{E}_{31}^- + \mathcal{E}_{42}^- \\
V^{(0)}(x,t) &= v_1^{(0)}\mathcal{E}_{11}^+ + v_2^{(0)}\mathcal{E}_{22}^+ \\
V^{(1)}(x,t) &= v_1^{(1)}\mathcal{E}_{14}^- + v_2^{(1)}\mathcal{E}_{31}^- + v_3^{(1)}\mathcal{E}_{42}^- \\
V^{(2)}(x,t) &= v_1^{(2)}\mathcal{E}_{12}^+ + v_2^{(2)}\mathcal{E}_{23}^+, \qquad K = 20(\mathcal{E}_{21}^- - 2\mathcal{E}_{32}^- + \mathcal{E}_{15}^-).
\end{aligned} \tag{17}$$

We continue analogously. The condition $[L, M] = 0$ leads to a set of recurrent relations (see [1, 8, 16]) which allow us to determine $V^{(k)}(x,t)$ in terms of the potential $Q(x,t)$ and its x-derivatives.

After the transformation $x \mapsto 2x$ the equations are

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= -2\frac{\partial}{\partial x}\left(3\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + (3u_2 + 6u_1)\frac{\partial u_2}{\partial x} + 3u_2^2 u_1 - 2u_1^3\right) \\ \frac{\partial u_2}{\partial t} &= -2\frac{\partial}{\partial x}\left(4\frac{\partial^2 u_2}{\partial x^2} + 3\frac{\partial^2 u_1}{\partial x^2} - (6u_1 + 3u_2)\frac{\partial u_1}{\partial x} + 3u_1^2 u_2 - 2u_2^3\right).\end{aligned} \tag{18}$$

The Hamiltonian formulation follows from (11) and we find for H [5]

$$\begin{aligned}H = \int_{-\infty}^{\infty} \mathrm{d}x\Bigg(& u_1^4 + u_2^4 - 3u_1^2 u_2^2 + 4\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial x}\right)^2 \\ & + 3u_2^2\left(\frac{\partial u_1}{\partial x}\right) - 6u_1^2\left(\frac{\partial u_2}{\partial x}\right) + 6\left(\frac{\partial u_1}{\partial x}\right)\left(\frac{\partial u_2}{\partial x}\right)\Bigg).\end{aligned} \tag{19}$$

3. Derivation of the Equations Related to $A_5^{(2)}$

Now we consider the twisted affine Kac-Moody algebra $A_5^{(2)}$ case. Its rank is three, the Coxeter number is $h = 10$ and its exponents are $1, 3, 5, 7, 9$, see [4, 5]. Then the Coxeter automorphism is given by

$$C(X) = C_2 V(X) C_2^{-1} \tag{20}$$

where V is the external automorphism of the algebra $A_5 \simeq \mathfrak{sl}(6)$ generated by the symmetry of its Dynkin diagram and C_2 is an element of the Cartan subgroup defined below. More precisely

$$V(X) = -S_2 X^T S_2^{-1}, \quad S_2 = E_{1,6} - E_{2,5} + E_{3,4} - E_{4,3} + E_{5,2} - E_{6,1}. \tag{21}$$

Note that in this Section the matrices E_{kj} are 6×6 matrices equal to $(E_{k,j})_{np} = \delta_{kn}\delta_{jp}$, besides $S_2^2 = -\mathbb{1}$.

In analogy with the previous Section we introduce

$$\mathcal{E}_{ij}^{\pm} = E_{i,j} \mp S_2 E_{ij}^T S_2^{-1} = E_{i,j} \mp (-1)^{i+j} E_{7-j,7-i} \tag{22}$$

which obviously satisfy

$$V(\mathcal{E}_{ij}^{+}) = \mathcal{E}_{ij}^{+}, \qquad V(\mathcal{E}_{ij}^{-}) = -\mathcal{E}_{ij}^{-}. \tag{23}$$

It is easy to check that $\mathcal{E}_{ij}^{+}$ provide a basis for the subalgebra $\mathfrak{sp}(6)$ of $A_5^{(2)}$. The Cartan subgroup element C_2 is defined by

$$C_2 = \mathrm{diag}\,(1, \omega, \omega^2, \omega^3, \omega^4, 1). \tag{24}$$

The basis is as follows

$$
\begin{aligned}
&\mathfrak{g}^{(0)} \equiv \operatorname{span}\{\mathcal{E}_{11}^{+}, \mathcal{E}_{22}^{+}, \mathcal{E}_{33}^{+}\}, \quad &&\mathfrak{g}^{(1)} \equiv \operatorname{span}\{\mathcal{E}_{21}^{+}, \mathcal{E}_{32}^{+}, \mathcal{E}_{43}^{+}, \mathcal{E}_{15}^{-}\} \\
&\mathfrak{g}^{(2)} \equiv \operatorname{span}\{\mathcal{E}_{31}^{+}, \mathcal{E}_{42}^{+}, \mathcal{E}_{14}^{-}\}, \quad &&\mathfrak{g}^{(3)} \equiv \operatorname{span}\{\mathcal{E}_{14}^{+}, \mathcal{E}_{25}^{+}, \mathcal{E}_{13}^{-}, \mathcal{E}_{24}^{-}\} \\
&\mathfrak{g}^{(4)} \equiv \operatorname{span}\{\mathcal{E}_{51}^{+}, \mathcal{E}_{12}^{-}, \mathcal{E}_{23}^{-}\}, \quad &&\mathfrak{g}^{(5)} \equiv \operatorname{span}\{\mathcal{E}_{16}^{+}, \mathcal{E}_{61}^{+}, \mathcal{E}_{33}^{-} - \mathcal{E}_{11}^{-}, \mathcal{E}_{33}^{-} - \mathcal{E}_{22}^{-}\} \\
&\mathfrak{g}^{(6)} \equiv \operatorname{span}\{\mathcal{E}_{15}^{+}, \mathcal{E}_{21}^{-}, \mathcal{E}_{32}^{-}\}, \quad &&\mathfrak{g}^{(7)} \equiv \operatorname{span}\{\mathcal{E}_{41}^{+}, \mathcal{E}_{52}^{+}, \mathcal{E}_{31}^{-}, \mathcal{E}_{42}^{-}\} \\
&\mathfrak{g}^{(8)} \equiv \operatorname{span}\{\mathcal{E}_{13}^{+}, \mathcal{E}_{24}^{+}, \mathcal{E}_{41}^{-}\}, \quad &&\mathfrak{g}^{(9)} \equiv \operatorname{span}\{\mathcal{E}_{12}^{+}, \mathcal{E}_{23}^{+}, \mathcal{E}_{34}^{+}, \mathcal{E}_{51}^{-}\}.
\end{aligned} \tag{25}
$$

The grading condition is like always

$$
\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}\right] \subset \mathfrak{g}^{(k+l)} \tag{26}
$$

where $k + l$ is taken modulo 10.
We take a Lax pair of the form

$$
\begin{aligned}
L &= \mathrm{i}\partial_x + Q(x,t) - \lambda J \\
M &= \mathrm{i}\partial_t + V^{(0)}(x,t) + \lambda V^{(1)}(x,t) + \lambda^2 V^{(2)}(x,t) - \lambda^3 K
\end{aligned} \tag{27}
$$

where

$$
Q(x,t) \in \mathfrak{g}^{(0)}, \qquad V^{(k)}(x,t) \in \mathfrak{g}^{(k)}, \qquad K \in \mathfrak{g}^{(3)}, \qquad J \in \mathfrak{g}^{(1)}. \tag{28}
$$

This means

$$
\begin{aligned}
Q(x,t) &= \mathrm{i} \sum_{j=1}^{3} q_j(x,t) \mathcal{E}_{jj}^{+}, \qquad\qquad J = \mathcal{E}_{21}^{+} + \mathcal{E}_{32}^{+} + \frac{1}{2}\mathcal{E}_{43}^{+} + \frac{1}{2}\mathcal{E}_{15}^{-} \\
V^{(0)}(x,t) &= \sum_{j=1}^{3} v_j^{(0)} \mathcal{E}_{jj}^{+} \\
V^{(1)}(x,t) &= v_1^{(1)}\mathcal{E}_{21}^{+} + v_2^{(1)}\mathcal{E}_{32}^{+} + \frac{1}{2} v_3^{(1)}\mathcal{E}_{43}^{+} + \frac{1}{2} v_4^{(1)}\mathcal{E}_{15}^{-} \\
V^{(2)}(x,t) &= -v_1^{(2)}\mathcal{E}_{31}^{+} - v_1^{(2)}\mathcal{E}_{42}^{+} - \frac{1}{2} v_3^{(2)}\mathcal{E}_{14}^{-}, \qquad\qquad K = bJ^3.
\end{aligned} \tag{29}
$$

The condition $[L,M] = 0$ leads to a set of recurrent relations (see [1, 8, 16]) which allow us to determine $V^{(k)}(x,t)$ in terms of the potential $Q(x,t)$ and its x-derivatives. For $V^{(2)}(x,t)$ we find, skipping the details, the result

$$
v_1^{(2)} = -\mathrm{i}b(q_1 + q_2 + q_3), \qquad v_2^{(2)} = -\mathrm{i}bq_2, \qquad v_3^{(2)} = -\mathrm{i}b(q_1 - q_2 - q_3).
$$

For $V^{(1)}(x,t)$ we find

$$\begin{aligned}
v_1^{(1)} &= 2b\left(q_1q_2 - \frac{\partial q_1}{\partial x}\right) + f \\
v_2^{(1)} &= b\left(q_3^2 - q_1^2 + q_1q_2 + q_2q_3 + \frac{\partial}{\partial x}\left(q_3 + q_2 - q_1\right)\right) + f \\
v_3^{(1)} &= b\left(q_3^2 - q_2^2 - q_1^2 + q_1q_2 + \frac{\partial}{\partial x}\left(q_3 + 2q_2 - q_1\right)\right) + f \\
v_4^{(1)} &= f
\end{aligned} \tag{30}$$

where $f(x,t)$ is some arbitrary function. Using a well known technique from the theory of recursion operators [1, 9, 16] we find from the equations for $V^{(0)}(x,t)$ also $f(x,t)$

$$f = \frac{b}{5}\left(2q_2^2 + 2q_1^2 - 3q_3^2 - 5q_1q_2 + \frac{\partial}{\partial x}(5q_1 - 4q_2 - 3q_3)\right) \tag{31}$$

and

$$\begin{aligned}
v_1^{(0)} &= \frac{\mathrm{i}b}{5}\left(-5\frac{\partial^2 q_1}{\partial x^2} + 3q_1\frac{\partial}{\partial x}(3q_2 + q_3) - 2q_1^3 + 3q_1(q_2^2 + q_3^2)\right) \\
v_2^{(0)} &= \frac{\mathrm{i}b}{5}\left(\frac{\partial^2}{\partial x^2}(4q_2 + 3q_3) + 3q_2\frac{\partial q_3}{\partial x} - 9q_1\frac{\partial q_1}{\partial x} + 6q_3\frac{\partial q_3}{\partial x} - 2q_2^3 + 3q_2(q_1^2 + q_3^2)\right) \\
v_3^{(0)} &= \frac{\mathrm{i}b}{5}\left(\frac{\partial^2}{\partial x^2}(q_3 + 3q_2) - 6q_3\frac{\partial q_2}{\partial x} - 3q_1\frac{\partial q_1}{\partial x} - 3q_2\frac{\partial q_2}{\partial x} - 2q_3^3 + 3q_3(q_1^2 + q_2^2)\right).
\end{aligned}$$

And finally, the λ-independent terms in the Lax representation provide the equations

$$\begin{aligned}
\alpha\frac{\partial q_1}{\partial t} &= \frac{\partial}{\partial x}\left(-5\frac{\partial^2 q_1}{\partial x^2} + 3q_1\frac{\partial}{\partial x}(3q_2 + q_3) - 2q_1^3 + 3q_1(q_2^2 + q_3^2\right) \\
\alpha\frac{\partial q_2}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}(4q_2 + 3q_3) + 3q_2\frac{\partial q_3}{\partial x} - 9q_1\frac{\partial q_1}{\partial x} + 6q_3\frac{\partial q_3}{\partial x} - 2q_2^3 + 3q_2(q_1^2 + q_3^2)\right) \\
\alpha\frac{\partial q_3}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}(q_3 + 3q_2) - 6q_3\frac{\partial q_2}{\partial x} - 3q_1\frac{\partial q_1}{\partial x} - 3q_2\frac{\partial q_2}{\partial x} - 2q_3^3 + 3q_3(q_1^2 + q_2^2)\right)
\end{aligned}$$

where $\alpha = \dfrac{5}{b}\cdot$

We find for the corresponding Hamiltonian (11)

$$\begin{aligned}
H = \frac{1}{\alpha}\int_{-\infty}^{\infty} \mathrm{d}x\Bigg(&-\frac{1}{2}\sum_{i=1}^{3} q_i^4 + \frac{3}{2}\sum_{\substack{i=1\\ i<j}}^{3}\sum_{j=1}^{3} q_i^2q_j^2 + \frac{5}{2}\left(\frac{\partial q_1}{\partial x}\right)^2 - 2\left(\frac{\partial q_2}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial q_3}{\partial x}\right)^2 \\
&+\frac{\partial q_2}{\partial x}\left(\frac{9}{2}q_1^2 - 3q_3^2\right) + \frac{3}{2}\frac{\partial q_3}{\partial x}(q_1^2 + q_2^2) - 3\left(\frac{\partial q_2}{\partial x}\right)\left(\frac{\partial q_3}{\partial x}\right)\Bigg).
\end{aligned} \tag{32}$$

4. On the Spectral Properties of the Lax Operators

4.1. General Theory

Here we will outline the general approach of constructing the fundamental analytic solutions (FAS) of the Lax operators L with deep reductions [2,7,14–16]. Next we will detail these results for the three different Lax operators considered above.

Our first remark is about the fact, that after a simple similarity transformation, which diagonalizes the relevant matrix J, each of the above Lax operators will take the form

$$\tilde{L} \equiv \mathrm{i}\frac{\partial \tilde{\chi}}{\partial x} + (\tilde{Q}(x,t) - \lambda \tilde{J})\tilde{\chi}(x,t,\lambda) = 0 \tag{33}$$

where $\tilde{J}$ is a diagonal matrix with complex eigenvalues.

The main ingredient needed for solving the direct and the inverse scattering problem of $\tilde{L}$ are the Jost solutions.

It is well known that the Lax operators of the form (33) with generic complex-valued J allow Jost solutions only for potentials on compact support [2]. An important theorem proved by Beals and Coifman [2] states that any smooth potential $\tilde{Q}(x,t)$ can be approximated with an arbitrary precision by a potential on finite support. Then one can introduce the Jost solutions by

$$\lim_{x\to-\infty} \tilde{\phi}_-(x,t,\lambda)\mathrm{e}^{\mathrm{i}J\lambda x} = \mathbb{1}, \qquad \lim_{x\to\infty} \tilde{\phi}_+(x,t,\lambda)\mathrm{e}^{\mathrm{i}J\lambda x} = \mathbb{1}. \tag{34}$$

Then the scattering matrix is introduced by

$$T(\lambda,t) = \widehat{\tilde{\phi}}_+(x,t,\lambda)\tilde{\phi}_-(x,t,\lambda) \tag{35}$$

where by “hat” we denote matrix inverse.

The next step of [2] was to prove that one can construct piece-wise FAS $\tilde{\chi}_\nu(x,t,\lambda)$ which allows analytic extension in a certain sector Ω_ν in the complex λ-plane. These results were generalized to any simple Lie algebra in [7, 14]. The result is that sector Ω_ν has as boundaries the rays starting from the origin $l_{\nu-1}$ and l_ν, see the Figures below. The rays l_ν are determined by the solution of the linear equations

$$\Im \lambda \alpha(\tilde{J}) = 0. \tag{36}$$

In what follows we will outline the construction of FAS for the operator $\mathfrak{L}$ which is defined by

$$\mathfrak{L} \equiv \mathrm{i}\frac{\partial \xi}{\partial x} + \tilde{Q}(x,t)\xi(x,t,\lambda) - \lambda[\tilde{J}, \xi(x,t,\lambda)] = 0. \tag{37}$$

Obviously the fundamental solutions of $\tilde{L}$ and $\mathfrak{L}$ are related by

$$\xi_{\pm}(x,t,\lambda) = \tilde{\phi}_{\pm}(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\tilde{J}x}. \tag{38}$$

The Jost solutions $\xi_{\pm}(x,t,\lambda)$ must satisfy Volterra type integral equations

$$\begin{aligned}\xi_+(x,t,\lambda) &= \mathbb{1} + \mathrm{i}\int_{\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}\lambda\tilde{J}(x-y)}Q(y,t)\xi_+(y,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\tilde{J}(x-y)}\\ \xi_-(x,t,\lambda) &= \mathbb{1} + \mathrm{i}\int_{\infty}^{x} \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}\lambda\tilde{J}(x-y)}Q(y,t)\xi_-(y,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\tilde{J}(x-y)}.\end{aligned} \tag{39}$$

Let us now formulate the basic properties of $\xi_\nu(x,t,\lambda)$ – the FAS of $\mathfrak{L}$:

1. The continuous spectrum of $\mathfrak{L}$ fills up the rays l_ν.
2. Due to the $\mathbb{Z}_h$ symmetry the rays l_ν close angles equal to $\pi/2h$ or π/h depending on the choice of the algebra.
3. To each ray l_ν we associate a subset of roots δ_ν of $\mathfrak{g}$ which satisfy the condition (36). Thus to each ray l_ν we associate a subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$ generated by $E_\alpha, E_{-\alpha}, H_\alpha$ for $\alpha \in \delta_\nu$.
4. In each of the sectors Ω_ν we can calculate the limits for $x \to \pm\infty$ along the lines l_ν, more specifically

$$\begin{aligned}\lim_{x\to-\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\xi_\nu(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} &= S_\nu^+(t,\lambda), \qquad \lambda \in l_\nu \mathrm{e}^{+\mathrm{i}0}\\ \lim_{x\to\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\xi_\nu(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} &= T_\nu^-(t,\lambda)D_\nu^+(\lambda)\end{aligned} \tag{40}$$

   and

$$\begin{aligned}\lim_{x\to-\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\xi_\nu(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} &= S_{\nu+1}^-(t,\lambda), \qquad \lambda \in l_{\nu+1} \mathrm{e}^{-\mathrm{i}0}\\ \lim_{x\to-\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\xi_\nu(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} &= T_{\nu+1}^+(t,\lambda)D_{\nu+1}^-(\lambda)\end{aligned} \tag{41}$$

   where $S_\nu^\pm$, $T_\nu^\pm$ and $D_\nu^\pm$ are given by

$$\begin{aligned}S_\nu^\pm(\lambda,t) &= \exp\Big(\sum_{\alpha\in\delta_\nu} s_\alpha^\pm(\lambda,t)E_{\pm\alpha}\Big), \qquad D_\nu^\pm(\lambda) = \exp\Big(\sum_{\alpha\in\delta_\nu} d_{\nu,\alpha}^\pm H_\alpha\Big)\\ T_\nu^\pm(\lambda,t) &= \exp\Big(\sum_{\alpha\in\delta_\nu} t_{\nu,\alpha}^\pm(\lambda,t)E_{\pm\alpha}\Big).\end{aligned} \tag{42}$$

   Obviously they take values in the subgroup $\mathcal{G}_\nu$ whose Lie algebra $\mathfrak{g}_\nu$ has as positive roots the subset of roots related to l_ν, see the Table 1.

5. The time-dependence of $S_\nu^\pm$, $T_\nu^\pm$ and $D_\nu^\pm$ is determined by the M operator as

$$
\begin{aligned}
&\mathrm{i}\frac{\partial S_\nu^\pm}{\partial t} - \lambda^3[K, S_\nu^\pm(\lambda,t)] = 0, \qquad \mathrm{i}\frac{\partial D_\nu^\pm}{\partial t} = 0 \\
&\mathrm{i}\frac{\partial T_\nu^\pm}{\partial t} - \lambda^3[K, T_\nu^\pm(\lambda,t)] = 0
\end{aligned} \tag{43}
$$

where K determines the leading term of the M-operator.

6. The asymptotics $S_0^\pm$, $T_0^\pm$ and $D_0^\pm$ and $S_1^\pm$, $T_1^\pm$ and $D_1^\pm$ can be considered as independent. All the others are obtained from them by the $\mathbb{Z}_h$ symmetry

$$
\begin{aligned}
S_{2\nu}^\pm(\lambda) &= C^\nu(S_0^\pm(\lambda\omega^\nu), \qquad & S_{2\nu+1}^\pm(\lambda) &= C^\nu(S_1^\pm(\lambda\omega^\nu) \\
T_{2\nu}^\pm(\lambda) &= C^\nu(T_0^\pm(\lambda\omega^\nu), \qquad & T_{2\nu+1}^\pm(\lambda) &= C^\nu(T_1^\pm(\lambda\omega^\nu) \\
D_{2\nu}^\pm(\lambda) &= C^\nu(D_0^\pm(\lambda\omega^\nu), \qquad & D_{2\nu+1}^\pm(\lambda) &= C^\nu(D_1^\pm(\lambda\omega^\nu).
\end{aligned} \tag{44}
$$

As a consequence of the above properties we prove the following lemma, which generalizes the results of Zakharov and Shabat [22] for this type of algebras.

Lemma 1. 1. *The* FAS $\xi_\nu(x,t,\lambda)$ *of* $\mathfrak{L}$ *are solutions of the* RHP

$$
\xi_{\nu+1}(x,t,\lambda) = \xi_\nu(x,t,\lambda)G_\nu(x,\lambda), \qquad G_\nu(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}\hat{S}_{\nu+1}^- S_{\nu+1}^+ \mathrm{e}^{\mathrm{i}\lambda Jx}
$$

which allows canonical normalization

$$
\lim_{\lambda\to\infty} \xi_\nu(x,t,\lambda) = \mathbb{1}. \tag{45}
$$

2. *The corresponding potential* $\tilde{Q}(x,t)$ *is reconstructed from* $\xi_\nu(x,t,\lambda)$ *by*

$$
\tilde{Q}(x,t) = \lim_{\lambda\to\infty} \lambda\left(\tilde{J} - \xi_\nu(x,t,\lambda)\tilde{J}\hat{\xi}_\nu(x,t,\lambda)\right) \tag{46}
$$

where $\xi_\nu(x,t,\lambda)$ *is the unique regular solution of the* RHP (1), [7, 15, 21].

Proof: 1) follows easily from equations (40), (41) and from the fact, that the FAS is determined uniquely by its asymptotic for $x \to \pm\infty$.

2) follows from the fact that $\xi_\nu(x,t,\lambda)$ is a fundamental solution of $\mathfrak{L}$. Multiply equation (37) by $\hat{\xi}_\nu(x,t,\lambda)$ on the right, take the limit $\lambda \to \infty$ and use the canonical normalization (45).

■

We will formulate the specific properties for the three algebras independently.

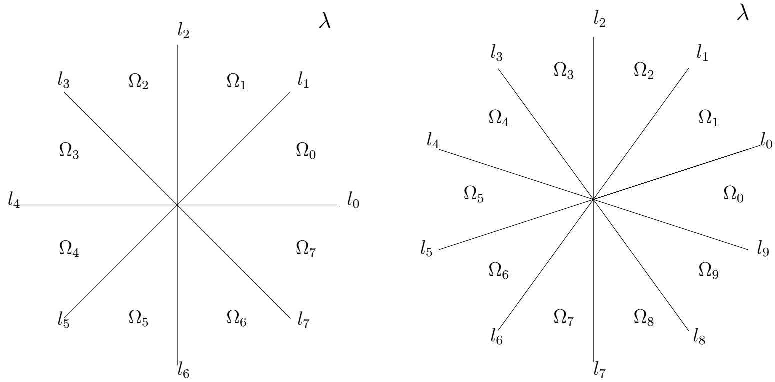

Figure 1. The continuous spectrum and the analyticity sectors of the FAS for the Lax operators: the case of $B_2^{(1)}$ – left panel; the case of $A_4^{(2)}$ – right panel.

Table 1. Subsets of positive roots related to the lines $l_\nu \cup l_{\nu+4}$, $\nu = 0, \dots, 3$ for the algebra $B_2^{(1)}$.

$l_0 \cup l_4$	$l_1 \cup l_5$	$l_2 \cup l_6$	$l_3 \cup l_7$
e_1	$e_1 - e_2$	e_2	$e_1 + e_2$

4.2. $B_2^{(1)}$

Here $h = 4$ and $\tilde{J} = \sqrt{2}\,\mathrm{diag}\,(1, \mathrm{i}, 0, -\mathrm{i}, -1)$. The rays l_ν are defined by $l_\nu\colon \arg\lambda = \nu\pi/4$, thus they close angles $\pi/4$. The sectors Ω_ν, $\nu = 0, \dots, 7$ are shown on Figure 1, left panel. The set of roots δ_ν related to each l_ν are given in Table 1.

4.3. $A_4^{(2)}$

Similarly for $A_4^{(2)}$ we have $h = 10$ and $\tilde{J} = \mathrm{diag}\,(\omega, \omega^3, -1, \omega^{-3}, \omega^{-1})$ with $\omega = \exp(2\pi\mathrm{i}/10)$. The rays l_ν are defined by $l_\nu\colon \arg\lambda = (2\nu+1)\pi/10$, $\nu = 0, \dots, 9$, thus they close angles $\pi/5$. The sectors Ω_ν, $\nu = 0, \dots, 9$ are shown in Fig. 1, right panel. The set of roots δ_ν related to each l_ν are given in Table 2.

Table 2. Subsets of positive roots related to the lines $l_\nu \cup l_{\nu+5}$ $\nu = 0, \dots, 4$ for the algebra $A_4^{(2)}$.

$l_0 \cup l_5$	$l_1 \cup l_6$	$l_2 \cup l_7$	$l_3 \cup l_8$	$l_4 \cup l_9$
$e_1 - e_2, e_3 - e_5$	$e_2 - e_5, e_3 - e_4$	$e_1 - e_5, e_2 - e_4$	$e_1 - e_4, e_2 - e_3$	$e_1 - e_3, e_4 - e_5$

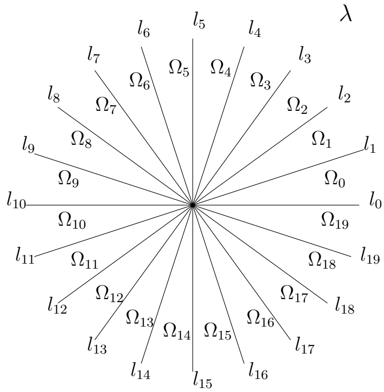

Figure 2. The continuous spectrum and the analyticity sectors of the FAS for the Lax operators for the case of $A_5^{(2)}$.

Table 3. Subsets of positive roots related to the lines $l_\nu \cup l_{\nu+10}$, $\nu = 0, \dots, 9$ for the algebra $A_5^{(2)}$.

$l_0 \cup l_{10}$ $e_1 - e_6$	$l_1 \cup l_{11}$ $e_1 - e_3, e_4 - e_5$	$l_2 \cup l_{12}$ $e_3 - e_6$	$l_3 \cup l_{13}$ $e_1 - e_2, e_3 - e_5$	$l_4 \cup l_{14}$ $e_5 - e_6$
$l_5 \cup l_{15}$ $e_2 - e_5, e_3 - e_4$	$l_6 \cup l_{16}$ $e_2 - e_6$	$l_7 \cup l_{17}$ $e_1 - e_5, e_2 - e_4$	$l_8 \cup l_{18}$ $e_4 - e_6$	$l_9 \cup l_{19}$ $e_1 - e_4, e_2 - e_3$

4.4. $A_5^{(2)}$

For $A_5^{(2)}$ we also have $h = 10$ but now $\tilde{J} = \operatorname{diag}(\omega, \omega^3, -1, \omega^{-3}, \omega^{-1})$ with $\omega = \exp(2\pi \mathrm{i}/10)$. The rays l_ν now are defined by $l_\nu \colon \arg\lambda = \nu\pi/10$, $\nu = 0, \dots, 19$, thus they close angles $\pi/10$. The sectors Ω_ν, $\nu = 0, \dots, 19$ are shown on Fig. 2. The set of roots δ_ν related to each l_ν are given in Table 3.

We end this Section by the following lemma

Lemma 2. *Each of the subalgebras $\mathfrak{g}_\nu$ related to the ray l_ν is a direct sum of $\mathfrak{sl}(2)$ subalgebras.*

Proof: Let us prove our lemma for the algebra $A_5^{(2)}$. First we consider the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ related to the rays l_0 and l_1. From Table 3 we find that the algebra $\mathfrak{g}_0$ is generated by E_α, $E_{-\alpha}$ and H_α, where α takes the values $e_1 - e_4$, $e_2 - e_3$ and $e_5 - e_6$. These three roots are mutually orthogonal, which means that $\mathfrak{g}_0 \equiv \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Similarly, the algebra $\mathfrak{g}_1$ is generated by E_β, $E_{-\beta}$ and H_β, where β takes the values $e_1 - e_3$ and $e_4 - e_6$, which are orthogonal to each other. Therefore $\mathfrak{g}_1 \equiv \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Next we use the $\mathbb{Z}_{r+1}$ symmetry, which in particular means that the set of the roots δ_ν related to the ray l_ν by the Coxeter transformation C as follows

$$\delta_{2\nu} = C^\nu(\delta_0), \qquad \delta_{2\nu+1} = C^\nu(\delta_1). \tag{47}$$

It remains to use the fact that the Coxeter transformation is an orthogonal transformation of the space of roots, so it obviously preserves the angles between any two roots.

The other cases are proved analogously. ■

5. Discussion and Conclusion

We have derived several systems of equations which are related to the affine Kac-Moody algebras $B_2^{(1)}$, $A_4^{(2)}$ and $A_5^{(2)}$ respectively. They admit a Lax representation and can be solved using Inverse scattering method. We also outlined the spectral properties of their Lax operators and formulated the corresponding RHP. This can be used to derive their soliton solutions via the dressing Zakharov-Shabat method. Lemma 2 can be used to prove the complete integrability of these mKdV equations. One can also develop the spectral theory of the relevant recursion operators following the ideas of [7, 16, 17, 19] which can be used as a ground for uniform deriving of all fundamental properties of the NLEE.

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