



## PARAMETRIC REALIZATION OF THE LORENTZ TRANSFORMATION GROUP IN PSEUDO-EUCLIDEAN SPACES

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**Abstract.** The Lorentz transformation group  $SO(m, n)$ ,  $m, n \in \mathbb{N}$ , is a group of Lorentz transformations of order  $(m, n)$ , that is, a group of special linear transformations in a pseudo-Euclidean space  $\mathbb{R}^{m, n}$  of signature  $(m, n)$  that leave the pseudo-Euclidean inner product invariant. A parametrization of  $SO(m, n)$  is presented, giving rise to the composition law of Lorentz transformations of order  $(m, n)$  in terms of parameter composition. The parameter composition, in turn, gives rise to a novel group-like structure that  $\mathbb{R}^{m, n}$  possesses, called a bi-gyrogroup. Bi-gyrogroups form a natural generalization of gyrogroups where the latter form a natural generalization of groups. Like the abstract gyrogroup, the abstract bi-gyrogroup can play a universal computational role which extends far beyond the domain of pseudo-Euclidean spaces.

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## 1. Introduction

A pseudo-Euclidean space  $\mathbb{R}^{m,n}$  of signature  $(m, n)$  where the numbers  $m, n \in \mathbb{N}$ , is an  $(m + n)$ -dimensional space with the pseudo-Euclidean inner product of signature  $(m, n)$ . A Lorentz transformation of order  $(m, n)$  is a special linear transformation  $\Lambda \in \text{SO}(m, n)$  in  $\mathbb{R}^{m,n}$  that leaves the pseudo-Euclidean inner product invariant. It is special in the sense that the determinant of the  $(m + n) \times (m + n)$  real matrix  $\Lambda$  is 1, and the determinant of its first  $m$  rows and columns is positive [9, p. 478]. The group of all Lorentz transformations of order  $(m, n)$  is also known as the special pseudo-orthogonal group, denoted by  $\text{SO}(m, n)$ .

A Lorentz transformation without rotations is called a boost. In general, two successive boosts are not equivalent to a boost. Rather, they are equivalent to a boost associated with two rotations, called a left rotation and a right rotation, or collectively, a *bi-rotation*. The two rotations of a bi-rotation are nontrivial if both

$m > 1$  and  $n > 1$ . The special case when  $m = 1$  and  $n > 1$  was studied in 1988 in [18]. The study in [18] resulted in the discovery of two novel algebraic structures that became known as a gyrogroup and a gyrovector space. Subsequent study of gyrovector spaces reveals in [19, 20, 22–25, 27] that gyrovector spaces form the algebraic setting for hyperbolic geometry, just as vector spaces form the algebraic setting for Euclidean geometry. The aim of this paper is to extend the study of the parametric realization of the Lorentz group in [18] from  $m = 1$  to  $m \geq 1$ , and to reveal the resulting new algebraic structure, called a *bi-gyrogroup*.

In order to emphasize that when  $m > 1$  and  $n > 1$  a successive application of two boosts generates a bi-rotation, a Lorentz boost of order  $(m, n)$ ,  $m, n > 1$ , is called a *bi-boost*. The composition law of two bi-boosts gives rise in this article to a *bi-gyrocommutative bi-gyrogroup* operation, just as the composition of two boosts gives rise in [18] to a gyrocommutative gyrogroup operation, as demonstrated in [19]. Accordingly, a bi-gyrogroup of order  $(m, n)$ ,  $m, n \in \mathbb{N}$ , is a group-like structure that specializes to a gyrogroup when either  $m = 1$  or  $n = 1$ .

We show in Theorem 8 that a Lorentz transformation  $\Lambda$  of order  $(m, n)$  possesses the unique parametrization  $\Lambda = \Lambda(P, O_n, O_m)$ , where

1.  $P \in \mathbb{R}^{n \times m}$  is any real  $n \times m$  matrix, where
2.  $O_n \in \text{SO}(n)$  is any  $n \times n$  special orthogonal matrix, taking  $P$  into  $O_n P$ , and, similarly, where
3.  $O_m \in \text{SO}(m)$  is any  $m \times m$  special orthogonal matrix, taking  $P$  into  $P O_m$ .

In the special case when  $m = 1$ , the Lorentz transformation of order  $(m, n)$  specializes to the Lorentz transformation of special relativity theory ( $n = 3$  in physical applications), where the parameter  $P$  is a vector that represents relativistic proper velocities.

The parametrization of the Lorentz transformation  $\Lambda$  enables in Theorem 21 the Lorentz transformation composition (or, product) law to be expressed in terms of parameter composition. Under the resulting parameter composition, the parameter  $O_n$  of  $\Lambda$ , called a *left rotation* (of  $P \in \mathbb{R}^{n \times m}$ ), forms a group. The group that the left rotations form is the special orthogonal group  $\text{SO}(n)$ . Similarly, under the parameter composition, the parameter  $O_m$  of  $\Lambda$ , called a *right rotation* (of  $P \in \mathbb{R}^{n \times m}$ ), forms a group. The group that the right rotations form is the special orthogonal group  $\text{SO}(m)$ . The pair  $(O_n, O_m) \in \text{SO}(n) \times \text{SO}(m)$  is called a bi-rotation, taking  $P \in \mathbb{R}^{n \times m}$  into  $O_n P O_m \in \mathbb{R}^{n \times m}$ .

Contrasting the left and right rotation parameters, the parameter  $P$  does not form a group under parameter composition. Rather, it forms a novel algebraic structure, called a *bi-gyrocommutative bi-gyrogroup*, defined in Definition 53.

A bi-gyrocommutative bi-gyrogroup is a group-like structure that generalizes the gyrocommutative gyrogroup structure. The latter, in turn, is a group-like structure that forms a natural generalization of the commutative group.

The concept of the gyrogroup emerged from the 1988 study of the parametrization of the Lorentz group in [18]. Presently, the gyrogroup concept plays a universal computational role, which extends far beyond the domain of special relativity, as noted by Chatelin in [1, p. 523] and in references therein and as evidenced, for instance, from [2, 4–7, 11, 14–16] and [12, 21, 26, 28]. In a similar way, the concept of the bi-gyrogroup emerges in this paper from the study of the parametrization of the Lorentz group  $SO(m, n)$ ,  $m, n \in \mathbb{N}$ . Hence, like gyrogroups, bi-gyrogroups are capable of playing a universal computational role that extends far beyond the domain of Lorentz transformations in pseudo-Euclidean spaces.

## 2. Lorentz Transformations of Order $(m, n)$

Let  $\mathbb{R}^{m,n}$  be an arbitrary  $(m+n)$ -dimensional pseudo-Euclidean space of a signature  $(m, n)$ ,  $m, n \in \mathbb{N}$ , with an orthonormal basis  $e_i$ ,  $i = 1, \dots, m+n$

$$e_i \cdot e_j = \epsilon_i \delta_{ij} \quad (1)$$

where

$$\epsilon_i = \begin{cases} +1, & i = 1, \dots, m \\ -1, & i = m+1, \dots, m+n. \end{cases} \quad (2)$$

The inner product  $\mathbf{x} \cdot \mathbf{y}$  of two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$

$$\mathbf{x} = \sum_{i=1}^{m+n} x_i e_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{m+n} y_i e_i \quad (3)$$

is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{m+n} \epsilon_i x_i y_i = \sum_{i=1}^m x_i y_i - \sum_{i=m+1}^{m+n} x_i y_i. \quad (4)$$

Let  $I_m$  be the  $m \times m$  identity matrix, and let  $\eta$  be the  $(m+n) \times (m+n)$  diagonal matrix

$$\eta = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (5)$$

where  $0_{m,n}$  is the  $m \times n$  zero matrix. Then, the matrix representation of the inner product (4) is

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^t \eta \mathbf{y} \quad (6)$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are the column vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+n} \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+n} \end{pmatrix} \quad (7)$$

and exponent  $t$  denotes transposition.

Let  $\Lambda$  be an  $(m+n) \times (m+n)$  matrix that leaves the inner product (6) invariant. Then, for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$

$$(\Lambda \mathbf{x})^t \eta \Lambda \mathbf{y} = \mathbf{x}^t \eta \mathbf{y} \quad (8)$$

implying  $\mathbf{x}^t \Lambda^t \eta \Lambda \mathbf{y} = \mathbf{x}^t \eta \mathbf{y}$ , so that

$$\Lambda^t \eta \Lambda = \eta. \quad (9)$$

The determinant of the matrix equation (9) yields

$$(\det \Lambda)^2 = 1 \quad (10)$$

noting that  $\det(\Lambda^t \eta \Lambda) = (\det \Lambda^t)(\det \eta)(\det \Lambda)$  and  $\det \Lambda^t = \det \Lambda$ . Hence

$$\det \Lambda = \pm 1. \quad (11)$$

The special transformations  $\Lambda$  that can be reached continuously from the identity transformation in  $\mathbb{R}^{m,n}$  constitute the special pseudo-orthogonal group  $\text{SO}(m, n)$ , also known as the (*generalized*) *Lorentz transformation group* of order  $(m, n)$ . Each element  $\Lambda$  of  $\text{SO}(m, n)$  is a Lorentz transformation of order  $(m, n)$ . It has determinant one

$$\det \Lambda = 1 \quad (12)$$

and the determinant of its first  $m$  rows and columns is positive [9, p. 478]. The Lorentz transformation of order  $(1, 3)$  turns out to be the common homogeneous, proper, orthochronous Lorentz transformation of Einstein's special theory of relativity [22].

Let  $\mathbb{R}^{m \times n}$  be the set of all  $m \times n$  real matrices. Following Dragon [3], in order to parametrize the special pseudo-orthogonal group  $\text{SO}(m, n)$ , we partition each  $(m+n) \times (m+n)$  matrix  $\Lambda \in \text{SO}(m, n)$  into four blocks consisting of the

submatrices i)  $A \in \mathbb{R}^{m \times m}$ , ii)  $\hat{A} \in \mathbb{R}^{n \times n}$ , iii)  $B \in \mathbb{R}^{n \times m}$ , and iv)  $\hat{B} \in \mathbb{R}^{m \times n}$ , so that

$$\Lambda = \begin{pmatrix} A & \hat{B} \\ B & \hat{A} \end{pmatrix}. \quad (13)$$

By means of (13) and (5), the matrix equation (9) takes the form

$$\begin{pmatrix} A^t & B^t \\ \hat{B}^t & \hat{A}^t \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \begin{pmatrix} A & \hat{B} \\ B & \hat{A} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (14)$$

or, equivalently

$$\begin{pmatrix} A^t A - B^t B & A^t \hat{B} - B^t \hat{A} \\ \hat{B}^t A - \hat{A}^t B & \hat{B}^t \hat{B} - \hat{A}^t \hat{A} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (15)$$

implying

$$A^t A = I_m + B^t B, \quad \hat{A}^t \hat{A} = I_n + \hat{B}^t \hat{B}, \quad A^t \hat{B} = B^t \hat{A}. \quad (16)$$

The symmetric matrix  $B^t B$  is diagonalizable by an orthogonal matrix with non-negative diagonal elements [10, pp. 171, 396-398, 402]. Hence, the eigenvalues of  $I_m + B^t B$  are not smaller than 1, so that  $(\det A)^2 = \det(I_m + B^t B) \geq 1$ . This, in turn, implies that  $A$  is invertible. Similarly,  $(\det \hat{A})^2 = \det(I_n + \hat{B}^t \hat{B}) \geq 1$ , so that  $\hat{A}$  is invertible.

An invertible real matrix  $A$  can be uniquely decomposed into the product of an orthogonal matrix  $O \in \text{SO}(m)$ ,  $O^t = O^{-1}$ , and a positive-definite symmetric matrix  $S$ ,  $S^t = S$ , with positive eigenvalues [8, p. 286]

$$A = OS. \quad (17)$$

Following (17) we have

$$A^t A = S^t O^t O S = S^2 \quad (18)$$

with positive eigenvalues  $\lambda_i > 0$ ,  $i = 1, \dots, m$ . Hence

$$S = \sqrt{A^t A} \quad (19)$$

has the positive eigenvalues  $\sqrt{\lambda_i}$  and the same eigenvectors as  $S^2$ .

The matrix  $S$  given by (19) satisfies (17) since  $AS^{-1}$  is orthogonal, as it should be, by (17). Indeed

$$(AS^{-1})^t AS^{-1} = (S^{-1})^t A^t AS^{-1} = S^{-1} S^2 S^{-1} = I_m. \quad (20)$$

Similarly,  $\hat{A}$  is invertible and possesses the decomposition

$$\hat{A} = \hat{O}\hat{S} \quad (21)$$

where  $\hat{O} \in SO(n)$  is an orthogonal matrix and  $\hat{S}$  is a positive-definite symmetric matrix.

By means of (17) and (21), the block matrix (13) possesses the decomposition

$$\Lambda = \begin{pmatrix} O & 0_{m,n} \\ 0_{n,m} & \hat{O} \end{pmatrix} \begin{pmatrix} S & \hat{P} \\ P & \hat{S} \end{pmatrix} \quad (22)$$

where the submatrices  $P$  and  $\hat{P}$  are to be determined in (24) below.

Following (22) and (13), along with (17) and (21), we have

$$\Lambda = \begin{pmatrix} OS & O\hat{P} \\ \hat{O}P & \hat{O}\hat{S} \end{pmatrix} = \begin{pmatrix} A & \hat{B} \\ B & \hat{A} \end{pmatrix} \quad (23)$$

so that  $\hat{O}P = B$  and  $O\hat{P} = \hat{B}$ , that is

$$P = \hat{O}^{-1}B, \quad \hat{P} = O^{-1}\hat{B}. \quad (24)$$

In (22),  $S$  and  $\hat{S}$  are invertible symmetric matrices, and  $O$  and  $\hat{O}$  are orthogonal matrices with determinant one.

By means of (9) and (22) we have the matrix equation

$$\begin{pmatrix} S^t & P^t \\ \hat{P}^t & \hat{S}^t \end{pmatrix} \begin{pmatrix} O^t & 0_{m,n} \\ 0_{n,m} & \hat{O}^t \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \begin{pmatrix} O & 0_{m,n} \\ 0_{n,m} & \hat{O} \end{pmatrix} \begin{pmatrix} S & \hat{P} \\ P & \hat{S} \end{pmatrix} \\ = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix}. \quad (25)$$

Noting that

$$\begin{pmatrix} O^t & 0_{m,n} \\ 0_{n,m} & \hat{O}^t \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \begin{pmatrix} O & 0_{m,n} \\ 0_{n,m} & \hat{O} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (26)$$

the matrix equation (25) yields

$$\begin{pmatrix} S^t & P^t \\ \hat{P}^t & \hat{S}^t \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \begin{pmatrix} S & \hat{P} \\ P & \hat{S} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (27)$$

so that

$$\begin{pmatrix} S^t & -P^t \\ \hat{P}^t & -\hat{S}^t \end{pmatrix} \begin{pmatrix} S & \hat{P} \\ P & \hat{S} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix} \quad (28)$$

and hence

$$\begin{pmatrix} S^t S - P^t P & S^t \hat{P} - P^t \hat{S} \\ \hat{P}^t S - \hat{S}^t P & \hat{P}^t \hat{P} - \hat{S}^t \hat{S} \end{pmatrix} = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -I_n \end{pmatrix}. \quad (29)$$

Noting that  $S^t S = S^2$  and  $\hat{S}^t \hat{S} = \hat{S}^2$ , (29) yields the equations

$$S^2 = I_m + P^t P, \quad \hat{S}^2 = I_n + \hat{P}^t \hat{P}, \quad S^t \hat{P} = P^t \hat{S} \quad (30)$$

Besides, because the matrix  $S$  is symmetric, the third and the first equations in (30) imply

$$\hat{P} = S^{-1} P^t \hat{S}, \quad S^{-2} = (I_m + P^t P)^{-1}. \quad (31)$$

Inserting (31) into the second equation in (30)

$$\begin{aligned} \hat{S}^2 &= I_n + \hat{P}^t \hat{P} = I_n + (\hat{S}^t P (S^t)^{-1}) S^{-1} P^t \hat{S} \\ &= I_n + \hat{S} P S^{-2} P^t \hat{S} = I_n + \hat{S} P (I_m + P^t P)^{-1} P^t \hat{S}. \end{aligned} \quad (32)$$

Multiplying both sides of (32) by  $\hat{S}^{-2}$ , we have

$$I_n = \hat{S}^{-2} + P (I_m + P^t P)^{-1} P^t. \quad (33)$$

Let  $\omega$  be an eigenvector of the matrix  $PP^t$ , and let  $\lambda$  be its associated eigenvalue

$$PP^t \omega = \lambda \omega. \quad (34)$$

If  $P^t \omega$  is not zero,  $P^t \omega \neq \mathbf{0}$ , then it is an eigenvector of  $P^t P$  with the same eigenvalue  $\lambda$

$$(P^t P) P^t \omega = \lambda P^t \omega. \quad (35)$$

Adding  $P^t \omega$  to both sides of (35) we have

$$(I_m + P^t P) P^t \omega = (1 + \lambda) P^t \omega \quad (36)$$

so that

$$\frac{1}{1 + \lambda} P^t \omega = (I_m + P^t P)^{-1} P^t \omega. \quad (37)$$

Multiplying both sides of (37) by  $P$

$$\frac{1}{1 + \lambda} PP^t \omega = P (I_m + P^t P)^{-1} P^t \omega \quad (38)$$

so that, by means of (34)

$$P (I_m + P^t P)^{-1} P^t \omega = \frac{\lambda}{1 + \lambda} \omega \quad (39)$$



for any eigenvector  $\boldsymbol{\omega}$  of  $PP^t$  for which  $P^t\boldsymbol{\omega} \neq 0$ . Equation (39) remains valid also when  $P^t\boldsymbol{\omega} = 0$  since in this case  $\lambda = 0$  by (34).

By means of (33) and (39) we have

$$\boldsymbol{\omega} = \hat{S}^{-2}\boldsymbol{\omega} + P(I_m + P^tP)^{-1}P^t\boldsymbol{\omega} = \hat{S}^{-2}\boldsymbol{\omega} + \frac{\lambda}{1+\lambda}\boldsymbol{\omega} \quad (40)$$

so that

$$\hat{S}^{-2}\boldsymbol{\omega} = \frac{1}{1+\lambda}\boldsymbol{\omega} \quad (41)$$

and, by equation (34)

$$\hat{S}^2\boldsymbol{\omega} = (1+\lambda)\boldsymbol{\omega} = I_n\boldsymbol{\omega} + \lambda\boldsymbol{\omega} = I_n\boldsymbol{\omega} + PP^t\boldsymbol{\omega} = (I_n + PP^t)\boldsymbol{\omega} \quad (42)$$

for any eigenvector  $\boldsymbol{\omega}$  of  $PP^t$ .

The eigenvectors  $\boldsymbol{\omega}$  constitute a basis of  $\mathbb{R}^n$ . Hence, it follows from (42) that

$$\hat{S}^2 = I_n + PP^t \quad (43)$$

and, hence

$$\hat{S} = \sqrt{I_n + PP^t}. \quad (44)$$

Following (30)–(31) and (44) we have

$$\hat{P} = S^{-1}P^t\hat{S} = \sqrt{I_m + P^tP}^{-1}P^t\sqrt{I_n + PP^t}. \quad (45)$$

Employing (45) and the eigenvectors  $\boldsymbol{\omega}$  of  $PP^t$ , we will show in (50) below that  $\hat{P} = P^t$ .

As in (34),  $\boldsymbol{\omega}$  is an eigenvector of the matrix  $PP^t$ ,  $P^t\boldsymbol{\omega} \neq 0$ , with its associated eigenvalue  $\lambda > 0$ , implying (37). Following (37), the matrix  $(I_m + P^tP)^{-1}$  possesses an eigenvector  $P^t\boldsymbol{\omega}$  with its associated eigenvalue  $1/(1+\lambda)$ . Hence, the matrix  $\sqrt{I_m + P^tP}^{-1}$  possesses the same eigenvector  $P^t\boldsymbol{\omega}$  with its associated eigenvalue  $1/\sqrt{1+\lambda}$

$$\sqrt{I_m + P^tP}^{-1}P^t\boldsymbol{\omega} = \frac{1}{\sqrt{1+\lambda}}P^t\boldsymbol{\omega}. \quad (46)$$

Similarly, the matrix  $I_n + PP^t$  satisfies the equation  $(I_n + PP^t)\boldsymbol{\omega} = (1+\lambda)\boldsymbol{\omega}$ , so that it possesses an eigenvector  $\boldsymbol{\omega}$  with its associated eigenvalue  $1+\lambda$ . Hence, the matrix  $\sqrt{I_n + PP^t}$  possesses the same eigenvector  $\boldsymbol{\omega}$  with its associated eigenvalue  $\sqrt{1+\lambda}$

$$\sqrt{I_n + PP^t}\boldsymbol{\omega} = \sqrt{1+\lambda}\boldsymbol{\omega}. \quad (47)$$

Hence, by (47) and (46)

$$\sqrt{I_m + P^t P}^{-1} P^t \sqrt{I_n + P P^t} \boldsymbol{\omega} = \sqrt{1 + \lambda} \sqrt{I_m + P^t P}^{-1} P^t \boldsymbol{\omega} = P^t \boldsymbol{\omega} \quad (48)$$

for any eigenvector  $\boldsymbol{\omega}$  of  $P P^t$  for which  $P^t \boldsymbol{\omega} \neq 0$ . Equation (48) remains valid also for  $\boldsymbol{\omega}$  with  $P^t \boldsymbol{\omega} = 0$  since in this case  $\lambda = 0$  by (34).

The eigenvectors  $\boldsymbol{\omega}$  constitute a basis of  $\mathbb{R}^n$ . Hence, it follows from (48) that

$$\sqrt{I_m + P^t P}^{-1} P^t \sqrt{I_n + P P^t} = P^t \quad (49)$$

so that, by (45) and (49)

$$\hat{P} = P^t. \quad (50)$$

Following (50) and the first two equations in (30) we have

$$\hat{P} = P^t, \quad S = \sqrt{I_m + P^t P}, \quad \hat{S} = \sqrt{I_n + P P^t}. \quad (51)$$

Inserting (51) into (22) and denoting  $O$  and  $\hat{O}$  by  $O_m$  and  $O_n$ , respectively, we obtain the  $(m+n) \times (m+n)$  matrix  $\Lambda$  parametrized by the three matrix parameters (i)  $P \in \mathbb{R}^{n \times m}$ , (ii)  $O_m \in \text{SO}(m)$ , and (iii)  $O_n \in \text{SO}(n)$

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix}. \quad (52)$$

**Lemma 1.** *The following commuting relations hold for all  $P \in \mathbb{R}^{n \times m}$*

$$P \sqrt{I_m + P^t P} = \sqrt{I_n + P P^t} P \quad (53)$$

$$P^t \sqrt{I_n + P P^t} = \sqrt{I_m + P^t P} P^t \quad (54)$$

$$P P^t \sqrt{I_n + P P^t} = \sqrt{I_n + P P^t} P P^t \quad (55)$$

$$P^t P \sqrt{I_m + P^t P} = \sqrt{I_m + P^t P} P^t P. \quad (56)$$

**Proof:** The commuting relation (54) follows from (45), noting that  $\hat{P} = P^t$ . The commuting relation (53) is obtained from (54) by matrix transposition. The commuting relation (56) is obtained by successive applications of (54) and (53). Finally, the commuting relation (55) is obtained by successive applications of (53) and (54). ■

### 3. Parametric Representation of $SO(m, n)$

The block orthogonal matrix in (52) can be uniquely resolved as a commuting product of two orthogonal block matrices

$$\begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix}. \quad (57)$$

The first and the second orthogonal matrices on the right side of (57) represent, respectively, i) a right rotation  $O_m \in SO(m)$  of  $\mathbb{R}^{n \times m}$ ,  $O_m : P \mapsto PO_m$ , and ii) a left rotation  $O_n \in SO(n)$  of  $\mathbb{R}^{n \times m}$ ,  $O_n : P \mapsto O_n P$ . Hence, the orthogonal matrix on the left side of (57), which represents the composition of the rotations  $O_m$  and  $O_n$ , is said to be a *bi-rotation* of the pseudo-Euclidean space  $\mathbb{R}^{m,n}$  about its origin.

By means of (57), (52) can be written as

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix}. \quad (58)$$

**Lemma 2.** *The commuting relations*

$$\begin{aligned} \sqrt{I_m + P^t P} O_m &= O_m \sqrt{I_m + (P O_m)^t (P O_m)} \\ O_n \sqrt{I_n + P P^t} &= \sqrt{I_n + (O_n P)(O_n P)^t} O_n \end{aligned} \quad (59)$$

hold for all  $P \in \mathbb{R}^{n \times m}$ ,  $O_m \in SO(m)$  and  $O_n \in SO(n)$ .

**Proof:**

$$\begin{aligned} I_m + P^t P &= O_m I_m O_m^t + O_m O_m^t P^t P O_m O_m^t \\ &= O_m (I_m + O_m^t P^t P O_m) O_m^t \\ &= O_m \sqrt{I_m + O_m^t P^t P O_m} \sqrt{I_m + O_m^t P^t P O_m} O_m^t \\ &= O_m \sqrt{I_m + O_m^t P^t P O_m} O_m^t O_m \sqrt{I_m + O_m^t P^t P O_m} O_m^t \\ &= (O_m \sqrt{I_m + O_m^t P^t P O_m} O_m^t)^2. \end{aligned} \quad (60)$$

Hence

$$\sqrt{I_m + P^t P} = O_m \sqrt{I_m + O_m^t P^t P O_m} O_m^t \quad (61)$$

implying the first matrix identity in (59).

Similarly

$$\begin{aligned}
I_n + PP^t &= O_n^t I_n O_n + O_n^t O_n P P^t O_n^t O_n \\
&= O_n^t (I_n + O_n P P^t O_n^t) O_n \\
&= O_n^t \sqrt{I_n + O_n P P^t O_n^t} \sqrt{I_n + O_n P P^t O_n^t} O_n \\
&= O_n^t \sqrt{I_n + O_n P P^t O_n^t} O_n O_n^t \sqrt{I_n + O_n P P^t O_n^t} O_n \\
&= (O_n^t \sqrt{I_n + O_n P P^t O_n^t} O_n)^2.
\end{aligned} \tag{62}$$

Hence

$$\sqrt{I_n + PP^t} = O_n^t \sqrt{I_n + O_n P P^t O_n^t} O_n \tag{63}$$

implying the second matrix identity in (59). ■

**Lemma 3.** *The commuting relations*

$$\begin{aligned}
\sqrt{I_m + P^t P}^{-1} O_m &= O_m \sqrt{I_m + (P O_m)^t (P O_m)}^{-1} \\
O_n \sqrt{I_n + P P^t}^{-1} &= \sqrt{I_n + (O_n P)(O_n P)^t}^{-1} O_n
\end{aligned} \tag{64}$$

hold for all  $P \in \mathbb{R}^{n \times m}$ ,  $O_m \in \text{SO}(m)$  and  $O_n \in \text{SO}(n)$ .

**Proof:** Inverting the first matrix identity in (59), we have the matrix identity

$$O_m^{-1} \sqrt{I_m + P^t P}^{-1} = \sqrt{I_m + (P O_m)^t (P O_m)}^{-1} O_m^{-1} \tag{65}$$

which implies the first equation in the Lemma.

Similarly, inverting the second matrix identity in (59), we obtain a matrix identity that implies the second equation in the Lemma. ■

**Lemma 4.** *The commuting relation*

$$\begin{aligned}
\begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} &\begin{pmatrix} \sqrt{I_m + (P O_m)^t (P O_m)} & (P O_m)^t \\ P O_m & \sqrt{I_n + (P O_m)(P O_m)^t} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix}
\end{aligned} \tag{66}$$

holds for all  $P \in \mathbb{R}^{n \times m}$  and  $O_m \in \text{SO}(m)$ .

**Proof:** Let  $J_1$  and  $J_2$  denote the left side and the right side of (66), respectively. Clearly

$$J_2 = \begin{pmatrix} \sqrt{I_m + P^t P} O_m & P^t \\ P O_m & \sqrt{I_n + P P^t} \end{pmatrix} \quad (67)$$

and, by means of the first commuting relation in Lemma 2

$$\begin{aligned} J_1 &= \begin{pmatrix} O_m \sqrt{I_m + (P O_m)^t (P O_m)} & O_m (P O_m)^t \\ P O_m & \sqrt{I_n + (P O_m)(P O_m)^t} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{I_m + P^t P} O_m & P^t \\ P O_m & \sqrt{I_n + P P^t} \end{pmatrix}. \end{aligned} \quad (68)$$

Hence,  $J_1 = J_2$ , and the proof is complete. ■

**Lemma 5.** *The commuting relation*

$$\begin{aligned} &\begin{pmatrix} \sqrt{I_m + (O_n P)^t (O_n P)} & (O_n P)^t \\ O_n P & \sqrt{I_n + (O_n P)(O_n P)^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \\ &= \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \end{aligned} \quad (69)$$

holds for all  $P \in \mathbb{R}^{n \times m}$  and  $O_n \in \text{SO}(n)$ .

**Proof:** Let  $J_3$  and  $J_4$  denote the left side and the right side of (69), respectively. Clearly

$$J_4 = \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ O_n P & O_n \sqrt{I_n + P P^t} \end{pmatrix} \quad (70)$$

and, by means of the second commuting relation in Lemma 2

$$\begin{aligned} J_3 &= \begin{pmatrix} \sqrt{I_m + (O_n P)^t (O_n P)} & (O_n P)^t O_n \\ O_n P & \sqrt{I_n + (O_n P)(O_n P)^t} O_n \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ O_n P & O_n \sqrt{I_n + P P^t} \end{pmatrix}. \end{aligned} \quad (71)$$

Hence,  $J_3 = J_4$ , and the proof is complete. ■

By (58) and Lemma 5

$$\begin{aligned} \Lambda &= \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \\ &= \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + (O_n P)^t (O_n P)} & (O_n P)^t \\ O_n P & \sqrt{I_n + (O_n P)(O_n P)^t} \end{pmatrix} \quad (72) \\ &\quad \times \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \end{aligned}$$

where  $P$ ,  $O_m$  and  $O_n$  are generic elements of  $\mathbb{R}^{n \times m}$ ,  $\text{SO}(m)$  and  $\text{SO}(n)$ , respectively, forming the three matrix parameters that determine  $\Lambda \in \text{SO}(m, n)$ .

The matrix parameter  $P$  of  $\Lambda$  in (58) is a generic element of  $\mathbb{R}^{n \times m}$ , and the orthogonal matrix  $O_n \in \text{SO}(n)$  maps  $\mathbb{R}^{n \times m}$  onto itself bijectively,  $O_n : P \rightarrow O_n P$ . Hence, the generic element  $O_n P \in \mathbb{R}^{n \times m}$  in (72) can, equivalently, be replaced by the generic element  $P \in \mathbb{R}^{n \times m}$ , thus obtaining from (72) the parametric representation of the generic Lorentz transformation  $\Lambda$

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \quad (73)$$

called the *bi-gyration decomposition* of  $\Lambda$ .

The generic Lorentz transformation matrix  $\Lambda$  of order  $(m+n) \times (m+n)$  is expressed in (73) as the product of the following three matrices in (74)–(76).

**The bi-boost:** The  $(m+n) \times (m+n)$  matrix  $B(P)$

$$B(P) := \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \quad (74)$$

is parametrized by  $P \in \mathbb{R}^{n \times m}$ ,  $m, n \in \mathbb{N}$ . In order to emphasize that  $B(P)$  is associated in (73) with a bi-rotation  $(O_m, O_n)$ , we call it a *bi-boost*. If  $m = 1$  and  $n = 3$ , the bi-boost descends to the common boost of a Lorentz transformation in special relativity theory, studied for instance in [18, 19, 22, 24].

**The right rotation:** The  $(m+n) \times (m+n)$  block orthogonal matrix

$$\rho(O_m) := \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \in \mathbb{R}^{(m+n) \times (m+n)} \quad (75)$$

is parametrized by  $O_m \in \text{SO}(m)$ . For  $m > 1$ ,  $O_m$  is an  $m \times m$  orthogonal matrix, destined to be *right-applied* to the  $n \times m$  matrices  $P$ ,  $P \rightarrow PO_m$ . Hence,  $\rho(O_m)$  is called a right rotation of the bi-boost  $B(P)$ .

**The left rotation:** The  $(m+n) \times (m+n)$  block orthogonal matrix

$$\lambda(O_n) := \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathbb{R}^{(m+n) \times (m+n)} \quad (76)$$

is parametrized by  $O_n \in \text{SO}(n)$ . For  $n > 1$ ,  $O_n$  is an  $n \times n$  orthogonal matrix, destined to be *left-applied* to the  $n \times m$  matrices  $P$ ,  $P \rightarrow O_n P$ . Hence,  $\lambda(O_n)$  is called a left rotation of the bi-boost  $B(P)$ .

A left and a right rotation are called collectively a *bi-rotation*. Suggestively, the term *bi-boost* emphasizes that the generic bi-boost is associated with a generic bi-rotation  $(O_n, O_m) \in \text{SO}(n) \times \text{SO}(m)$ .

With the notation in (74)–(76), the results of Lemma 5 and Lemma 4 can be written as commuting relations between bi-boosts and left and right rotations, as the following lemma asserts.

**Lemma 6.** *The commuting relations*

$$\begin{aligned} \lambda(O_n)B(P) &= B(O_n P)\lambda(O_n) \\ B(P)\rho(O_m) &= \rho(O_m)B(PO_m) \end{aligned} \quad (77)$$

$$\lambda(O_n)B(P)\rho(O_m) = \rho(O_m)B(O_n P O_m)\lambda(O_n)$$

hold for any  $P \in \mathbb{R}^{n \times m}$ ,  $O_m \in \text{SO}(m)$  and  $O_n \in \text{SO}(n)$ .

**Proof:** The first matrix identity in (77) is the result of Lemma 5, expressed in the notation in (74)–(76). Similarly, the second matrix identity in (77) is the result of Lemma 4, expressed in the notation in (74)–(76). The third matrix identity in (77) follows from the first and second matrix identities in (77), noting that  $\lambda(O_n)$  and  $\rho(O_m)$  commute. ■

With the notation in (74)–(76), the Lorentz transformation matrix  $\Lambda$  in (72), parametrized by  $P$ ,  $O_m$  and  $O_n$ , is given by the equation

$$\Lambda(O_m, P, O_n) = \rho(O_m)B(P)\lambda(O_n). \quad (78)$$

It proves useful to use the column notation

$$\Lambda(O_m, P, O_n) =: \begin{pmatrix} P \\ O_n \\ O_m \end{pmatrix} \quad (79)$$

so that a product of two Lorentz matrices is written as a product between two column triples. Thus, for instance, the product (or, composition) of the two Lorentz transformations  $\Lambda_1 = \Lambda(O_{n,1}, P_1, O_{m,1})$  and  $\Lambda_2 = \Lambda(O_{n,2}, P_2, O_{m,2})$  is written as

$$\Lambda_1 \Lambda_2 = \begin{pmatrix} P_1 \\ O_{n,1} \\ O_{m,1} \end{pmatrix} \begin{pmatrix} P_2 \\ O_{n,2} \\ O_{m,2} \end{pmatrix}. \quad (80)$$

The Lorentz transformation product law, written in column notation, will be presented in Theorem 21, p. 71, following the study of associated special left and right automorphisms, called left and right gyrations or, collectively, bi-gyrations.

Formalizing the main result of Sections 2 and 3, we have the following definition and theorem.

**Definition 7. (Special Pseudo-Orthogonal Group).** *A special pseudo-orthogonal transformation  $\Lambda$  in the pseudo-Euclidean space  $\mathbb{R}^{m,n}$  is a linear transformation in  $\mathbb{R}^{m,n}$ , also known as a (generalized, special) Lorentz transformation of order  $(m, n)$ , if relative to the basis (1)–(2), it leaves the inner product (4) invariant, has determinant  $\det \Lambda = 1$ , and the determinant of its first  $m$  rows and columns is positive. The group of all special pseudo-orthogonal transformations in  $\mathbb{R}^{m,n}$ , that is, the group of all Lorentz transformations of order  $(m, n)$ , is denoted by  $\text{SO}(m, n)$ .*

**Theorem 8. (Lorentz Transformation Bi-gyration Decomposition).** *A matrix  $\Lambda \in \mathbb{R}^{(m+n) \times (m+n)}$  is a Lorentz transformation of order  $(m, n)$  (that is, a special pseudo-orthogonal transformation in  $\mathbb{R}^{m,n}$ ),  $\Lambda \in \text{SO}(m, n)$ , if and only if it is given uniquely by the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \quad (81)$$

or, parametrically in short

$$\Lambda = \Lambda(O_m, P, O_n) = \rho(O_m) B(P) \lambda(O_n) = \begin{pmatrix} P \\ O_n \\ O_m \end{pmatrix}. \quad (82)$$

**Proof:** Result (81) is identical with (73). ■

**Theorem 9. (Lorentz Transformation Polar Decomposition).** *Any Lorentz Transformation matrix  $\Lambda \in \text{SO}(m, n)$  possesses the polar decomposition*

$$\Lambda = \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \quad (83)$$



**Proof:** By Lemma 4 and (72), we have

$$\begin{aligned}
 \Lambda &= \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m + (PO_m)^t(PO_m)} & (PO_m)^t \\ PO_m & \sqrt{I_n + (PO_m)(PO_m)^t} \end{pmatrix} \\
 &\times \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \\
 &= \begin{pmatrix} \sqrt{I_m + P^t P} & P^t \\ P & \sqrt{I_n + P P^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix}
 \end{aligned} \tag{84}$$

noting that  $P \in \mathbb{R}^{n \times m}$  is a generic main parameter of  $\Lambda \in \text{SO}(m, n)$  if and only if  $PO_m \in \mathbb{R}^{n \times m}$  is a generic main parameter of  $\Lambda$  for any  $O_m \in \text{SO}(m)$ . ■

#### 4. Inverse Lorentz Transformation

**Theorem 10. (The Inverse Bi-boost).** *The inverse of the bi-boost  $B(P)$ ,  $P \in \mathbb{R}^{n \times m}$ , is  $B(-P)$*

$$B(P)^{-1} = B(-P). \tag{85}$$

**Proof:** By Lemma 1 we have the commuting relations

$$P^t \sqrt{I_n + P P^t} = \sqrt{I_m + P^t P} P^t, \quad \sqrt{I_n + P P^t} P = P \sqrt{I_m + P^t P} \tag{86}$$

which, in the notation in (51), are

$$\hat{P} \hat{S} = S \hat{P}, \quad \hat{S} P = P S \tag{87}$$

and, clearly

$$S^2 - \hat{P} P = I_m, \quad \hat{S}^2 - P \hat{P} = I_n. \tag{88}$$

Hence, by (74) and (51)

$$\begin{aligned}
 B(P)B(-P) &= \begin{pmatrix} S & \hat{P} \\ P & \hat{S} \end{pmatrix} \begin{pmatrix} S & -\hat{P} \\ -P & \hat{S} \end{pmatrix} = \begin{pmatrix} S^2 - \hat{P} P & -S \hat{P} + \hat{P} \hat{S} \\ P S - \hat{S} P & -P \hat{P} + \hat{S}^2 \end{pmatrix} \\
 &= \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} = I_{m+n}
 \end{aligned} \tag{89}$$

as desired. ■

A Lorentz transformation matrix  $\Lambda$  of order  $(m+n) \times (m+n)$ ,  $m, n \geq 2$ , involves the bi-rotation  $\lambda(O_n)\rho(O_m)$ , as shown in (82). Bi-boosts are Lorentz transformations without bi-rotations, that is by (78)–(79), bi-boosts  $B(P)$  are

$$\Lambda(I_m, P, I_n) = \rho(I_m)B(P)\lambda(I_n) = B(P) = \begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix} \quad (90)$$

for any  $P \in \mathbb{R}^{n \times m}$ .

Rewriting (85) in the column notation, we have

$$\begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix}^{-1} = \begin{pmatrix} -P \\ I_n \\ I_m \end{pmatrix} \quad (91)$$

so that, accordingly

$$\begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} -P \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ I_m \end{pmatrix} \quad (92)$$

$(0_{n,m}, I_n, I_m)^t$  being the identity Lorentz transformation of order  $(m, n)$ .

**Theorem 11. (The Inverse Lorentz Transformation).** *The inverse of a Lorentz transformation  $\Lambda = (P, O_n, O_m)^t$  is given by the equation*

$$\begin{pmatrix} P \\ O_n \\ O_m \end{pmatrix}^{-1} = \begin{pmatrix} -O_n^t P O_m^t \\ O_n^t \\ O_m^t \end{pmatrix}. \quad (93)$$

**Proof:** The proof is given by the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} \Lambda(O_m, P, O_n)^{-1} &\stackrel{(1)}{\cong} \{\rho(O_m)B(P)\lambda(O_n)\}^{-1} \\ &\stackrel{(2)}{\cong} \lambda(O_n^t)B(-P)\rho(O_m^t) \\ &\stackrel{(3)}{\cong} B(-O_n^t P)\lambda(O_n^t)\rho(O_m^t) \\ &\stackrel{(4)}{\cong} B(-O_n^t P)\rho(O_m^t)\lambda(O_n^t) \\ &\stackrel{(5)}{\cong} \rho(O_m^t)B(-O_n^t P O_m^t)\lambda(O_n^t). \end{aligned} \quad (94)$$

Derivation of the numbered equalities in (94) follows

1. By (82).
2. Obvious, noting (85).
3. Follows from (2) by the first matrix identity in (77).
4. Follows from (3) by commuting  $\lambda(O_n^t)$  and  $\rho(O_m^t)$ .
5. Follows from (4) by the second matrix identity in (77).

■

## 5. Bi-Boost Parameter Recognition

Composing the bi-gyration decomposition (81) of the Lorentz transformation  $\Lambda \in \text{SO}(m, n)$  in Theorem 8, we have the Lorentz transformation

$$\Lambda = \begin{pmatrix} O_m \sqrt{I_m + P^t P} & O_m P^t O_n \\ P & \sqrt{I_n + P P^t} O_n \end{pmatrix} =: \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \quad (95)$$

parametrized by the three parameters

1.  $P \in \mathbb{R}^{n \times m}$ , an  $n \times m$  real matrix, called the *main parameter* of the Lorentz transformation  $\Lambda \in \text{SO}(m, n)$
2.  $O_n \in \text{SO}(n)$ , a left rotation of  $P$  (or, equivalently, a right rotation of  $P^t$ ) and
3.  $O_m \in \text{SO}(m)$ , a right rotation of  $P$  (or, equivalently, a left rotation of  $P^t$ ).

We naturally face the task of determining the matrix parameters  $P$ ,  $O_n$  and  $O_m$  of the  $\text{SO}(m, n)$  matrix  $\Lambda$  in (95) from its block entries  $E_{ij}$ ,  $i, j = 1, 2$ .

The matrix parameters  $O_m$  and  $O_n$  of  $\Lambda$  in (81) cannot be recognized from (95) straightforwardly by inspection. Fortunately, however, the matrix parameter  $P$  is recovered from (95) by straightforward inspection,  $P = E_{21}$ , thus obtaining the first equation in (96) below. Then, following (95) we have  $I_m + P^t P = I_m + E_{21}^t E_{21}$  and  $I_n + P P^t = I_n + E_{21} E_{21}^t$ , so that (95) yields the following *parameter recognition formulas*

$$\begin{aligned} P &= E_{21}, & E_{12} &= O_m P^t O_n \\ O_m &= E_{11} \sqrt{I_m + E_{21}^t E_{21}}^{-1}, & O_n &= \sqrt{I_n + E_{21} E_{21}^t}^{-1} E_{22}. \end{aligned} \quad (96)$$

In the parameter recognition formulas (96) the parameters  $P$ ,  $O_n$  and  $O_m$  of the composite Lorentz transformation  $\Lambda$  in (95) and in the decomposed Lorentz transformation  $\Lambda$  in (81) are recognized from the block entries  $E_{ij}$ ,  $i, j = 1, 2$ , of the composite Lorentz transformation (95). Our ability to recover the main parameter of a Lorentz transformation suggests the following definition of main parameter composition, called *bi-gyroaddition*.

**Definition 12. (Bi-gyroaddition, Bi-gyrogroupoid).** *Let  $\Lambda = B(P_1)B(P_2)$  be a Lorentz transformation given by the product of two bi-boosts parametrized by  $P_1, P_2 \in \mathbb{R}^{n \times m}$ . Then, the main parameter,  $P_{12}$ , of  $\Lambda$  is said to be the composition of  $P_1$  and  $P_2$ ,*

$$P_{12} = P_1 \oplus P_2 \quad (97)$$

*giving rise to a binary operation,  $\oplus$ , called bi-gyroaddition, in the space  $\mathbb{R}^{n \times m}$  of all  $n \times m$  real matrices. Being a groupoid of the parameter  $P \in \mathbb{R}^{n \times m}$ , the resulting groupoid  $(\mathbb{R}^{n \times m}, \oplus)$  is called the parameter bi-gyrogroupoid.*

Definition 12 encourages us to the study of the bi-boost composition law in Section 6.

## 6. Bi-Boost Composition Parameters

In general, the product of two bi-boosts is not a bi-boost. However, the product of two bi-boosts is an element of the Lorentz group  $SO(m, n)$  and, hence, by Theorem 8, can be parametrized, as shown in Section 5. Following (74), let

$$B(P_k) = \begin{pmatrix} \sqrt{I_m + P_k^t P_k} & P_k^t \\ P_k & \sqrt{I_n + P_k P_k^t} \end{pmatrix} \quad (98)$$

$k = 1, 2$ , be two bi-boosts, so that their product is

$$\begin{aligned} \Lambda &= B(P_1)B(P_2) \\ &= \begin{pmatrix} \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} + P_1^t P_2 & \sqrt{I_m + P_1^t P_1} P_2^t + P_1^t \sqrt{I_n + P_2 P_2^t} \\ P_1 \sqrt{I_m + P_2^t P_2} + \sqrt{I_n + P_1 P_1^t} P_2 & P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \end{pmatrix} \\ &=: \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}. \quad (99) \end{aligned}$$

By the parameter recognition formulas (96), the main parameter

$$P_{12} = P_1 \oplus P_2 \quad (100)$$

and the left and right rotation parameters  $O_{n,12}$  and  $O_{m,12}$  of the bi-boost product  $\Lambda = B(P_1)B(P_2)$  in (99) are given by

$$\begin{aligned} P_{12} &= P_1 \oplus P_2 = E_{21}, & E_{12} &= O_{m,12} P_{12}^t O_{n,12} \\ O_{n,12} &= \sqrt{I_n + E_{21} E_{21}^t}^{-1} E_{22}, & O_{m,12} &= E_{11} \sqrt{I_m + E_{21}^t E_{21}}^{-1} \end{aligned} \quad (101)$$

where  $E_{ij}$ ,  $i, j = 1, 2$ , are defined by the last equation in (99).

Hence, by (78)

$$\Lambda = B(P_1)B(P_2) = \rho(O_{m,12})B(P_1 \oplus P_2)\lambda(O_{n,12}). \quad (102)$$

Following Definition 12, we view  $\oplus$  as a binary operation between elements  $P \in \mathbb{R}^{n \times m}$ , thus obtaining the *bi-gyrogroupoid*  $(\mathbb{R}^{n \times m}, \oplus)$  that will give rise to a group-like structure called a *bi-gyrogroup*. Accordingly, the binary operation  $\oplus$  is the bi-gyrooperation of  $\mathbb{R}^{n \times m}$ , called *bi-gyroaddition*, and  $P_1 \oplus P_2$  is the *bi-gyrosum* of  $P_1$  and  $P_2$  in  $\mathbb{R}^{n \times m}$ .

It is now convenient to rename the right rotation  $O_{m,12}$  and the left rotation  $O_{n,12}$  in (101)–(102) as a right and a left *gyrations*. In symbols

$$O_{m,12} =: \text{rgyr}[P_1, P_2] \in \text{SO}(m), \quad O_{n,12} =: \text{lgyr}[P_1, P_2] \in \text{SO}(n). \quad (103)$$

We call  $\text{rgyr}[P_1, P_2]$  the *right gyration* generated by  $P_1$  and  $P_2$ , and call  $\text{lgyr}[P_1, P_2]$  the *left gyration* generated by  $P_1$  and  $P_2$ . The pair of a left and a right gyration, each generated by  $P_1$  and  $P_2$ , is viewed collectively as the *bi-gyration* generated by  $P_1$  and  $P_2$ .

The bi-boost product (102) is now written as

$$B(P_1)B(P_2) = \rho(\text{rgyr}[P_1, P_2])B(P_1 \oplus P_2)\lambda(\text{lgyr}[P_1, P_2]) \quad (104)$$

demonstrating that the product of two bi-boosts generated by  $P_1$  and  $P_2$  is a bi-boost generated by  $P_1 \oplus P_2$  along with a bi-gyration generated by  $P_1$  and  $P_2$ .

The bi-gyrosum  $P_1 \oplus P_2$  of  $P_1$  and  $P_2$ , and the bi-gyrations generated by  $P_1$  and  $P_2$  that appear in (104) are determined from (99)–(103)

$$\begin{aligned}
 P_1 \oplus P_2 &= P_1 \sqrt{I_m + P_2^t P_2} + \sqrt{I_n + P_1 P_1^t} P_2 \\
 \text{rgyr}[P_1, P_2] &= \left\{ P_1^t P_2 + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} \right\} \\
 &\quad \times \sqrt{I_m + (P_1 \oplus P_2)^t (P_1 \oplus P_2)}^{-1} \\
 \text{lgyr}[P_1, P_2] &= \sqrt{I_n + (P_1 \oplus P_2)(P_1 \oplus P_2)^t}^{-1} \\
 &\quad \times \left\{ P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\
 \text{rgyr}[P_1, P_2] (P_1 \oplus P_2)^t \text{lgyr}[P_1, P_2] &= \sqrt{I_m + P_1^t P_1 P_2^t} + P_1^t \sqrt{I_n + P_2 P_2^t} \\
 &\quad \stackrel{(1)}{\cong} P_1^t \oplus P_2^t \stackrel{(2)}{\cong} (P_2 \oplus P_1)^t.
 \end{aligned} \tag{105}$$

The equation marked by (1) in (105) follows immediately from the first equation in (105), replacing  $P_1, P_2 \in \mathbb{R}^{n \times m}$  by  $P_1^t, P_2^t \in \mathbb{R}^{m \times n}$ .

The equation marked by (2) in (105) is derived from the first equation in (105) in the following straightforward chain of equations.

$$\begin{aligned}
 (P_2 \oplus P_1)^t &= \left\{ P_2 \sqrt{I_m + P_1^t P_1} + \sqrt{I_n + P_2 P_2^t} P_1 \right\}^t \\
 &= \sqrt{I_m + P_1^t P_1 P_2^t} + P_1^t \sqrt{I_n + P_2 P_2^t} \\
 &= (P_1^t) \sqrt{I_n + (P_2^t)^t P_2^t} + \sqrt{I_m + (P_1^t)(P_1^t)^t (P_2^t)} \\
 &= P_1^t \oplus P_2^t.
 \end{aligned} \tag{106}$$

Formalizing results in (105), we obtain the following theorem.

**Theorem 13. (Bi-gyroaddition and Bi-gyration).** *The bi-gyroaddition and bi-gyration in the parameter bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$  are given by the equations*

$$\begin{aligned}
 P_1 \oplus P_2 &= P_1 \sqrt{I_m + P_2^t P_2} + \sqrt{I_n + P_1 P_1^t} P_2 \\
 \text{lgyr}[P_1, P_2] &= \sqrt{I_n + (P_1 \oplus P_2)(P_1 \oplus P_2)^t}^{-1} \\
 &\quad \times \left\{ P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\
 \text{rgyr}[P_1, P_2] &= \left\{ P_1^t P_2 + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} \right\} \\
 &\quad \times \sqrt{I_m + (P_1 \oplus P_2)^t (P_1 \oplus P_2)}^{-1}
 \end{aligned} \tag{107}$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

The following corollary results immediately from Theorem 13.

**Corollary 14. (Trivial Bi-gyration).**

$$\text{lgyr}[0_{n,m}, P] = \text{lgyr}[P, 0_{n,m}] = I_n, \quad \text{lgyr}[\ominus P, P] = \text{lgyr}[P, \ominus P] = I_n \tag{108}$$

$$\text{rgyr}[0_{n,m}, P] = \text{rgyr}[P, 0_{n,m}] = I_m, \quad \text{rgyr}[\ominus P, P] = \text{rgyr}[P, \ominus P] = I_m$$

for all  $P \in \mathbb{R}^{n \times m}$ .

The trivial bi-gyration

$$\text{lgyr}[P, P] = I_n, \quad \text{rgyr}[P, P] = I_m \tag{109}$$

for all  $P \in \mathbb{R}^{n \times m}$  cannot be derived immediately from Theorem 13. It will, therefore, be derived in (133) and (131), and formalized in Theorem 16, p. 66.

The bi-boost product (104), written in the column notation, takes the elegant form

$$B(P_1)B(P_2) = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \tag{110}$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ . When  $P_1 = P$  and  $P_2 = -P$ , (110) specializes to

$$B(P)B(-P) = \begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} -P \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P \oplus (-P) \\ \text{lgyr}[P, -P] \\ \text{rgyr}[P, -P] \end{pmatrix} \tag{111}$$

for all  $P \in \mathbb{R}^{n \times m}$ . But, the left side of (111) is also determined in (92), implying the identities

$$P \oplus (-P) = 0_{n,m}, \quad \text{lgyr}[P, -P] = I_n, \quad \text{rgyr}[P, -P] = I_m. \quad (112)$$

The first equation in (112) implies that

$$-P =: \ominus P \quad (113)$$

is the inverse of  $P$  with respect to the binary operation  $\oplus$  in  $\mathbb{R}^{n \times m}$ . Hence, we use the notations  $-P$  and  $\ominus P$  interchangeably. Furthermore, we naturally use the notation  $P_1 \oplus (-P_2) = P_1 \oplus (\ominus P_2) = P_1 \ominus P_2$ , and rewrite (112) as

$$P \ominus P = 0, \quad \text{lgyr}[P, \ominus P] = I_n, \quad \text{rgyr}[P, \ominus P] = I_m \quad (114)$$

in agreement with (108).

Similarly, we rewrite (91) as

$$\begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix}^{-1} = \begin{pmatrix} \ominus P \\ I_n \\ I_m \end{pmatrix} \quad (115)$$

that is

$$B(P)^{-1} = B(\ominus P) \quad (116)$$

for all  $P \in \mathbb{R}^{n \times m}$ .

The first equation in (105) implies that

$$(-P_1) \oplus (-P_2) = -(P_1 \oplus P_2). \quad (117)$$

Hence, following the definition of  $\ominus P$  in (113), and by (117), the bi-gyroaddition  $\oplus$  obeys the *gyroautomorphic inverse property*

$$\ominus(P_1 \oplus P_2) = \ominus P_1 \ominus P_2 \quad (118)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

It follows from the gyroautomorphic inverse property (118) and from (105) that bi-gyrations are *even*, that is

$$\text{lgyr}[-P_1, -P_2] = \text{lgyr}[P_1, P_2], \quad \text{rgyr}[-P_1, -P_2] = \text{lgyr}[P_1, P_2] \quad (119)$$

or, equivalently

$$\text{lgyr}[\ominus P_1, \ominus P_2] = \text{lgyr}[P_1, P_2], \quad \text{rgyr}[\ominus P_1, \ominus P_2] = \text{lgyr}[P_1, P_2]. \quad (120)$$



## 7. Automorphisms of the Parameter Bi-Gyrogroupoid

Left and right rotations turn out to be left and right automorphisms of the parameter bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$ . We recall that a groupoid,  $(S, +)$ , is a nonempty set,  $S$ , with a binary operation,  $+$ . A *left automorphism* of a groupoid  $(S, +)$  is a bijection  $f$  of  $S$ ,  $f : S \rightarrow S$ ,  $s \mapsto fs$ , that respects the binary operation, that is,  $f(s_1 + s_2) = fs_1 + fs_2$ . Similarly, a *right automorphism* of a groupoid  $(S, +)$  is a bijection  $f$  of  $S$ ,  $f : S \rightarrow S$ ,  $s \mapsto sf$ , that respects the binary operation, that is,  $(s_1 + s_2)f = s_1f + s_2f$ . The need to distinguish between left and right automorphisms of the bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$  is clear from Theorem 15 below.

**Theorem 15. (Left and Right Automorphisms of  $(\mathbb{R}^{n \times m}, \oplus)$ ).** *Any rotation  $O_n \in \text{SO}(n)$  is a left automorphism of the parameter bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$ , and any rotation  $O_m \in \text{SO}(m)$  is a right automorphism of the parameter bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$ , that is*

$$\begin{aligned} O_n(P_1 \oplus P_2) &= O_n P_1 \oplus O_n P_2 \\ (P_1 \oplus P_2) O_m &= P_1 O_m \oplus P_2 O_m \\ O_n(P_1 \oplus P_2) O_m &= O_n P_1 O_m \oplus O_n P_2 O_m \end{aligned} \quad (121)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ .

**Proof:** By the first equation in (107) and the second equation in (59)

$$\begin{aligned} O_n(P_1 \oplus P_2) &= O_n(P_1 \sqrt{I_m + P_2^t P_2} + \sqrt{I_n + P_1 P_1^t} P_2) \\ &= O_n P_1 \sqrt{I_m + (O_n P_2)^t (O_n P_2)} + O_n \sqrt{I_n + P_1 P_1^t} P_2 \\ &= O_n P_1 \sqrt{I_m + (O_n P_2)^t (O_n P_2)} \\ &\quad + \sqrt{I_n + (O_n P_1)(O_n P_1)^t} O_n P_2 = O_n P_1 \oplus O_n P_2 \end{aligned} \quad (122)$$

thus proving the first identity in (121).

Similarly, by the first equation in (107) and the first equation in (59)

$$\begin{aligned} (P_1 \oplus P_2) O_m &= (P_1 \sqrt{I_m + P_2^t P_2} + \sqrt{I_n + P_1 P_1^t} P_2) O_m \\ &= P_1 \sqrt{I_m + P_2^t P_2} O_m + \sqrt{I_n + P_1 P_1^t} P_2 O_m \\ &= P_1 O_m \sqrt{I_m + (P_2 O_m)^t (P_2 O_m)} \\ &\quad + \sqrt{I_n + (P_1 O_m)(P_1 O_m)^t} P_2 O_m = P_1 O_m \oplus P_2 O_m \end{aligned} \quad (123)$$

thus proving the second identity in (121). The third identity in (121) follows immediately from the first two identities in (121). ■

Left gyrations,  $\text{lgyr}[P_1, P_2]$ , and right gyrations,  $\text{rgyr}[P_1, P_2]$ ,  $P_1, P_2 \in \mathbb{R}^{n \times m}$ , are, by (103) and Theorem 15, left and right automorphisms of  $(\mathbb{R}^{n \times m}, \oplus)$ .

Accordingly, left and right gyrations are also called left and right gyroautomorphisms of  $(\mathbb{R}^{n \times m}, \oplus)$  or, collectively, *bi-gyroautomorphisms* of the parameter bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$ .

Since  $-P = \ominus P$ , we clearly have the identities

$$O_n(\ominus P) = \ominus O_n P, \quad (\ominus P)O_m = \ominus P O_m, \quad O_n(\ominus P)O_m = \ominus O_n P O_m \quad (124)$$

for all  $P \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ .

## 8. The Bi-Boost Square

We are now in the position to determine the parameters of the squared bi-boost. If we use the convenient notation

$$b_m := \sqrt{I_m + P^t P}, \quad b_n := \sqrt{I_n + P P^t} \quad (125)$$

$P \in \mathbb{R}^{n \times m}$ , then, by (74)

$$B(P) = \begin{pmatrix} b_m & P^t \\ P & b_n \end{pmatrix} \quad (126)$$

and the squared bi-boost  $B(P)$  leads to the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} B(P)^2 &\stackrel{(1)}{\cong} \begin{pmatrix} b_m & P^t \\ P & b_n \end{pmatrix} \begin{pmatrix} b_m & P^t \\ P & b_n \end{pmatrix} \stackrel{(2)}{\cong} \begin{pmatrix} b_m^2 + P^t P & b_m P^t + P^t b_n \\ P b_m + b_n P & b_n^2 + P P^t \end{pmatrix} \\ &\stackrel{(3)}{\cong} \begin{pmatrix} I_m + 2P^t P & 2P^t b_n \\ 2b_n P & I_n + 2P P^t \end{pmatrix} \stackrel{(4)}{\cong} : \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}. \end{aligned} \quad (127)$$

Derivation of the numbered equalities in (127) follows

1. This equation follows from (126).
2. Follows from Item (1) by block matrix multiplication.
3. Results from (125) and the commuting relations (54) and (53).

4. This equation defines  $E_{ij}, i, j = 1, 2$ .

Hence, by the parameter recognition formulas (101), along with (103), we have

$$P \oplus P = E_{21}, \quad E_{12} = \text{rgyr}[P, P](P \oplus P)^t \text{lgyr}[P, P] \tag{128}$$

$$\text{rgyr}[P, P] = E_{11} \sqrt{I_m + E_{21}^t E_{21}}^{-1}, \quad \text{lgyr}[P, P] = \sqrt{I_n + E_{21} E_{21}^t}^{-1} E_{22}$$

where  $E_{ij}$  are given by Item (4) of (127).

Following the first equation in (128) and the definition of  $E_{21}$  in Item (4) of (127), and (53), we have the equations

$$E_{21} = P \oplus P = 2b_n P = 2P b_m. \tag{129}$$

Let us consider the following chain of equations, some of which are numbered for subsequent explanation

$$\begin{aligned} E_{11} &\stackrel{(1)}{=} I_m + 2P^t P = \{I_m + 4P^t P + 4(P^t P)^2\}^{\frac{1}{2}} \\ &= \{I_m + 4(I_m + P^t P)P^t P\}^{\frac{1}{2}} = \{I_m + 4b_m^2 P^t P\}^{\frac{1}{2}} \\ &\stackrel{(2)}{=} \{I_m + 4P^t b_n^2 P\}^{\frac{1}{2}} = \{I_m + 2(b_n P)^t 2b_n P\}^{\frac{1}{2}} \\ &\stackrel{(3)}{=} \sqrt{I_m + E_{21}^t E_{21}}. \end{aligned} \tag{130}$$

Derivation of the numbered equalities in (130) follows

1. This equation follows from the definition of  $E_{11}$  in Item (4) of (127).
2. This equation is obtained from its predecessor by two successive applications of the commuting relation (54).
3. Follows from (129).

We see from (130) and the second equation in (128) that the right gyration generated by  $P$  and  $P$  is trivial

$$\text{rgyr}[P, P] = I_m \tag{131}$$

for all  $P \in \mathbb{R}^{n \times m}$ .

Similarly to (130), let us consider the following chain of equations, some of which are numbered for subsequent explanation

$$\begin{aligned}
 E_{22} &\stackrel{(1)}{\equiv} I_n + 2PP^t = \{I_n + 4PP^t + 4(PP^t)^2\}^{\frac{1}{2}} \\
 &= \{I_n + 4(I_n + PP^t)PP^t\}^{\frac{1}{2}} = \{I_n + 4b_n^2 PP^t\}^{\frac{1}{2}} \\
 &\stackrel{(2)}{\equiv} \{I_n + 4Pb_m^2 P^t\}^{\frac{1}{2}} = \{I_n + 2Pb_m 2(Pb_m)^t\}^{\frac{1}{2}} \\
 &\stackrel{(3)}{\equiv} \sqrt{I_n + E_{21}E_{21}^t}.
 \end{aligned} \tag{132}$$

Derivation of the numbered equalities in (132) follows

1. This equation follows from the definition of  $E_{22}$  in Item (4) of (127).
2. This equation is obtained from its predecessor by two successive applications of the commuting relation (53).
3. Follows from (129).

We see from (132) and the third equation in (128) that the left gyration generated by  $P$  and  $P$  is trivial

$$\text{lgyr}[P, P] = I_n \tag{133}$$

for all  $P \in \mathbb{R}^{n \times m}$ .

It follows from (128)–(133) that

$$\begin{aligned}
 E_{11} &= \sqrt{I_m + (P \oplus P)^t (P \oplus P)}, & E_{21} &= P \oplus P \\
 E_{22} &= \sqrt{I_n + (P \oplus P)(P \oplus P)^t}, & E_{12} &= (P \oplus P)^t.
 \end{aligned} \tag{134}$$

Hence, by the extreme sides of (127)

$$B(P)^2 = B(P \oplus P) \tag{135}$$

so that a the square of a bi-boost is, again, a bi-boost.

As a byproduct of squaring the bi-boost, we have obtained the results in (131) and (133), which we formalize in the following theorem.

**Theorem 16. (A trivial bi-gyration).**

$$\text{lgyr}[P, P] = I_n, \quad \text{rgyr}[P, P] = I_m \tag{136}$$

for all  $P \in \mathbb{R}^{n \times m}$ .

## 9. Commuting Relations Between Bi-Gyrations and Bi-Rotations

Remarkably, bi-gyrations  $(\text{lgyr}[P_1, P_2], \text{rgyr}[P_1, P_2]) \in \text{SO}(n) \times \text{SO}(m)$  and bi-rotations  $(O_n, O_m) \in \text{SO}(n) \times \text{SO}(m)$  possess an elegant commuting relation, stated in the following theorem.

**Theorem 17. (Bi-gyration – bi-rotation commuting relation).**

$$O_n \text{lgyr}[P_1, P_2] = \text{lgyr}[O_n P_1, O_n P_2] O_n \quad (137)$$

and

$$\text{rgyr}[P_1, P_2] O_m = O_m \text{rgyr}[P_1 O_m, P_2 O_m] \quad (138)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ .

**Proof:** The matrix identity (137) is proved in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 O_n \text{lgyr}[P_1, P_2] &\stackrel{(1)}{\cong} O_n \sqrt{I_n + (P_1 \oplus P_2)(P_1 \oplus P_2)^t}^{-1} \\
 &\quad \times \left\{ P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\
 &\stackrel{(2)}{\cong} \sqrt{I_n + (O_n P_1 \oplus O_n P_2)(O_n P_1 \oplus O_n P_2)^t}^{-1} O_n \\
 &\quad \times \left\{ P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\
 &\stackrel{(3)}{\cong} \sqrt{I_n + (O_n P_1 \oplus O_n P_2)(O_n P_1 \oplus O_n P_2)^t}^{-1} \\
 &\quad \times \left\{ O_n P_1 P_2^t + O_n \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\
 &\stackrel{(4)}{\cong} \sqrt{I_n + (O_n P_1 \oplus O_n P_2)(O_n P_1 \oplus O_n P_2)^t}^{-1} \\
 &\quad \times \left\{ (O_n P_1)(O_n P_2)^t \right. \\
 &\quad \left. + \sqrt{I_n + (O_n P_1)(O_n P_1)^t} \sqrt{I_n + (O_n P_2)(O_n P_2)^t} \right\} O_n \\
 &\stackrel{(5)}{\cong} \text{lgyr}[O_n P_1, O_n P_2] O_n.
 \end{aligned} \quad (139)$$

Derivation of the numbered equalities in (139) follows

1. This equation follows from the third equation in (105).
2. Follows from Item (1) by Lemma 3, p. 50, and Theorem 15, p. 63.
3. Follows from Item (2) by the linearity of  $O_n$ .
4. Follows from Item (3) by the obvious matrix identity

$$O_n P_1 P_2^t = (O_n P_1)(O_n P_2)^t O_n$$

and from Lemma 2, p. 49.

5. Follows from Item (4) by the linearity of  $O_n$  and by the third equation in (105).

The proof of the matrix identity (138) in (140) below is similar to the proof of the matrix identity (137) in (139)

$$\begin{aligned}
 \text{lgyr}[P_1, P_2]O_m &\stackrel{(1)}{\cong} \left\{ P_1^t P_2 + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} \right\} \\
 &\quad \times \sqrt{I_m + (P_1 \oplus P_2)^t (P_1 \oplus P_2)}^{-1} O_m \\
 &\stackrel{(2)}{\cong} \left\{ P_1^t P_2 + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} \right\} O_m \\
 &\quad \times \sqrt{I_m + (P_1 O_m \oplus P_2 O_m)^t (P_1 O_m \oplus P_2 O_m)}^{-1} \\
 &\stackrel{(3)}{\cong} \left\{ P_1^t P_2 O_m + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} O_m \right\} \\
 &\quad \times \sqrt{I_m + (P_1 O_m \oplus P_2 O_m)^t (P_1 O_m \oplus P_2 O_m)}^{-1} \\
 &\stackrel{(4)}{\cong} \left\{ O_m (P_1 O_m)^t (P_2 O_m) \right. \\
 &\quad \left. + O_m \sqrt{I_m + (P_1 O_m)^t (P_1 O_m)} \sqrt{I_m + (P_2 O_m)^t (P_2 O_m)} \right\} \\
 &\quad \times \sqrt{I_m + (P_1 O_m \oplus P_2 O_m)^t (P_1 O_m \oplus P_2 O_m)}^{-1} \\
 &\stackrel{(5)}{\cong} O_m \text{rgyr}[P_1 O_m, P_2 O_m].
 \end{aligned} \tag{140}$$

Derivation of the numbered equalities in (140) follows

1. This equation follows from the second equation in (105).

2. Follows from Item (1) by Lemma 3, p. 50, and Theorem 15, p.63.
3. Follows from Item (2) by the linearity of  $O_m$ .
4. Follows from Item (3) by the obvious matrix identity

$$P_1^t P_2 O_m = O_m (P_1 O_m)^t (P_2 O_m)$$

and from Lemma 2, p. 49, and from the linearity of  $O_m$ .

5. Follows from Item (4) by the linearity of  $O_m$  and by the second equation in (105).

■

The following corollary results immediately from Theorem 17.

**Corollary 18.** *Let  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ . Then*

$$\text{lgyr}[O_n P_1, O_n P_2] = \text{lgyr}[P_1, P_2] \quad (141)$$

*if and only if  $O_n$  and  $\text{lgyr}[P_1, P_2]$  commute, that is, if and only if*

$$O_n \text{lgyr}[P_1, P_2] = \text{lgyr}[P_1, P_2] O_n.$$

*Similarly*

$$\text{rgyr}[P_1 O_m, P_2 O_m] = \text{rgyr}[P_1, P_2] \quad (142)$$

*if and only if  $O_m$  and  $\text{rgyr}[P_1, P_2]$  commute, that is, if and only if*

$$O_m \text{rgyr}[P_1, P_2] = \text{rgyr}[P_1, P_2] O_m.$$

**Example 19.** *The left (right) gyration  $\text{lgyr}[P_1, P_2]$  ( $\text{rgyr}[P_1, P_2]$ ) commutes with itself. Hence, by Corollary 18*

$$\begin{aligned} \text{lgyr}[\text{lgyr}[P_1, P_2] P_1, \text{lgyr}[P_1, P_2] P_2] &= \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1 \text{rgyr}[P_1, P_2], P_2 \text{rgyr}[P_1, P_2]] &= \text{rgyr}[P_1, P_2]. \end{aligned} \quad (143)$$

Left gyrations are invariant under parameter right rotations  $O_m \in \text{SO}(m)$ , and right gyrations are invariant under parameter left rotations  $O_n \in \text{SO}(n)$ , as the following theorem asserts.

**Theorem 20. (Bi-gyration Invariance Relation).**

$$\text{lgyr}[P_1 O_m, P_2 O_m] = \text{lgyr}[P_1, P_2] \quad (144)$$

$$\text{rgyr}[O_n P_1, O_n P_2] = \text{rgyr}[P_1, P_2] \quad (145)$$

*for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ .*

**Proof:** The proof follows straightforwardly from the second and the third equations in (105), p. 60, and from Theorem 15, p. 63, noting that  $(P_1 O_m)(P_2 O_m)^t = P_1 P_2^t$  and  $(O_n P_1)^t(O_n P_2) = P_1^t P_2$  for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ . ■

## 10. Product of Lorentz Transformations

Let  $\Lambda_1$  and  $\Lambda_2$  be two Lorentz transformations of order  $(m, n)$ ,  $m, n \in \mathbb{N}$ , so that, according to (82)

$$\begin{aligned}\Lambda_1 &= \Lambda(O_{n,1}, P_1, O_{m,1}) = \rho(O_{m,1})B(P_1)\lambda(O_{n,1}) \\ \Lambda_2 &= \Lambda(O_{n,2}, P_2, O_{m,2}) = \rho(O_{m,2})B(P_2)\lambda(O_{n,2}).\end{aligned}\tag{146}$$

The product  $\Lambda_1 \Lambda_2$  of  $\Lambda_1$  and  $\Lambda_2$  is obtained in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}\Lambda_1 \Lambda_2 &\stackrel{(1)}{\cong} \rho(O_{m,1})B(P_1)\lambda(O_{n,1})\rho(O_{m,2})B(P_2)\lambda(O_{n,2}) \\ &\stackrel{(2)}{\cong} \rho(O_{m,1})B(P_1)\rho(O_{m,2})\lambda(O_{n,1})B(P_2)\lambda(O_{n,2}) \\ &\stackrel{(3)}{\cong} \rho(O_{m,1})\rho(O_{m,2})B(P_1 O_{m,2})B(O_{n,1} P_2)\lambda(O_{n,1})\lambda(O_{n,2}) \\ &\stackrel{(4)}{\cong} \rho(O_{m,1} O_{m,2})B(P_1 O_{m,2})B(O_{n,1} P_2)\lambda(O_{n,1} O_{n,2}) \\ &\stackrel{(5)}{\cong} \rho(O_{m,1} O_{m,2}) \\ &\quad \times \rho(\text{rgyr}[P_1 O_{m,2}, O_{n,1} P_2])B(P_1 O_{m,2} \oplus O_{n,1} P_2) \\ &\quad \quad \lambda(\text{lgyr}[P_1 O_{m,2}, O_{n,1} P_2]) \times \lambda(O_{n,1} O_{n,2}) \\ &\stackrel{(6)}{\cong} \rho(O_{m,1} O_{m,2} \text{rgyr}[P_1 O_{m,2}, O_{n,1} P_2]) \\ &\quad \times B(P_1 O_{m,2} \oplus O_{n,1} P_2) \\ &\quad \times \lambda(\text{lgyr}[P_1 O_{m,2}, O_{n,1} P_2] O_{n,1} O_{n,2}).\end{aligned}\tag{147}$$

Derivation of the numbered equalities in (147) follows

1. This equation follows from (146).
2. Follows from (1) since  $\lambda(O_{n,1})$  and  $\rho(O_{m,2})$  commute.



3. Follows from (2) by Lemma 6, p. 53.
4. Follows from (3) by the matrix identities  $\rho(O_{m,1})\rho(O_{m,2}) = \rho(O_{m,1}O_{m,2})$  and  $\lambda(O_{n,1})\lambda(O_{n,2}) = \lambda(O_{n,1}O_{n,2})$ , which are obvious.
5. Follows from (4) by the bi-boost composition law (104), p. 59.
6. Obvious (Similar to the argument in Item (4)).

In the column notation (79), the result of (147) gives the product law of Lorentz transformations in the following theorem.

**Theorem 21. (Lorentz transformation product law).** *The product of two Lorentz transformations  $\Lambda_1 = (P_1, O_{n,1}, O_{m,1})^t$  and  $\Lambda_2 = (P_2, O_{n,2}, O_{m,2})^t$  of order  $(m, n)$ ,  $m, n \in \mathbb{N}$ , is given by*

$$\Lambda_1\Lambda_2 = \begin{pmatrix} P_1 \\ O_{n,1} \\ O_{m,1} \end{pmatrix} \begin{pmatrix} P_2 \\ O_{n,2} \\ O_{m,2} \end{pmatrix} = \begin{pmatrix} P_1O_{m,2} \oplus O_{n,1}P_2 \\ \text{lgYr}[P_1O_{m,2}, O_{n,1}P_2]O_{n,1}O_{n,2} \\ O_{m,1}O_{m,2}\text{rYr}[P_1O_{m,2}, O_{n,1}P_2] \end{pmatrix}. \quad (148)$$

**Example 22.** *In the special case when  $P_1 = P_2 = 0_{n,m}$  and  $O_{m,1} = O_{m,2} = I_m$ , the parameter composition law (148) yields the equation*

$$\begin{pmatrix} 0_{n,m} \\ O_{n,1} \\ I_m \end{pmatrix} \begin{pmatrix} 0_{n,m} \\ O_{n,2} \\ I_m \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ O_{n,1}O_{n,2} \\ I_m \end{pmatrix} \quad (149)$$

*demonstrating that under the parameter composition law (148) the parameter  $O_n$  forms the spacial orthogonal group  $\text{SO}(n)$ .*

**Example 23.** *In the special case when  $P_1 = P_2 = 0_{n,m}$  and  $O_{n,1} = O_{n,2} = I_n$ , the parameter composition law (148) yields the equation*

$$\begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1} \end{pmatrix} \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,2} \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1}O_{m,2} \end{pmatrix} \quad (150)$$

*demonstrating that under the parameter composition law (148) the parameter  $O_m$  forms the spacial orthogonal group  $\text{SO}(m)$ .*

**Example 24.** In the special case when  $O_{n,1} = O_{n,2} = I_n$  and  $O_{m,1} = O_{m,2} = I_m$  the parameter composition law (148) yields the equation

$$\begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix}. \quad (151)$$

Clearly, under the parameter composition law (148) the parameter  $P \in \mathbb{R}^{n \times m}$  does not form a group. Indeed, following the parametrization of the generalized Lorentz group  $\text{SO}(m, n)$  in (146), we face the task of determining the composition law of the parameter  $P \in \mathbb{R}^{n \times m}$  along with the resulting group-like structure of the parameter set  $\mathbb{R}^{n \times m}$ . We will find in the sequel that the group-like structure of  $\mathbb{R}^{n \times m}$  that results from the composition law of the parameter  $P$  is a natural generalization of the gyrocommutative gyrogroup structure, called a bi-gyrocommutative bi-gyrogroup.

The Lorentz transformation product (148) represents matrix multiplication. As such, it is associative and, clearly, its inverse obeys the identity

$$(\Lambda_1 \Lambda_2)^{-1} = \Lambda_2^{-1} \Lambda_1^{-1}. \quad (152)$$

## 11. The Bi-Gyrocommutative Law

Bi-boosts are Lorentz transformations without bi-rotations. Let

$$B(P_k) = (P_k, I_n, I_m)^t \quad (153)$$

$P_k \in \mathbb{R}^{n \times m}$ ,  $k = 1, 2$ , be two bi-boosts in  $\mathbb{R}^{(m+n) \times (m+n)}$ . Then, by (148) with  $O_{n,1} = O_{n,2} = I_n$  and  $O_{m,1} = O_{m,2} = I_m$  (or by (110)), and by (93) with  $O_n = \text{lgyr}[P_1, P_2]$  and  $O_m = \text{rgyr}[P_1, P_2]$

$$\begin{aligned} (B(P_1)B(P_2))^{-1} &= \left\{ \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \right\}^{-1} = \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -\text{lgyr}^{-1}[P_1, P_2](P_1 \oplus P_2)\text{rgyr}^{-1}[P_1, P_2] \\ \text{lgyr}^{-1}[P_1, P_2] \\ \text{rgyr}^{-1}[P_1, P_2] \end{pmatrix}. \end{aligned} \quad (154)$$

Here  $\text{lgyr}^{-1}[P_1, P_2] = (\text{lgyr}[P_1, P_2])^{-1}$  and  $\text{rgyr}^{-1}[P_1, P_2] = (\text{rgyr}[P_1, P_2])^{-1}$ .

Calculating  $(B(P_1)B(P_2))^{-1}$  in a different way, as indicated in (152), yields

$$\begin{aligned} (B(P_1)B(P_2))^{-1} &= B(P_2)^{-1}B(P_1)^{-1} = \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix}^{-1} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -P_2 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} -P_1 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} (-P_2) \oplus (-P_1) \\ \text{lgyr}[-P_2, -P_1] \\ \text{rgyr}[-P_2, -P_1] \end{pmatrix}. \end{aligned} \quad (155)$$

Hence, the extreme right sides of (154)–(155) are equal, implying the equality of their respective entries, giving rise to the three equations in (156)–(157) below.

The second and third entries of the extreme right sides of (154)–(155), along with the even property (119) of bi-gyrations, imply the *bi-gyration inversion law*

$$\begin{aligned} \text{lgyr}^{-1}[P_1, P_2] &= \text{lgyr}[-P_2, -P_1] = \text{lgyr}[P_2, P_1] \\ \text{rgyr}^{-1}[P_1, P_2] &= \text{rgyr}[-P_2, -P_1] = \text{rgyr}[P_2, P_1]. \end{aligned} \quad (156)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

The first entry of the extreme right sides of (154)–(155), along with (156) and the gyroautomorphic inverse property (118), yields

$$\begin{aligned} (-P_2) \oplus (-P_1) &= -\text{lgyr}^{-1}[P_1, P_2](P_1 \oplus P_2)\text{rgyr}^{-1}[P_1, P_2] \\ &= -\text{lgyr}[-P_2, -P_1](P_1 \oplus P_2)\text{rgyr}[-P_2, -P_1] \\ &= \text{lgyr}[-P_2, -P_1]\{-(P_1 \oplus P_2)\}\text{rgyr}[-P_2, -P_1] \\ &= \text{lgyr}[-P_2, -P_1]\{(-P_1) \oplus (-P_2)\}\text{rgyr}[-P_2, -P_1] \end{aligned} \quad (157)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Renaming  $-P_1$  and  $-P_2$  as  $P_2$  and  $P_1$ , the extreme sides of (157) give the *bi-gyrocommutative law* of the bi-gyroaddition  $\oplus$

$$P_1 \oplus P_2 = \text{lgyr}[P_1, P_2](P_2 \oplus P_1)\text{rgyr}[P_1, P_2] \quad (158)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Instructively, a short derivation of the bi-gyrocommutative law of  $\oplus$  is presented below. Transposing the extreme sides of the fourth matrix equation in (105), noting that by (156)

$$\begin{aligned} \text{lgyr}[P_1, P_2]^t &= \text{lgyr}^{-1}[P_1, P_2] = \text{lgyr}[P_2, P_1] \\ \text{rgyr}[P_1, P_2]^t &= \text{rgyr}^{-1}[P_1, P_2] = \text{rgyr}[P_2, P_1] \end{aligned} \quad (159)$$

and renaming the pair  $(P_1, P_2)$  as  $(P_2, P_1)$ , we obtain the matrix identity

$$P_1 \oplus P_2 = \text{lgyr}[P_1, P_2](P_2 \oplus P_1) \text{rgyr}[P_1, P_2] \quad (160)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

The matrix identity (160) gives the *bi-gyrocommutative law* of the binary operation  $\oplus$  in  $\mathbb{R}^{n \times m}$ , according to which  $P_1 \oplus P_2$  equals  $P_2 \oplus P_1$  bi-gyrated by the bi-gyration  $(\text{lgyr}[P_1, P_2], \text{rgyr}[P_1, P_2])$  generated by  $P_1$  and  $P_2$ , for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Formalizing the result in (158) and in (160) we obtain the following theorem.

**Theorem 25. (Bi-gyrocommutative Law).** *The binary operation  $\oplus$  in  $\mathbb{R}^{n \times m}$  possesses the bi-gyrocommutative law*

$$P_1 \oplus P_2 = \text{lgyr}[P_1, P_2](P_2 \oplus P_1) \text{rgyr}[P_1, P_2] \quad (161)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

When  $m = 1$  right gyrations are trivial,  $\text{rgyr}[P_1, P_2] = I_m$ . Hence, in the special case when  $m = 1$ , the bi-gyrocommutative law (161) of bi-gyrogroup theory descends to the gyrocommutative law of gyrogroup theory found, for instance, in [19, 20, 22–25, 27].

Formalizing the results in (159) we obtain the following theorem.

**Theorem 26. (Bi-gyration Inversion Law).** *The bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$  possesses the left gyration inversion law*

$$\text{lgyr}^{-1}[P_1, P_2] = \text{lgyr}[P_2, P_1] \quad (162a)$$

and the right gyration inversion law

$$\text{rgyr}^{-1}[P_1, P_2] = \text{rgyr}[P_2, P_1] \quad (162b)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Identities (162a)–(162b) form the *inversive symmetric property* of bi-gyrations.

## 12. The Bi-Gyroassociative Law

Matrix multiplication is associative. Hence

$$(\Lambda_1 \Lambda_2) \Lambda_3 = \Lambda_1 (\Lambda_2 \Lambda_3). \quad (163)$$

On the one hand, by (110) and (148)

$$\begin{aligned}
 (B(P_1)B(P_2))B(P_3) &= \left\{ \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \right\} \begin{pmatrix} P_3 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \begin{pmatrix} P_3 \\ I_n \\ I_m \end{pmatrix} \\
 &= \begin{pmatrix} (P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_3 \\ \text{lgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_3] \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \text{rgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_3] \end{pmatrix}.
 \end{aligned} \tag{164}$$

On the other hand, similarly, by (110) and (148),

$$\begin{aligned}
 B(P_1)(B(P_2)B(P_3)) &= \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \left\{ \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_3 \\ I_n \\ I_m \end{pmatrix} \right\} = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \oplus P_3 \\ \text{lgyr}[P_2, P_3] \\ \text{rgyr}[P_2, P_3] \end{pmatrix} \\
 &= \begin{pmatrix} P_1 \text{rgyr}[P_2, P_3] \oplus (P_2 \oplus P_3) \\ \text{lgyr}[P_1 \text{rgyr}[P_2, P_3], P_2 \oplus P_3] \text{lgyr}[P_2, P_3] \\ \text{rgyr}[P_2, P_3] \text{rgyr}[P_1 \text{rgyr}[P_2, P_3], P_2 \oplus P_3] \end{pmatrix}.
 \end{aligned} \tag{165}$$

Hence, by (163)–(165), corresponding entries of the extreme right sides of (164) and (165) are equal, giving rise to the *bi-gyroassociative law*

$$(P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_3 = P_1 \text{rgyr}[P_2, P_3] \oplus (P_2 \oplus P_3) \tag{166}$$

and to the bi-gration identities

$$\text{lgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_3] \text{lgyr}[P_1, P_2] = \text{lgyr}[P_1 \text{rgyr}[P_2, P_3], P_2 \oplus P_3] \text{lgyr}[P_2, P_3] \tag{167}$$

$$\text{rgyr}[P_1, P_2] \text{rgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_3] = \text{rgyr}[P_2, P_3] \text{rgyr}[P_1 \text{rgyr}[P_2, P_3], P_2 \oplus P_3]$$

for all  $P_1, P_2, P_3 \in \mathbb{R}^{n \times m}$ .

Formalizing the result in (166) we obtain the following theorem.

**Theorem 27. (Bi-gyroassociative law in  $(\mathbb{R}^{n \times m}, \oplus)$ ).** *The bi-gyroaddition  $\oplus$  in  $\mathbb{R}^{n \times m}$  possesses the bi-gyroassociative law*

$$(P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_3 = P_1 \text{rgyr}[P_2, P_3] \oplus (P_2 \oplus P_3) \tag{168}$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Note that in the bi-gyroassociative law (168),  $P_1$  and  $P_2$  are grouped together on the left side, while  $P_2$  and  $P_3$  are grouped together on the right side.

When  $m = 1$  right gyrations are trivial,  $\text{rgyr}[P_1, P_2] = I_{m=1} = 1$ . Hence, in the special case when  $m = 1$ , the bi-gyroassociative law (168) descends to the gyroassociative law of gyrogroup theory found, for instance, in [19, 20, 22–25, 27]. The bi-gyroassociative law gives rise to the left and right cancellation laws in the following theorem.

**Theorem 28. (Left and right cancellation laws in  $(\mathbb{R}^{n \times m}, \oplus)$ ).** *The bi-gyrogroupoid  $(\mathbb{R}^{n \times m}, \oplus)$  possesses the left and right cancellation laws*

$$P_2 = \ominus P_1 \text{rgyr}[P_1, P_2] \oplus (P_1 \oplus P_2) \quad (169)$$

and

$$P_1 = (P_1 \oplus P_2) \ominus \text{lgyr}[P_1, P_2] P_2 \quad (170)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** The left cancellation law (169) follows from the bi-gyroassociative law (168) with  $P_1 = \ominus P_2$ , noting that  $\text{lgyr}[\ominus P_2, P_2]$  is trivial by (114), p. 62. The right cancellation law (170) follows from the bi-gyroassociative law (168) with  $P_3 = \ominus P_2$ , noting that  $\text{rgyr}[P_2, \ominus P_2]$  is trivial. ■

The bi-gyroassociative law gives rise to the left and right bi-gyroassociative laws in the following theorem.

**Theorem 29. (Left and right bi-gyroassociative law in  $(\mathbb{R}^{n \times m}, \oplus)$ ).** *The bi-gyroaddition  $\oplus$  in  $\mathbb{R}^{n \times m}$  possesses the left bi-gyroassociative law*

$$P_1 \oplus (P_2 \oplus P_3) = (P_1 \text{rgyr}[P_3, P_2] \oplus P_2) \oplus \text{lgyr}[P_1 \text{rgyr}[P_3, P_2], P_2] P_3 \quad (171)$$

and the right bi-gyroassociative law

$$(P_1 \oplus P_2) \oplus P_3 = P_1 \text{rgyr}[P_2, \text{lgyr}[P_2, P_1] P_3] \oplus (P_2 \oplus \text{lgyr}[P_2, P_1] P_3) \quad (172)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** The left bi-gyroassociative law (171) is obtained from the bi-gyroassociative law (168) by replacing  $P_1$  by  $P_1 \text{rgyr}[P_3, P_2]$  and noting the bi-gyration inversion law (156). Similarly, the right bi-gyroassociative law (172) is obtained from the bi-gyroassociative law (168) by replacing  $P_3$  by  $\text{lgyr}[P_2, P_1] P_3$  and noting the bi-gyration inversion law (156). ■

### 13. Bi-Gyration Reduction Properties

A reduction property of a gyration  $\text{lgyr}[P_1, P_2]$  or  $\text{rgyr}[P_1, P_2]$  is a property enabling the gyration to be expressed as a gyration that involves  $P_1 \oplus P_2$ . Several reduction properties are derived in Subsections 13.1 – 13.4 below.

#### 13.1. Bi-Gyration Reduction Properties I

When  $P_3 = \ominus P_2$ , (167) specializes to

$$\begin{aligned} \text{lgyr}[P_1 \oplus P_2, \ominus \text{lgyr}[P_1, P_2]P_2] \text{lgyr}[P_1, P_2] &= I_n \\ \text{rgyr}[P_1, P_2] \text{rgyr}[P_1 \oplus P_2, \ominus \text{lgyr}[P_1, P_2]P_2] &= I_m \end{aligned} \quad (173)$$

or, equivalently by bi-gyration inversion, (156)

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[\ominus \text{lgyr}[P_1, P_2]P_2, P_1 \oplus P_2] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[\ominus \text{lgyr}[P_1, P_2]P_2, P_1 \oplus P_2]. \end{aligned} \quad (174)$$

Similarly, when  $P_2 = \ominus P_1$ , (167) specializes to

$$\begin{aligned} I_n &= \text{lgyr}[P_1 \text{rgyr}[\ominus P_1, P_3], \ominus P_1 \oplus P_3] \text{lgyr}[\ominus P_1, P_2] \\ I_m &= \text{rgyr}[\ominus P_1, P_3] \text{rgyr}[P_1 \text{rgyr}[\ominus P_1, P_3], \ominus P_1 \oplus P_3] \end{aligned} \quad (175)$$

or, equivalently by bi-gyration inversion, (156) and renaming  $P_3$  as  $\ominus P_2$

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]]. \end{aligned} \quad (176)$$

Formalizing the results in (174) and (176) we obtain the following theorem.

**Theorem 30. (Left and Right Gyration Reduction Properties).**

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[\ominus \text{lgyr}[P_1, P_2]P_2, P_1 \oplus P_2] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[\ominus \text{lgyr}[P_1, P_2]P_2, P_1 \oplus P_2] \end{aligned} \quad (177)$$

and

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \end{aligned} \quad (178)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

### 13.2. Bi-Gyration Reduction Properties II

In general, the product of bi-boosts in a pseudo-orthogonal group  $\text{SO}(m, n)$  is a Lorentz transformation which is not a boost. In some special cases, however, the product of bi-boosts is again a bi-boost, as shown below.

Let  $P_1, P_2 \in \mathbb{R}^{n \times m}$ , and let  $J(P_1, P_2)$  be the bi-boost symmetric product

$$J(P_1, P_2) = B(P_1)B(P_2)B(P_1) = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \quad (179)$$

which is symmetric with respect to the central bi-boost factor  $(P_2, I_n, I_m)^t$ . Then, by the Lorentz product law (148)

$$\begin{aligned} J(P_1, P_2) &= \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \\ &= \begin{pmatrix} (P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_1 \\ \text{lgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_1] \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \text{rgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_1] \end{pmatrix} =: \begin{pmatrix} P_3 \\ O_n \\ O_m \end{pmatrix}. \end{aligned} \quad (180)$$

By means of (91), p. 56, it is clear that

$$J(P_1, P_2)^{-1} = J(-P_1, -P_2). \quad (181)$$

Hence, by the gyroautomorphic inverse property (118), p. 62, and by the bi-gyration even property, (119), p. 62, it is clear from (180) that

$$J(P_1, P_2)^{-1} = J(-P_1, -P_2) = \begin{pmatrix} -P_3 \\ O_n \\ O_m \end{pmatrix}. \quad (182)$$

But, it follows from the inverse Lorentz transformation (93), p. 56, that

$$J(P_1, P_2)^{-1} = \begin{pmatrix} -O_n^{-1}P_3O_m^{-1} \\ O_n^{-1} \\ O_m^{-1} \end{pmatrix}. \quad (183)$$

Comparing the right sides of (183) and (182), we find that  $O_n = O_n^{-1}$  and  $O_m = O_m^{-1}$ , implying  $O_n = I_n$  and  $O_m = I_m$ . Hence, the bi-boost product  $J(P_1, P_2)$  is,



again, a bi-boost, so that by (180)

$$J(P_1, P_2) = \begin{pmatrix} ((P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_1) \\ I_n \\ I_m \end{pmatrix}. \quad (184)$$

Following (184) and (180) we have the bi-gyration identities

$$\begin{aligned} \text{lgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_1] \text{lgyr}[P_1, P_2] &= I_n \\ \text{rgyr}[P_1, P_2] \text{rgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_1] &= I_m \end{aligned} \quad (185)$$

implying, by the bi-gyration inversion law (162)

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \end{aligned} \quad (186)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

The results in (184)–(185) can readily be extended to the symmetric product of any number of bi-boosts that appear symmetrically with respect to a central factor. Thus, for instance, the symmetric bi-boost product  $J$

$$J = B(P_k)B(P_{k-1}) \dots B(P_2)B(P_1)B(P_0)B(P_1)B(P_2) \dots B(P_{k-1})B(P_k) \quad (187)$$

is symmetric with respect to the central factor  $B(P_0)$ , for any  $k \in \mathbb{N}$ , and all  $P_i \in \mathbb{R}^{n \times m}$ ,  $i = 0, 1, 2, \dots, k$ . In particular, the bi-boost product  $J$  in (187) is, again, a bi-boost.

We now manipulate the first bi-gyration identity in (185) into an elegant form that will be elevated to the status of a theorem in Theorem 31 below. Let us consider the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} I_n &\stackrel{(1)}{\cong} \text{lgyr}[P_1 \oplus P_2, \text{lgyr}[P_1, P_2]P_1] \text{lgyr}[P_1, P_2] \\ &\stackrel{(2)}{\cong} \text{lgyr}[P_1, P_2] \text{lgyr}[\text{lgyr}[P_2, P_1](P_1 \oplus P_2), P_1] \\ &\stackrel{(3)}{\cong} \text{lgyr}[P_1, P_2] \text{lgyr}[\text{lgyr}[P_2, P_1](P_1 \oplus P_2) \text{rgyr}[P_2, P_1], P_1 \text{rgyr}[P_2, P_1]] \\ &\stackrel{(4)}{\cong} \text{lgyr}[P_1, P_2] \text{lgyr}[P_2 \oplus P_1, P_1 \text{rgyr}[P_2, P_1]]. \end{aligned} \quad (188)$$

Derivation of the numbered equalities in (188) follows

1. This equation is the first equation in the first bi-gyration identity in (185).
2. Follows from the commuting relation (137), p. 67, with  $O_n = \text{lgyr}[P_1, P_2]$ , noting that  $\text{lgyr}[P_2, P_1] = \text{lgyr}^{-1}[P_1, P_2]$ .
3. Follows from the bi-gyration invariance relation (144), p. 69.
4. Follows from the bi-gyrocommutative law (161), p. 74.

By (188) and the bi-gyration inversion law (162)

$$\text{lgyr}[P_2, P_1] = \text{lgyr}[P_2 \oplus P_1, P_1 \text{rgyr}[P_2, P_1]]. \quad (189)$$

Renaming  $(P_1, P_2)$  in (189) as  $(P_2, P_1)$ , we obtain the first identity in the following theorem.

**Theorem 31. (Left Gyration Reduction Properties).**

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[P_1 \oplus P_2, P_2 \text{rgyr}[P_1, P_2]] \quad (190)$$

and

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \quad (191)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** The bi-gyration identity (190) is identical with (189). The bi-gyration identity (191) is obtained from (190) by applying the bi-gyration inversion law (162) followed by renaming  $(P_1, P_2)$  as  $(P_2, P_1)$ . ■

When  $m = 1$  right gyrations are trivial,  $\text{rgyr}[P_1, P_2] = I_{m=1} = 1$ . Hence, in the special case when  $m = 1$ , the bi-gyration reduction properties (190)–(191) descend to the gyration properties of gyrogroup theory found, for instance, in [20].

The bi-gyration identity (191) involves both left and right gyrations. We manipulate it into an identity that involves only left gyrations in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} \text{lgyr}[P_1, P_2] &\stackrel{(1)}{\cong} \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \\ &\stackrel{(2)}{\cong} \text{lgyr}[\text{lgyr}[P_1, P_2] P_1 \text{rgyr}[P_2, P_1], \text{lgyr}[P_1, P_2](P_2 \oplus P_1)] \\ &\stackrel{(3)}{\cong} \text{lgyr}[\text{lgyr}[P_1, P_2] P_1 \text{rgyr}[P_2, P_1] \text{rgyr}[P_1, P_2], \\ &\quad \text{lgyr}[P_1, P_2](P_2 \oplus P_1) \text{rgyr}[P_1, P_2]] \\ &\stackrel{(4)}{\cong} \text{lgyr}[\text{lgyr}[P_1, P_2] P_1, P_1 \oplus P_2]. \end{aligned} \quad (192)$$

Derivation of the numbered equalities in (192) follows

1. This equation is the bi-gyration identity 191.
2. Follows from Result (141) of Corollary 18, p. 69, noting that, by Item (1), the left gyrations  $\text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1]$  and  $\text{lgyr}[P_1, P_2]$  commute since they are equal.
3. Follows from (144), p. 69.
4. Follows from Item (3) by applying both the bi-gyration inversion law (162) and the bi-gyrocommutative law (161), p. 74.

By means of the bi-gyration inversion law (162), the second bi-gyration identity in (185) gives rise to the bi-gyration identity

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \quad (193)$$

leading to the following theorem.

**Theorem 32. (Right gyration reduction properties).**

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \quad (194)$$

and

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[P_2 \oplus P_1, \text{lgyr}[P_2, P_1]P_2] \quad (195)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** The bi-gyration identity (194) is identical with (193). The bi-gyration identity (195) is obtained from (194) by applying the bi-gyration inversion law (162) followed by renaming  $(P_1, P_2)$  as  $(P_2, P_1)$ . ■

The bi-gyration identity (195) involves both left and right gyrations. We manipulate it into an identity that involves only right gyrations in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} \text{rgyr}[P_1, P_2] &\stackrel{(1)}{\cong} \text{rgyr}[P_2 \oplus P_1, \text{lgyr}[P_2, P_1]P_2] \\ &\stackrel{(2)}{\cong} \text{rgyr}[(P_2 \oplus P_1) \text{rgyr}[P_1, P_2], \text{lgyr}[P_2, P_1]P_2 \text{rgyr}[P_1, P_2]] \\ &\stackrel{(3)}{\cong} \text{rgyr}[\text{lgyr}[P_1, P_2](P_2 \oplus P_1) \text{rgyr}[P_1, P_2], \\ &\quad \text{lgyr}[P_1, P_2] \text{lgyr}[P_2, P_1]P_2 \text{rgyr}[P_1, P_2]] \\ &\stackrel{(4)}{\cong} \text{rgyr}[P_1 \oplus P_2, P_2 \text{rgyr}[P_1, P_2]]. \end{aligned} \quad (196)$$

Derivation of the numbered equalities in (196) follows

1. This equation is the bi-gyration identity 195.
2. Follows from Result (142) of Corollary 18, p. 69, noting that, by Item 1, the right gyrations  $\text{rgyr}[P_2 \oplus P_1, \text{lgyr}[P_2, P_1]P_2]$  and  $\text{rgyr}[P_1, P_2]$  commute since they are equal.
3. Follows from (145), p. 69.
4. Follows from Item (3) by applying both the bi-gyration inversion law (162) and the bi-gyrocommutative law (161), p. 74.

Formalizing the results in (192) and (196) we obtain the following theorem.

**Theorem 33. (Bi-gyration reduction properties).**

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \quad (197)$$

and

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[P_1 \oplus P_2, P_2 \text{rgyr}[P_1, P_2]] \quad (198)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

### 13.3. Bi-Gyration reduction properties III

As in Subsection 13.2, let  $P_1, P_2 \in \mathbb{R}^{n \times m}$ , and let  $J(P_1, P_2)$  be the bi-boost symmetric product

$$J(P_1, P_2) = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \quad (199)$$

which is symmetric with respect to the central bi-boost factor  $(P_2, I_n, I_m)^t$ . Then

$$\begin{aligned} J(P_1, P_2) &= \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \oplus P_1 \\ \text{lgyr}[P_2, P_1] \\ \text{rgyr}[P_2, P_1] \end{pmatrix} \\ &= \begin{pmatrix} P_1 \text{rgyr}[P_2, P_1] \oplus (P_2 \oplus P_1) \\ \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \text{lgyr}[P_2, P_1] \\ \text{rgyr}[P_2, P_1] \text{rgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \end{pmatrix} =: \begin{pmatrix} P_3 \\ O_n \\ O_m \end{pmatrix}. \end{aligned} \quad (200)$$

By means of (91), p. 56, it is clear that

$$J(P_1, P_2)^{-1} = J(-P_1, -P_2). \quad (201)$$

Hence, by the gyroautomorphic inverse property (118), p. 62, and by the bi-gyration even property, (119), p. 62, it is clear from (200) that

$$J(P_1, P_2)^{-1} = J(-P_1, -P_2) = \begin{pmatrix} -P_3 \\ O_n \\ O_m \end{pmatrix}. \quad (202)$$

But, it follows from the inverse Lorentz transformation (93), p. 56, that

$$J(P_1, P_2)^{-1} = \begin{pmatrix} -O_n^{-1}P_3O_m^{-1} \\ O_n^{-1} \\ O_m^{-1} \end{pmatrix}. \quad (203)$$

Comparing the right sides of (203) and (202), we find that  $O_m = I_m$  and  $O_n = I_n$ . Hence, the bi-boost product  $J(P_1, P_2)$  is, again, a bi-boost, so that by (200)

$$J(P_1, P_2) = \begin{pmatrix} P_1 \text{rgyr}[P_2, P_1] \oplus (P_2 \oplus P_1) \\ I_n \\ I_m \end{pmatrix}. \quad (204)$$

Following (204) and (200) we have the bi-gyration identities

$$\begin{aligned} \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \text{lgyr}[P_2, P_1] &= I_n \\ \text{rgyr}[P_2, P_1] \text{rgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] &= I_m \end{aligned} \quad (205)$$

implying

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \end{aligned} \quad (206)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

The first entries of (180) and (200) imply the interesting identity

$$(P_1 \oplus P_2) \oplus \text{lgyr}[P_1, P_2]P_1 = P_1 \text{rgyr}[P_2, P_1] \oplus (P_2 \oplus P_1). \quad (207)$$

### 13.4. Bi-Gyration Reduction Properties IV

Let  $(P_1, I_n, I_m)^t$  and  $(P_2, I_n, I_m)^t$  be two given bi-boosts in the pseudo-Euclidean space  $\mathbb{R}^{m,n}$ , and let the bi-boost  $(X, O_n, O_m)^t$  be given by the equation

$$\begin{pmatrix} X \\ O_n \\ O_m \end{pmatrix} = \begin{pmatrix} \ominus P_1 \\ I_n \\ I_m \end{pmatrix}^{-1} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix}. \quad (208)$$

Then the following two consequences of (208) are equivalent

$$\begin{pmatrix} X \\ O_n \\ O_m \end{pmatrix} = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \quad (209)$$

and

$$\begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} \ominus P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} X \\ O_n \\ O_m \end{pmatrix} = \begin{pmatrix} \ominus P_1 O_m \oplus X \\ \text{lgyr}[\ominus P_1 O_m, X] O_n \\ O_m \text{rgyr}[\ominus P_1 O_m, X] \end{pmatrix}. \quad (210)$$

The matrix equation (210) in  $\mathbb{R}^{m,n}$  implies

$$O_n = \text{lgyr}[X, \ominus P_1 O_m], \quad O_m = \text{rgyr}[X, \ominus P_1 O_m] \quad (211)$$

so that, by the first entry of the matrix equation (209)

$$O_n = \text{lgyr}[P_1 \oplus P_2, \ominus P_1 O_m], \quad O_m = \text{rgyr}[P_1 \oplus P_2, \ominus P_1 O_m]. \quad (212)$$

Inserting  $O_n$  and  $O_m$  from the second and the third entries of the matrix equation (209) into (212), we obtain the reduction properties

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_1 \oplus P_2, \ominus P_1 \text{lgyr}[P_1, P_2]] \end{aligned} \quad (213)$$

thus recovering (178).

As a first example, the first reduction property in (213) gives rise to the reduction property

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1] \quad (214)$$

in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 \text{lgyr}[P_1, P_2] &\stackrel{(1)}{\cong} \text{lgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \\
 &\stackrel{(2)}{\cong} \text{lgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1 \text{rgyr}[P_1, P_2] \text{rgyr}[P_2, P_1]] \\
 &\stackrel{(3)}{\cong} \text{lgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1]. \tag{215}
 \end{aligned}$$

Derivation of the numbered equalities in (215) follows

1. This is the first identity in (213).
2. Item (2) is derived from Item (1) by applying Identity (144) of Theorem 20, p. 69, with  $O_m = \text{rgyr}[P_2, P_1]$ .
3. Item 3 follows immediately from Item 2 by the bi-gyration inversion law (156), p. 73.

As a second example, the second reduction property in (213) gives rise to the reduction property

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1] \tag{216}$$

in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 \text{rgyr}[P_1, P_2] &\stackrel{(1)}{\cong} \text{rgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]] \\
 &\stackrel{(2)}{\cong} \text{rgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1 \text{rgyr}[P_1, P_2] \text{rgyr}[P_2, P_1]] \\
 &\stackrel{(3)}{\cong} \text{rgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1]. \tag{217}
 \end{aligned}$$

Derivation of the numbered equalities in (217) follows

1. This is the second identity in (213).
2. The right gyrations  $\text{rgyr}[P_2, P_1]$  and  $\text{rgyr}[P_1, P_2]$  commute since they are inverse of each other. Hence, by Item 1, the right gyrations  $\text{rgyr}[P_2, P_1]$  and  $\text{rgyr}[P_1 \oplus P_2, \ominus P_1 \text{rgyr}[P_1, P_2]]$  commute. The latter commutativity, in turn, implies Item 2 by Corollary 18, p. 69, with  $O_m = \text{rgyr}[P_2, P_1]$ .

3. Item 3 follows immediately from Item 2 by the bi-gyration inversion law (156), p. 73.

Formalizing the results in (214) and (216) we obtain the following interesting theorem.

**Theorem 34.** For all  $P_1, P_2 \in \mathbb{R}^{n \times m}$

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \oplus' P_2, \ominus' P_1] \\ \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_1 \oplus' P_2, \ominus' P_1] \end{aligned} \quad (218)$$

where  $\oplus'$  is a binary operation in  $\mathbb{R}^{n \times m}$  given by

$$P_1 \oplus' P_2 = (P_1 \oplus P_2) \text{rgyr}[P_2, P_1]. \quad (219)$$

It follows from (219) that

$$\ominus' P = \ominus P = -P \quad (220)$$

for all  $P \in \mathbb{R}^{n \times m}$ .

## 14. Bi-Gyrogroups

Theorem 34 indicates that it will prove useful to replace the binary operation  $\oplus$  in  $\mathbb{R}^{n \times m}$  by the *bi-gyrogroup operation*  $\oplus'$  in  $\mathbb{R}^{n \times m}$  in Definition 35 below.

**Definition 35. (Bi-gyrogroup Operation, Bi-gyrogroups).** Let  $(\mathbb{R}^{n \times m}, \oplus)$  be a *bi-gyrogroupoid* (Definition 12, p. 58). The *bi-gyrogroup binary operation*  $\oplus'$  in  $\mathbb{R}^{n \times m}$  is given by

$$P_1 \oplus' P_2 = (P_1 \oplus P_2) \text{rgyr}[P_2, P_1] \quad (221)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ . The resulting groupoid  $(\mathbb{R}^{n \times m}, \oplus')$  is called a *bi-gyrogroup*.

Following (221) we have, by right gyration inversion, (162b), p. 74

$$P_1 \oplus P_2 = (P_1 \oplus' P_2) \text{rgyr}[P_1, P_2] \quad (222)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

We will find in the sequel that the bi-gyrogroups  $(\mathbb{R}^{n \times m}, \oplus')$ , rather than the bi-gyrogroupoids  $(\mathbb{R}^{n \times m}, \oplus)$ , form the desired elegant algebraic structure that the parametrization of the Lorentz group  $\text{SO}(m, n)$  encodes. The point is that we must study bi-gyrogroupoids in order to pave the way to the study of bi-gyrogroups.



Accordingly, we note that the bi-gyrogroup operation  $\oplus'$  is determined in (221) in terms of the bi-gyrogroupoid operation  $\oplus$  and a right gyration. However, it can be determined equivalently by  $\oplus$  and a left gyration as well. Indeed, it follows from (221) and the bi-gyrocommutative law (161), p. 74, in  $(\mathbb{R}^{n \times m}, \oplus)$  that

$$P_1 \oplus' P_2 = \text{lgYr}[P_1, P_2](P_2 \oplus P_1) \quad (223)$$

and hence

$$P_1 \oplus P_2 = \text{lgYr}[P_1, P_2](P_2 \oplus' P_1) \quad (224)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Following Definition 35 of the bi-gyrogroup binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$ , it proves useful to express the bi-gyrations of  $\mathbb{R}^{n \times m}$  in terms of  $\oplus'$  rather than  $\oplus$ , in the following theorem.

**Theorem 36. (Bi-gyrogroup Bi-gyrations).** *The left and right bi-gyration in the bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  are given by the equations*

$$\begin{aligned} \text{lgYr}[P_1, P_2] &= \sqrt{I_n + (P_1 \oplus' P_2)(P_1 \oplus' P_2)^t}^{-1} \\ &\quad \times \left\{ P_1 P_2^t + \sqrt{I_n + P_1 P_1^t} \sqrt{I_n + P_2 P_2^t} \right\} \\ \text{rgYr}[P_1, P_2] &= \left\{ P_1^t P_2 + \sqrt{I_m + P_1^t P_1} \sqrt{I_m + P_2^t P_2} \right\} \\ &\quad \times \sqrt{I_m + (P_2 \oplus' P_1)^t (P_2 \oplus' P_1)}^{-1} \end{aligned} \quad (225)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** Noting that  $\text{rgYr}[P_1, P_2] \in \text{SO}(m)$ , the first equation in (225) follows from (222) and the second equation in (107), p. 61. Similarly, noting that  $\text{lgYr}[P_1, P_2] \in \text{SO}(n)$ , the second equation in (225) follows from (224) and the third equation in (107). ■

Note that the first equation in (225) and the second equation in (107), p. 61, are identically the same equations with a single exception: the binary operation  $\oplus$  in (107) is replaced by the binary operation  $\oplus'$  in (225). Note also that the order of gyrosummation in the second equation in (225) is  $P_2 \oplus' P_1$  rather than  $P_1 \oplus' P_2$ .

Clearly, the identity element of the groupoid  $(\mathbb{R}^{n \times m}, \oplus')$  is  $0_{n,m}$ , and the inverse  $\ominus' P$  of  $P \in (\mathbb{R}^{n \times m}, \oplus')$  is  $\ominus' P = \ominus P = -P$ , as stated in (220), noting that  $\text{rgYr}[\ominus P, P] = I_m$  is trivial according to Corollary 14, p. 61.

Following a study of bi-gyrogroups in the sequel we will present an axiomatic approach to bi-gyrogroups, which forms a natural extension of the axiomatic approach to groups and to gyrogroups.

**Theorem 37. (Bi-gyrogroup Left and Right Automorphisms).**

$$\begin{aligned}
O_n(P_1 \oplus' P_2) &= O_n P_1 \oplus' O_n P_2 \\
(P_1 \oplus' P_2) O_m &= P_1 O_m \oplus' P_2 O_m \\
O_n(P_1 \oplus' P_2) O_m &= O_n P_1 O_m \oplus' O_n P_2 O_m
\end{aligned} \tag{226}$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ ,  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ .

**Proof:** The first identity in (226) is proved in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
O_n(P_1 \oplus' P_2) &\stackrel{(1)}{\cong} O_n(P_1 \oplus P_2) \text{rgyr}[P_2, P_1] \\
&\stackrel{(2)}{\cong} (O_n P_1 \oplus O_n P_2) \text{rgyr}[P_2, P_1] \\
&\stackrel{(3)}{\cong} (O_n P_1 \oplus O_n P_2) \text{rgyr}[O_n P_2, O_n P_1] \\
&\stackrel{(4)}{\cong} O_n P_1 \oplus' O_n P_2.
\end{aligned} \tag{227}$$

Derivation of the numbered equalities in (227) follows

1. Follows from Definition 35.
2. Follows from the first identity in (121), p. 63.
3. Follows from (145), p. 69.
4. Follows from Definition 35.

The second identity in (226) is proved in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
(P_1 \oplus' P_2) O_m &\stackrel{(1)}{\cong} (P_1 \oplus P_2) \text{rgyr}[P_2, P_1] O_m \\
&\stackrel{(2)}{\cong} (P_1 \oplus P_2) O_m \text{rgyr}[P_2 O_m, P_1 O_m] \\
&\stackrel{(3)}{\cong} (P_1 O_m \oplus P_2 O_m) \text{rgyr}[P_2 O_m, P_1 O_m] \\
&\stackrel{(4)}{\cong} P_1 O_m \oplus' P_2 O_m.
\end{aligned} \tag{228}$$

Derivation of the numbered equalities in (228) follows

1. Follows from Definition 35.
2. Follows from in (138), p. 67.
3. Follows from the second identity in (121), p. 63.
4. Follows from Definition 35.

Finally, the third identity in (226) follows immediately from the first two identities in (226). ■

The maps  $O_n : P \mapsto O_n P$ ,  $O_m : P \mapsto P O_m$ , and  $(O_n, O_m) : P \mapsto O_n P O_m$  of  $\mathbb{R}^{n \times m}$  onto itself are bijective. Hence, by Theorem 37

1. the map  $O_n : P \mapsto O_n P$  is a left automorphism of the bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$
2. the map  $O_m : P \mapsto P O_m$  is a right automorphism of the bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  and
3. the map  $(O_n, O_m) : P \mapsto O_n P O_m$  is a bi-automorphism of the bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  (A bi-automorphism being an automorphism consisting of a left and a right automorphism).

**Theorem 38. (Left cancellation law in the groupoid  $(\mathbb{R}^{n \times m}, \oplus')$ ).** *The bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  possesses the left cancellation law*

$$\ominus' P_1 \oplus' (P_1 \oplus' P_2) = P_2 \quad (229)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** The proof is provided by the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 \ominus' P_1 \oplus' (P_1 \oplus' P_2) &\stackrel{(1)}{\cong} \ominus P_1 \oplus' (P_1 \oplus P_2) \text{rgyr}[P_2, P_1] \\
 &\stackrel{(2)}{\cong} (\ominus P_1 \oplus (P_1 \oplus P_2) \text{rgyr}[P_2, P_1]) \text{rgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], \ominus P_1] \\
 &\stackrel{(3)}{\cong} (\ominus P_1 \oplus (P_1 \oplus P_2) \text{rgyr}[P_2, P_1]) \text{rgyr}[P_1, P_2] \quad (230) \\
 &\stackrel{(4)}{\cong} \ominus P_1 \text{rgyr}[P_1, P_2] \oplus (P_1 \oplus P_2) \\
 &\stackrel{(5)}{\cong} P_2.
 \end{aligned}$$

Derivation of the numbered equalities in (230) follows

1. Follows from (220) and from Definition 35 of  $\oplus'$  applied to  $P_1 \oplus' P_2$ .
2. Follows from Definition 35 of  $\oplus'$ .
3. Follows from (216), p. 85.
4. Follows from the second identity in (121) of Theorem 15, p. 63, applied with  $O_m = \text{rgyr}[P_2, P_1]$ , and from the bi-gyration inversion law (156), p. 73.
5. Follows from the left cancellation law (170), p. 76, in  $(\mathbb{R}^{n \times m}, \oplus)$ .

■

**Lemma 39.** *Let  $O_n \in \text{SO}(n)$  and  $O_m \in \text{SO}(m)$ ,  $n, m \in \mathbb{N}$ . Then*

$$O_n P O_m = P \quad (231)$$

for all  $P \in \mathbb{R}^{n \times m}$  if and only if  $O_n = I_n$  and  $O_m = I_m$ .

**Proof:** If  $O_n = I_n$  and  $O_m = I_m$ , then obviously (231) is true for all  $P \in \mathbb{R}^{n \times m}$ . Conversely, assuming  $O_n P O_m = P$ , or equivalently

$$O_n^t P = P O_m \quad (232)$$

$O_n \in \text{SO}(n)$ ,  $O_m \in \text{SO}(m)$ , for all  $P \in \mathbb{R}^{n \times m}$ , we will show that  $O_n = I_n$  and  $O_m = I_m$ .

Let

$$O_n^t = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad O_m = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{m1} & \cdots & b_{mm} \end{pmatrix}. \quad (233)$$

Furthermore, let  $P_{ij} \in \mathbb{R}^{n \times m}$  be the matrix

$$P_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad (234)$$

with one at the  $ij$ -entry and zeros elsewhere,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ .

Then the matrix product  $O_n^t P_{ij}$

$$O_n^t P_{ij} = \begin{pmatrix} 0 & \dots & a_{1i} & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & a_{ii} & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & a_{ni} & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad (235)$$

is a matrix with  $j$ -th column  $(a_{1i}, \dots, a_{ii}, \dots, a_{ni})^t$  and zeros elsewhere. Shown explicitly in (235) are the first column, the  $j$ -th column and the  $m$ -th column of the matrix  $O_n^t P_{ij}$ , along with its first row,  $i$ -th row and  $n$ -th row.

Similarly, the matrix product  $P_{ij} O_m$

$$P_{ij} O_m = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & \\ b_{1j} & \dots & b_{jj} & \dots & b_{jm} \\ \vdots & & & & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad (236)$$

is a matrix with  $i$ -th row  $(b_{1j}, \dots, b_{jj}, \dots, b_{jm})$  and zeros elsewhere. Shown explicitly in (236) are the first column, the  $j$ -th column and the  $m$ -th column of the matrix  $P_{ij} O_m$ , along with its first row,  $i$ -th row and  $n$ -th row.

It follows from (232) that (235) and (236) are equal. Hence, by comparing entries of the matrices in (235)–(236) we have

$$a_{ii} = b_{jj} \quad (237)$$

and

$$a_{ii_1} = 0, \quad b_{jj_1} = 0 \quad (238)$$

for all  $i, i_1 = 1, \dots, n$ , and all  $j, j_1 = 1, \dots, m, i_1 \neq i$  and  $j_1 \neq j$ .

By (237)–(238) and (233) we have

$$O_n^t = \lambda I_n, \quad O_m = \lambda I_m. \quad (239)$$

Moreover,  $\lambda = 1$  since, by assumption,  $O_n \in SO(n)$  and  $O_m \in SO(m)$ . Hence,  $O_n = I_n$  and  $O_m = I_m$ , as desired. ■

The following Lemma 40 is an immediate consequence of Lemma 39.

**Lemma 40.** *Let  $O_{n,k} \in \text{SO}(n)$  and  $O_{m,k} \in \text{SO}(m)$ ,  $n, m \in \mathbb{N}$ ,  $k = 1, 2$ . Then*

$$O_{n,1}PO_{m,1} = O_{n,2}PO_{m,2} \quad (240)$$

for all  $P \in \mathbb{R}^{n \times m}$  if and only if  $O_{n,1} = O_{n,2}$  and  $O_{m,1} = O_{m,2}$ .

## 15. Bi-Gyration Decomposition and Polar Decomposition

In this section we present manipulations that lead to the bi-gyroassociative and bi-gyrocommutative laws of the binary operation  $\oplus'$  in Theorems 41 and 42 below.

The product of two bi-boosts,  $B(P_1)$  and  $B(P_2)$ ,  $P_1, P_2 \in \mathbb{R}^{n \times m}$ , is a Lorentz transformation  $\Lambda = B(P_1)B(P_2) \in \text{SO}(m, n)$  that need not be a bi-boost. As such, it possesses the bi-gyration decomposition (73), p. 52, as well as the polar decomposition (83), p. 54, along with the bi-gyration in (103), p. 59.

The bi-gyration decomposition of the bi-boost product gives rise to the binary operation  $\oplus$  in  $\mathbb{R}^{n \times m}$  as follows. By (102), p. 59, the bi-boost product  $B(P_1)B(P_2)$  possesses the unique bi-gyration decomposition (104)

$$B(P_1)B(P_2) = \rho(\text{rgyr}[P_1, P_2])B(P_{12})\lambda(\text{lgyr}[P_1, P_2]) \quad (241)$$

where, by Definition 12, p. 58

$$P_{12} =: P_1 \oplus P_2. \quad (242)$$

Similarly, the polar decomposition of the bi-boost product gives rise to the binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  as follows. By (83), p. 54, and (103), the bi-boost product  $B(P_1)B(P_2)$  possesses the unique polar decomposition

$$B(P_1)B(P_2) = B(P''_{12})\rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2]) \quad (243)$$

where, by definition

$$P''_{12} =: P_1 \oplus'' P_2. \quad (244)$$

In order to see the relationship between the binary operations  $\oplus$  and  $\oplus'$  in  $\mathbb{R}^{n \times m}$  we employ the second identity in (77), p. 53, with  $O_m = \text{rgyr}[P_1, P_2]$  to manipulate the polar decomposition (243) into the equivalent bi-gyration decomposition

$$\begin{aligned} B(P_1)B(P_2) &= B(P''_{12})\rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2]) \\ &= \rho(\text{rgyr}[P_1, P_2])B(P''_{12}\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2]). \end{aligned} \quad (245)$$

Comparing (245) with (241), noting that the bi-gyration decomposition is unique, we find that  $P''_{12}\text{rgyr}[P_1, P_2] = P_{12}$ , or equivalently, by means of (242) and (244)

$$P_1 \oplus'' P_2 = (P_1 \oplus P_2)\text{rgyr}[P_2, P_1], \quad P_1 \oplus P_2 = (P_1 \oplus'' P_2)\text{rgyr}[P_1, P_2] \quad (246)$$

in agreement with the definition of  $\oplus'$  in Definition 35. Hence

$$\oplus'' = \oplus'. \quad (247)$$

It follows from (247) that the bi-gyrogroup operation  $\oplus' = \oplus''$  in Definition 35 stems from the polar decomposition (243), just as the bi-gyrogroupoid operation  $\oplus$  stems from the bi-gyration decomposition (241).

It is convenient here to temporarily use the short notation

$$L_{P_1, P_2} := \text{lgYr}[P_1, P_2], \quad R_{P_1, P_2} := \text{rgYr}[P_1, P_2] \quad (248)$$

in intermediate results, turning back to the full notation in final results, noting that  $L_{P_1, P_2}^{-1} = L_{P_2, P_1}$  and  $R_{P_1, P_2}^{-1} = R_{P_2, P_1}$ .

Identities (243) and (246) imply

$$\rho(R_{P_1, P_2})\lambda(L_{P_1, P_2}) = B(-(P_1 \oplus P_2)R_{P_2, P_1})B(P_1)B(P_2). \quad (249)$$

Identities (241) and (246) imply, by right gyration inversion, the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned} B(P_1 \oplus P_2)\lambda(L_{P_1, P_2}) &\stackrel{(1)}{\cong} \rho(R_{P_2, P_1})B(P_1)B(P_2) \\ &\stackrel{(2)}{\cong} B(P_1 R_{P_1, P_2})\rho(R_{P_2, P_1})B(P_2) \\ &\stackrel{(3)}{\cong} B(P_1 R_{P_1, P_2})B(P_2 R_{P_1, P_2})\rho(R_{P_2, P_1}). \end{aligned} \quad (250)$$

Derivation of the numbered equalities in (250) follows

1. This identity is obtained from (241) and (242) by using the right gyration inversion law in (156) according to which  $\rho(\text{rgYr}[P_1, P_2])^{-1} = \rho(R_{P_2, P_1})$ .
2. Follows from Item (1) by an application to  $B(P_1)$  of the second identity in (77), p. 53, with  $O_m = R_{P_2, P_1}$ , noting the right gyration inversion law,  $R_{P_1, P_2} R_{P_2, P_1} = I_m$ .
3. Like Item (2), Item (3) follows from an application to  $B(P_2)$  of the second identity in (77), p. 53, with  $O_m = R_{P_2, P_1}$ , noting the right gyration inversion law,  $R_{P_1, P_2} R_{P_2, P_1} = I_m$ .

By means of (250) and right gyration inversion we have

$$B(P_1 \oplus P_2) = B(P_1 R_{P_1, P_2})B(P_2 R_{P_1, P_2})\rho(R_{P_2, P_1})\lambda(L_{P_2, P_1}) \quad (251)$$

so that, by bi-boost inversion

$$\rho(R_{P_2, P_1})\lambda(L_{P_2, P_1}) = B(\ominus P_2 R_{P_1, P_2})B(\ominus P_1 R_{P_1, P_2})B(P_1 \oplus P_2). \quad (252)$$

Inverting both sides of (252) and noting that the matrices  $\lambda(L_{P_1, P_2})$  and  $\rho(R_{P_1, P_2})$  commute, we obtain the identity

$$\rho(R_{P_1, P_2})\lambda(L_{P_1, P_2}) = B(\ominus(P_1 \oplus P_2))B(P_1 R_{P_1, P_2})B(P_2 R_{P_1, P_2}). \quad (253)$$

Comparing (249) and (253), we obtain the identity

$$\begin{aligned} B(\ominus(P_1 \oplus P_2)R_{P_2, P_1})B(P_1)B(P_2) &= B(\ominus(P_1 \oplus P_2))B(P_1 R_{P_1, P_2})B(P_2 R_{P_1, P_2}) \\ &= \rho(R_{P_1, P_2})\lambda(L_{P_1, P_2}) \end{aligned} \quad (254)$$

which, in full notation, takes the form

$$\begin{aligned} &B(\ominus(P_1 \oplus P_2)\text{rgyr}[P_2, P_1])B(P_1)B(P_2) \\ &= B(\ominus(P_1 \oplus P_2))B(P_1\text{rgyr}[P_1, P_2])B(P_2\text{rgyr}[P_1, P_2]) \\ &= \rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2]). \end{aligned} \quad (255)$$

By Definition 35, the extreme sides of (255) yield the identity

$$\rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2]) = B(\ominus(P_1 \oplus' P_2))B(P_1)B(P_2) \quad (256)$$

so that for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$

$$\rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2])B(X) = B(\ominus(P_1 \oplus' P_2))B(P_1)B(P_2)B(X). \quad (257)$$

Let  $J_1$  ( $J_2$ ) denote the left (right) side of (257). Using the column notation in (79), p. 53, we manipulate the left side,  $J_1$ , of (257) as follows, where we apply the Lorentz transformation product law (148), p. 71, and note Corollary 14 on trivial bi-gyrations. Therefore

$$\begin{aligned} J_1 &= \rho(\text{rgyr}[P_1, P_2])\lambda(\text{lgyr}[P_1, P_2])B(X) \\ &= \begin{pmatrix} 0_{n,m} \\ \text{lgyr}[P_1, P_2] \\ I_m \end{pmatrix} \begin{pmatrix} 0_{n,m} \\ I_n \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \begin{pmatrix} X \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} \begin{pmatrix} X \\ I_n \\ I_m \end{pmatrix} \quad (258) \\ &= \begin{pmatrix} \text{lgyr}[P_1, P_2]X \\ \text{lgyr}[0_{n,m}, \text{lgyr}[P_1, P_2]X]\text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2]\text{rgyr}[0_{n,m}, \text{lgyr}[P_1, P_2]X] \end{pmatrix} = \begin{pmatrix} \text{lgyr}[P_1, P_2]X \\ \text{lgyr}[P_1, P_2] \\ \text{rgyr}[P_1, P_2] \end{pmatrix} =: \begin{pmatrix} A_1 \\ B_1 \\ C_1 \end{pmatrix}. \end{aligned}$$



Similarly, applying the Lorentz transformation product law (148) we manipulate the right side,  $J_2$ , of (257) as follows

$$\begin{aligned}
 J_2 &= B(\ominus(P_1 \oplus' P_2))B(P_1)B(P_2)B(X) \\
 &= \begin{pmatrix} \ominus(P_1 \oplus' P_2) \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} X \\ I_n \\ I_m \end{pmatrix} \\
 &= \begin{pmatrix} \ominus(P_1 \oplus' P_2) \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \oplus X \\ \text{lgyr}[P_2, X] \\ \text{rgyr}[P_2, X] \end{pmatrix} \tag{259} \\
 &= \begin{pmatrix} \ominus(P_1 \oplus' P_2) \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_1 \text{rgyr}[P_2, X] \oplus (P_2 \oplus X) \\ \text{lgyr}[P_1 \text{rgyr}[P_2, X], P_2 \oplus X] \text{lgyr}[P_2, X] \\ \text{rgyr}[P_2, X] \text{rgyr}[P_1 \text{rgyr}[P_2, X], P_2 \oplus X] \end{pmatrix}.
 \end{aligned}$$

In the following equations (260) we adjust each entry of the right column of the extreme right side of (259) to our needs.

By the second equation in (121), p. 63, with  $O_m = \text{rgyr}[P_2, X]$ , and the right gyration inversion law (162b), and by (221)–(222), we have

$$\begin{aligned}
 P_1 \text{rgyr}[P_2, X] \oplus (P_2 \oplus X) &= \{P_1 \oplus (P_2 \oplus X) \text{rgyr}[X, P_2]\} \text{rgyr}[P_2, X] \\
 &= \{P_1 \oplus (P_2 \oplus' X)\} \text{rgyr}[P_2, X] \tag{260a} \\
 &= \{P_1 \oplus' (P_2 \oplus' X)\} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X].
 \end{aligned}$$

By (144) with  $O_m = \text{rgyr}[X, P_2]$ , and the right gyration inversion law (162b), and by (221), we have

$$\begin{aligned}
 \text{lgyr}[P_1 \text{rgyr}[P_2, X], P_2 \oplus X] &= \text{lgyr}[P_1, (P_2 \oplus X) \text{rgyr}[X, P_2]] \\
 &= \text{lgyr}[P_1, P_2 \oplus' X]. \tag{260b}
 \end{aligned}$$

By (138) with  $O_m = \text{rgyr}[P_2, X]$ , and the right gyration inversion law (162b), and by (221), we have

$$\begin{aligned}
 &\text{rgyr}[P_2, X] \text{rgyr}[P_1 \text{rgyr}[P_2, X], P_2 \oplus X] \\
 &= \text{rgyr}[P_1, (P_2 \oplus X) \text{rgyr}[X, P_2]] \text{rgyr}[P_2, X] \tag{260c} \\
 &= \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X].
 \end{aligned}$$

By means of the equations in (260), the extreme right side of (259) can be written as

$$J_2 = \begin{pmatrix} \ominus(P_1 \oplus P_2) \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} \{P_1 \oplus (P_2 \oplus X)\} \text{rgyr}[P_1, P_2 \oplus X] \text{rgyr}[P_2, X] \\ \text{lgyr}[P_1, P_2 \oplus X] \text{lgyr}[P_2, X] \\ \text{rgyr}[P_1, P_2 \oplus X] \text{rgyr}[P_2, X] \end{pmatrix} =: \begin{pmatrix} A_2 \\ B_2 \\ C_2 \end{pmatrix}. \quad (261)$$

We now face the task of calculating  $A_2$ ,  $B_2$  and  $C_2$  by means of the Lorentz product law (148). Applying the Lorentz product law to (261), we calculate the second entry,  $B_2$ , of  $J_2$  and simplify it in the following chain of equations, which are numbered for subsequent explanation, and where we use the notation

$$O_m = \text{rgyr}[P_1, P_2 \oplus X] \text{rgyr}[P_2, X] \quad (262)$$

$$\begin{aligned} B_2 &\stackrel{(1)}{\cong} \text{lgyr}[\ominus(P_1 \oplus P_2)O_m, \{P_1 \oplus (P_2 \oplus X)\}O_m] \text{lgyr}[P_1, P_2 \oplus X] \text{lgyr}[P_2, X] \\ &\stackrel{(2)}{\cong} \text{lgyr}[\ominus(P_1 \oplus P_2), P_1 \oplus (P_2 \oplus X)] \text{lgyr}[P_1, P_2 \oplus X] \text{lgyr}[P_2, X]. \end{aligned} \quad (263)$$

Derivation of the numbered equalities in (263) follows

1. This equation is obtained by calculating the Lorentz transformation product in (261) by means of (148), selecting the resulting second entry, and using the notation in (262).
2. Follows from Item (1) by omitting the matrix  $O_m$  from the two entries of  $\text{lgyr}$  according to (144), p. 69.

By (257),  $J_1 = J_2$  and hence, by (258) and (261),  $B_2 = B_1$ , that is, by (263) and (258)

$$\text{lgyr}[\ominus(P_1 \oplus P_2), P_1 \oplus (P_2 \oplus X)] \text{lgyr}[P_1, P_2 \oplus X] \text{lgyr}[P_2, X] = \text{lgyr}[P_1, P_2] \quad (264)$$

for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$ .

Similarly, we calculate the third entry,  $C_2$ , of  $J_2$  and simplify it in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 C_2 &\stackrel{(1)}{\cong} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] \\
 &\quad \text{rgyr}[\ominus(P_1 \oplus' P_2) \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X], \\
 &\quad \{P_1 \oplus'(P_1 \oplus' X)\} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X]] \\
 &\stackrel{(2)}{\cong} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[\ominus(P_1 \oplus' P_2) \text{rgyr}[P_1, P_2 \oplus' X], \\
 &\quad \{P_1 \oplus'(P_2 \oplus' X)\} \text{rgyr}[P_1, P_2 \oplus' X]] \text{rgyr}[P_2, X] \\
 &\stackrel{(3)}{\cong} \text{rgyr}[\ominus(P_1 \oplus' P_2), P_1 \oplus'(P_2 \oplus' X)] \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X].
 \end{aligned} \tag{265}$$

Derivation of the numbered equalities in (265) follows

1. This equation is obtained by calculating the Lorentz transformation product in (261) by means of (148), and selecting the resulting third entry.
2. Follows from Item (1) by applying Identity (138), p. 67, with  $O_m$  given by  $O_m = \text{rgyr}[P_2, X]$ .
3. Follows from Item (2) by applying Identity (138), p. 67, with  $O_m$  given by  $O_m = \text{rgyr}[P_1, P_2 \oplus' X]$ .

By (257),  $J_1 = J_2$  and hence, by (258) and (261),  $C_2 = C_1$ , that is, by (265) and (258)

$$\text{rgyr}[\ominus(P_1 \oplus' P_2), P_1 \oplus'(P_2 \oplus' X)] \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] = \text{rgyr}[P_1, P_2] \tag{266}$$

for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$ .

We are now in a position to calculate the first entry,  $A_2$ , of  $J_2$  and simplify it in the following chain of equations, which are numbered for subsequent explanation

$$\begin{aligned}
 A_2 &\stackrel{(1)}{\cong} \ominus(P_1 \oplus' P_2) \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] \\
 &\quad \oplus \{P_1 \oplus'(P_2 \oplus' X)\} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] \\
 &\stackrel{(2)}{\cong} \{\ominus(P_1 \oplus' P_2) \oplus \{P_1 \oplus'(P_2 \oplus' X)\}\} \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] \\
 &\stackrel{(3)}{\cong} \{\ominus(P_1 \oplus' P_2) \oplus \{P_1 \oplus'(P_2 \oplus' X)\}\} \text{rgyr}[\ominus(P_1 \oplus' P_2), P_1 \oplus'(P_2 \oplus' X)] \\
 &\quad \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X] \\
 &\stackrel{(4)}{\cong} \{\ominus(P_1 \oplus' P_2) \oplus \{P_1 \oplus'(P_2 \oplus' X)\}\} \text{rgyr}[P_1, P_2].
 \end{aligned} \tag{267}$$

Derivation of the numbered equalities in (267) follows

1. This equation is obtained by calculating the Lorentz transformation product in (261) by means of (148), and selecting the resulting first entry.
2. Item (2) is obtained by using the second Identity in (121) with
 
$$O_m = \text{rgyr}[P_1, P_2 \oplus' X] \text{rgyr}[P_2, X].$$
3. The binary operation  $\oplus$  that appears in Item (2) is expressed here in terms of the binary operation  $\oplus'$  by means of (222).
4. Item (4) follows from Item (3) by Identity (266).

By (257),  $J_1 = J_2$  and hence, by (258) and (261),  $A_2 = A_1$ , that is, by (267) and (258)

$$\{\ominus(P_1 \oplus' P_2) \oplus' \{P_1 \oplus' (P_2 \oplus' X)\}\} \text{rgyr}[P_1, P_2] = \text{lgyr}[P_1, P_2] X. \quad (268)$$

Hence, by right gyration inversion

$$\ominus'(P_1 \oplus' P_2) \oplus' \{P_1 \oplus' (P_2 \oplus' X)\} = \text{lgyr}[P_1, P_2] X \text{rgyr}[P_2, P_1] \quad (269)$$

for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$ .

Left gyroadding  $(P_1 \oplus' P_2) \oplus'$  to both sides of (269) and applying the left cancellation law (229), we obtain the *left bi-gyroassociative law*

$$\begin{aligned} & (P_1 \oplus' P_2) \oplus' \text{lgyr}[P_1, P_2] X \text{rgyr}[P_2, P_1] \\ &= (P_1 \oplus' P_2) \oplus' \{\ominus'(P_1 \oplus' P_2) \oplus' \{P_1 \oplus' (P_2 \oplus' X)\}\} = P_1 \oplus' (P_2 \oplus' X). \end{aligned} \quad (270)$$

**Theorem 41. (Bi-gyrogroup left and right bi-gyroassociative law).** *The binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  possesses the left bi-gyroassociative law*

$$P_1 \oplus' (P_2 \oplus' X) = (P_1 \oplus' P_2) \oplus' \text{lgyr}[P_1, P_2] X \text{rgyr}[P_2, P_1] \quad (271)$$

*and the right bi-gyroassociative law*

$$(P_1 \oplus' P_2) \oplus' X = P_1 \oplus' (P_2 \oplus' \text{lgyr}[P_2, P_1] X \text{rgyr}[P_1, P_2]) \quad (272)$$

for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$ .

**Proof:** The left bi-gyroassociative law (271) is proved in (270).

The right bi-gyroassociative law (272) results from an application of the left bi-gyroassociative law to the right side of (272), by means of bi-gyration inversion

$$\begin{aligned} & P_1 \oplus' (P_2 \oplus' \text{lgyr}[P_2, P_1] X \text{rgyr}[P_1, P_2]) \\ &= (P_1 \oplus' P_2) \oplus' \text{lgyr}[P_1, P_2] \text{lgyr}[P_2, P_1] X \text{rgyr}[P_1, P_2] \text{rgyr}[P_2, P_1] \\ &= (P_1 \oplus' P_2) \oplus' X. \end{aligned} \quad (273)$$

■

## 16. Bi-Gyrocommutative Law

The bi-gyrocommutative law in  $(\mathbb{R}^{n \times m}, \oplus')$  is obtained in Section 15 by comparing the bi-gyration decomposition and the polar decomposition of the bi-boost product  $\Lambda = B(P_1)B(P_2)$ . In this section we derive the bi-gyrocommutative law in  $(\mathbb{R}^{n \times m}, \oplus')$  from its counterpart (161), p. 74, in  $(\mathbb{R}^{n \times m}, \oplus)$ .

**Theorem 42. (Bi-gyrocommutative law in  $(\mathbb{R}^{n \times m}, \oplus')$ ).** *The binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  possesses the bi-gyrocommutative law*

$$P_1 \oplus' P_2 = \text{lgyr}[P_1, P_2](P_2 \oplus' P_1) \text{rgyr}[P_2, P_1] \quad (274)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

**Proof:** By means of (222), p. 86, and right gyration inversion (162b), p. 74, the bi-gyrocommutative law (161), p. 74, in  $(\mathbb{R}^{n \times m}, \oplus)$  can be expressed in terms of  $\oplus'$  rather than  $\oplus$ , obtaining

$$\begin{aligned} (P_1 \oplus' P_2) \text{rgyr}[P_1, P_2] &= \text{lgyr}[P_1, P_2](P_2 \oplus' P_1) \text{rgyr}[P_2, P_1] \text{rgyr}[P_1, P_2] \\ &= \text{lgyr}[P_1, P_2](P_2 \oplus' P_1). \end{aligned} \quad (275)$$

Identity (274) of the Theorem follows immediately from (275) by right gyration inversion. ■

## 17. Gyrogroup Gyration

The bi-gyroassociative laws (271)–(272) and the bi-gyrocommutative law (274) suggest the following definition of gyrations in terms of left and right gyrations.

**Definition 43. (Gyrogroup gyrations).** *The gyrator*

$$\text{gyr} : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \rightarrow \text{Aut}(\mathbb{R}^{n \times m}, \oplus')$$

generates automorphisms called gyrations,  $\text{gyr}[P_1, P_2] \in \text{Aut}(\mathbb{R}^{n \times m}, \oplus')$ , given by the equation

$$\text{gyr}[P_1, P_2]X = \text{lgyr}[P_1, P_2]X \text{rgyr}[P_2, P_1] \quad (276)$$

for all  $P_1, P_2, X \in \mathbb{R}^{n \times m}$ , where left gyrations,  $\text{lgyr}[P_1, P_2]$ , and right gyrations,  $\text{rgyr}[P_2, P_1]$ , are given in (107), p. 61. The gyration  $\text{gyr}[P_1, P_2]$  is said to be the gyration generated by  $P_1, P_2 \in \mathbb{R}^{n \times m}$ . Being automorphisms of  $(\mathbb{R}^{n \times m}, \oplus')$ , gyrations are also called gyroautomorphisms.

Definition 43 will turn out rewarding, leading to the discovery that any bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  is a gyrocommutative gyrogroup.

**Theorem 44. (Gyrogroup gyroassociative and gyrocommutative laws).** *The binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  obeys the left and the right gyroassociative law*

$$P_1 \oplus' (P_2 \oplus' X) = (P_1 \oplus' P_2) \oplus' \text{gyr}[P_1, P_2]X \quad (277)$$

and

$$(P_1 \oplus' P_2) \oplus' X = P_1 \oplus' (P_2 \oplus' \text{gyr}[P_2, P_1]X) \quad (278)$$

and the gyrocommutative law

$$P_1 \oplus' P_2 = \text{gyr}[P_1, P_2](P_2 \oplus' P_1). \quad (279)$$

**Proof:** Identities (277)–(278) follow immediately from Definition 43 and the left and right bi-gyroassociative law (271)–(272). Similarly, (279) follow immediately from Definition 43 and the bi-gyrocommutative law (274). ■

**Lemma 45.** *The relation (276) between gyrations  $\text{gyr}[P_1, P_2]$  and corresponding bi-gyrations  $(\text{lgyr}[P_1, P_2], \text{rgyr}[P_2, P_1])$ ,  $P_1, P_2 \in (\mathbb{R}^{n \times m}, \oplus')$ , is bijective.*

**Proof:** Let  $P_k \in \mathbb{R}^{n \times m}$ ,  $k = 1, 2, 3, 4$ . Assuming

$$(\text{lgyr}[P_1, P_2], \text{rgyr}[P_2, P_1]) = (\text{lgyr}[P_3, P_4], \text{rgyr}[P_4, P_3]) \quad (280)$$

it clearly follows from (276) that

$$\text{gyr}[P_1, P_2] = \text{gyr}[P_3, P_4]. \quad (281)$$

Conversely, assuming (281), then

$$\text{gyr}[P_1, P_2]X = \text{gyr}[P_3, P_4]X \quad (282)$$

for all  $X \in \mathbb{R}^{n \times m}$ , so that by (276)

$$\text{lgyr}[P_1, P_2]X \text{rgyr}[P_2, P_1] = \text{lgyr}[P_3, P_4]X \text{rgyr}[P_4, P_3] \quad (283)$$

for all  $X \in \mathbb{R}^{n \times m}$ .

Noting that  $\text{lgyr}[P, Q] \in \text{SO}(n)$  and  $\text{rgyr}[P, Q] \in \text{SO}(m)$  for any  $P, Q \in \mathbb{R}^{n \times m}$ , (280) follows from (283) and Lemma 40, p. 92, and the proof is complete. ■

It is anticipated in Definition 43 that gyrations are automorphisms. The following theorem asserts that this is indeed the case.

**Theorem 46. (Gyroautomorphism).** *Gyrations  $\text{gyr}[P_1, P_2]$  of a bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  are automorphisms of the bi-gyrogroup.*

**Proof:** It follows from the bi-gyration inversion law in Theorem 26, p. 74, and from (276) that  $\text{gyr}[P_1, P_2]$  is invertible

$$\text{gyr}^{-1}[P_1, P_2] = \text{gyr}[P_2, P_1] \quad (284)$$

for all  $P_1, P_2 \in \mathbb{R}^{n \times m}$ .

Furthermore, noting that  $\text{lgyr}[P_1, P_2] \in \text{SO}(n)$  and  $\text{rgyr}[P_1, P_2] \in \text{SO}(m)$  it follows from (276) and the third identity in (226), p. 88, that

$$\text{gyr}[P_1, P_2](P \oplus' Q) = \text{gyr}[P_1, P_2]P \oplus' \text{gyr}[P_1, P_2]Q \quad (285)$$

for all  $P_1, P_2, P, Q \in \mathbb{R}^{n \times m}$ . Hence, by (284) and (285), gyrations of  $(\mathbb{R}^{n \times m}, \oplus')$  are automorphisms of  $(\mathbb{R}^{n \times m}, \oplus')$ , and the proof is complete. ■

**Theorem 47. (Left gyration reduction properties).** *Left gyrations of a bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  possess the left gyration left reduction property*

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[P_1 \oplus' P_2, P_2] \quad (286)$$

and the left gyration right reduction property

$$\text{lgyr}[P_1, P_2] = \text{lgyr}[P_1, P_2 \oplus' P_1]. \quad (287)$$

**Proof:** By (190), p. 80, (144), p. 69, with  $O_m = \text{rgyr}[P_2, P_1]$ , gyration inversion, and (221), p. 86, we have the following chain of equations

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \oplus P_2, P_2 \text{rgyr}[P_1, P_2]] \\ &= \text{lgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], P_2 \text{rgyr}[P_1, P_2] \text{rgyr}[P_2, P_1]] \\ &= \text{lgyr}[(P_1 \oplus P_2) \text{rgyr}[P_2, P_1], P_2] \\ &= \text{lgyr}[P_1 \oplus' P_2, P_2] \end{aligned} \quad (288)$$

thus proving (286).

By (191), p. 80, (145), p. 69, with  $O_m = \text{rgyr}[P_1, P_2]$ , gyration inversion, and (221), p. 86, we have the following chain of equations

$$\begin{aligned} \text{lgyr}[P_1, P_2] &= \text{lgyr}[P_1 \text{rgyr}[P_2, P_1], P_2 \oplus P_1] \\ &= \text{lgyr}[P_1 \text{rgyr}[P_2, P_1] \text{rgyr}[P_1, P_2], (P_2 \oplus P_1) \text{rgyr}[P_1, P_2]] \\ &= \text{lgyr}[P_1, (P_2 \oplus P_1) \text{rgyr}[P_1, P_2]] \\ &= \text{lgyr}[P_1, P_2 \oplus' P_1] \end{aligned} \quad (289)$$

thus proving (287). ■

**Theorem 48. (Right gyration reduction properties).** *Right gyrations of a bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  possess the right gyration left reduction property*

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[P_1 \oplus' P_2, P_2] \quad (290)$$

and the right gyration right reduction property

$$\text{rgyr}[P_1, P_2] = \text{rgyr}[P_1, P_2 \oplus' P_1]. \quad (291)$$

**Proof:** By (195), p. 81, (145), p. 69, with  $O_n = \text{lgyr}[P_1, P_2]$ , gyration inversion, and (223), p. 87, we have the following chain of equations

$$\begin{aligned} \text{rgyr}[P_1, P_2] &= \text{rgyr}[P_2 \oplus P_1, \text{lgyr}[P_2, P_1]P_2] \\ &= \text{rgyr}[\text{lgyr}[P_1, P_2](P_2 \oplus P_1), \text{lgyr}[P_1, P_2]\text{lgyr}[P_2, P_1]P_2] \\ &= \text{rgyr}[\text{lgyr}[P_1, P_2](P_2 \oplus P_1), P_2] \\ &= \text{rgyr}[P_1 \oplus' P_2, P_2] \end{aligned} \quad (292)$$

thus proving (290).

By (194), p. 81, (145), p. 69, with  $O_n = \text{lgyr}[P_2, P_1]$ , gyration inversion, and (223), p. 87, we have the following chain of equations

$$\begin{aligned} \text{rgyr}[P_1, P_2] &= \text{rgyr}[\text{lgyr}[P_1, P_2]P_1, P_1 \oplus P_2] \\ &= \text{rgyr}[\text{lgyr}[P_2, P_1]\text{lgyr}[P_1, P_2]P_1, \text{lgyr}[P_2, P_1](P_1 \oplus P_2)] \\ &= \text{rgyr}[P_1, \text{lgyr}[P_2, P_1](P_1 \oplus P_2)] \\ &= \text{rgyr}[P_1, P_2 \oplus' P_1] \end{aligned} \quad (293)$$

thus proving (291). ■

**Theorem 49. (Gyration reduction properties).** *The gyrations of any bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$ ,  $m, n \in \mathbb{N}$ , possess the left and right reduction property*

$$\text{gyr}[P_1, P_2] = \text{gyr}[P_1 \oplus' P_2, P_2] \quad (294)$$

and

$$\text{gyr}[P_1, P_2] = \text{gyr}[P_1, P_2 \oplus' P_1]. \quad (295)$$

**Proof:** Identities (294) and (295) follow from Definition 43 of  $\text{gyr}$  in terms of  $\text{lgyr}$  and  $\text{rgyr}$ , and from Theorems 47 and 48. ■



## 18. Gyrogroups

We are now in a position to present the definition of the abstract gyrocommutative gyrogroup, and prove that any bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  is a gyrocommutative gyrogroup.

Forming a natural generalization of groups, gyrogroups emerged in the 1988 study of the parametrization of the Lorentz group of Einstein's special relativity theory [18, 19]. Einstein velocity addition, thus, provides a concrete example of a gyrocommutative gyrogroup operation in the ball of all relativistically admissible velocities.

**Definition 50. (Gyrogroups).** *A groupoid  $(G, \oplus)$  is a gyrogroup if its binary operation satisfies the following axioms G1)–G5). In  $G$  there is at least one element,  $0$ , called a left identity, satisfying*

$$\text{G1)} \quad 0 \oplus a = a$$

*for all  $a \in G$ . There is an element  $0 \in G$  satisfying axiom G1) such that for each  $a \in G$  there is an element  $\ominus a \in G$ , called a left inverse of  $a$ , satisfying*

$$\text{G2)} \quad \ominus a \oplus a = 0.$$

*Moreover, for any  $a, b, c \in G$  there exists a unique element  $\text{gyr}[a, b]c \in G$  such that the binary operation obeys the left gyroassociative law*

$$\text{G3)} \quad a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c.$$

*The map  $\text{gyr}[a, b] : G \rightarrow G$  given by  $c \mapsto \text{gyr}[a, b]c$  is an automorphism of the groupoid  $(G, \oplus)$ , that is*

$$\text{G4)} \quad \text{gyr}[a, b] \in \text{Aut}(G, \oplus)$$

*and the automorphism  $\text{gyr}[a, b]$  of  $G$  is called the gyroautomorphism, or the gyration, of  $G$  generated by  $a, b \in G$ . The operator  $\text{gyr} : G \times G \rightarrow \text{Aut}(G, \oplus)$  is called the gyrator of  $G$ . Finally, the gyroautomorphism  $\text{gyr}[a, b]$  generated by any  $a, b \in G$  possesses the left reduction property*

$$\text{G5)} \quad \text{gyr}[a, b] = \text{gyr}[a \oplus b, b]$$

*called the reduction axiom.*

The gyrogroup axioms G1)–G5) in Definition 50 are classified into three classes:

1. The first pair of axioms, G1) and G2), is a reminiscent of the group axioms.
2. The last pair of axioms, G4) and G5), presents the gyrator axioms.
3. The middle axiom, G3), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation  $a \ominus b = a \oplus (\ominus b)$  in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

**Definition 51. (Gyrocommutative gyrogroups).** *A gyrogroup  $(G, \oplus)$  is gyrocommutative if its binary operation obeys the gyrocommutative law*

$$\text{G6) } \quad a \oplus b = \text{gyr}[a, b](b \oplus a)$$

for all  $a, b \in G$ .

**Theorem 52. (Gyrocommutative gyrogroup).** *Any bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$ ,  $n, m \in \mathbb{N}$ , is a gyrocommutative gyrogroup.*

**Proof:** We will validate each of the six gyrocommutative gyrogroup axioms G1)–G6) in Definitions 50 and 51.

1. The bi-gyrogroup  $(\mathbb{R}^{n \times m}, \oplus')$  possesses the left identity  $0_{n,m}$ , thus validating Axiom G1).
2. Every element  $P \in \mathbb{R}^{n \times m}$  possesses the left inverse  $\ominus' P := -P \in \mathbb{R}^{n \times m}$ , thus validating Axiom G2).
3. The binary operation  $\oplus'$  obeys the left gyroassociative law (277) by Theorem 44, thus validating Axiom G3).
4. The map  $\text{gyr}[P_1, P_2]$  is an automorphism of  $(\mathbb{R}^{n \times m}, \oplus')$  by Theorem 46, that is,  $\text{gyr}[P_1, P_2] \in \text{Aut}(\mathbb{R}^{n \times m}, \oplus')$ , thus validating Axiom G4).
5. The binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  possesses the left reduction property (294) by Theorem 49, thus validating Axiom G5).
6. The binary operation  $\oplus'$  in  $\mathbb{R}^{n \times m}$  possesses the gyrocommutative law (279) by Theorem 44, thus validating Axiom G6).

■

## 19. The Abstract Bi-Gyrogroup

Following the key features of the bi-gyrogroups  $(\mathbb{R}^{n \times m}, \oplus')$ , the abstract (bi-gyrocommutative) bi-gyrogroup is defined to be an abstract (gyrocommutative) gyrogroup the gyrations of which are bi-gyrations. In order to define bi-gyrations in the abstract context, we introduce the concept of bi-automorphisms of a groupoid.

An automorphism of a groupoid  $(S, +)$  is a bijective map  $f$  of  $S$  onto itself that respects the groupoid binary operation, that is,  $f(s_1 + s_2) = f(s_1) + f(s_2)$  for all  $s_1, s_2 \in S$ . An automorphism group,  $\text{Aut}_0(S, +)$ , of  $(S, +)$  is a group of automorphisms of  $(S, +)$  with group operation given by automorphism composition.

Let  $\text{Aut}_L(S, +)$  and  $\text{Aut}_R(S, +)$  be two automorphism groups of  $(S, +)$ , called a left and a right automorphism group of  $(S, +)$ , such that

$$\text{Aut}_L(S, +) \cap \text{Aut}_R(S, +) = I \quad (296)$$

$I$  being the identity automorphism of  $(S, +)$ .

Finally, let

$$\text{Aut}_0(S, +) = \text{Aut}_L(S, +) \times \text{Aut}_R(S, +) \quad (297)$$

be the direct product of  $\text{Aut}_L(S, +)$  and  $\text{Aut}_R(S, +)$ .

1. The application of  $f_L \in \text{Aut}_L(S, +)$  to  $s \in S$  is denoted by  $f_L(s)$  or  $f_L s$ .
2. The application of  $f_R \in \text{Aut}_R(S, +)$  to  $s \in S$  is denoted by  $(s)f_R$  or  $s f_R$ .
3. Accordingly, the application of  $(f_L, f_R) \in \text{Aut}_0(S, +)$  to  $s \in S$  is denoted by

$$(f_L, f_R)s = f_L s f_R \quad (298)$$

where we assume that the composed map in (298) is associative, that is

$$(f_L s) f_R = f_L (s f_R). \quad (299)$$

Furthermore, we assume that the composed map in (298) is unique, that is

$$f_{L,1} s f_{R,1} = f_{L,2} s f_{R,2} \implies f_{L,1} = f_{L,2} \quad \text{and} \quad f_{R,1} = f_{R,2} \quad (300)$$

for any  $f_{L,k} \in \text{Aut}_L(S, +)$ ,  $f_{R,k} \in \text{Aut}_R(S, +)$ ,  $k = 1, 2$ , and  $s \in S$ .

The automorphism group  $\text{Aut}_0(S, +) = \text{Aut}_L(S, +) \times \text{Aut}_R(S, +)$  is said to be a *bi-automorphism group* of the groupoid  $(S, +)$ .

Let us now consider the case when the groupoid  $(S, +)$  is a gyrogroup. A gyroautomorphism group  $\text{Aut}_0(S, +)$  of  $(S, +)$  is any automorphism group of  $(S, +)$  that contains the gyrations of  $(S, +)$ . If  $\text{Aut}_0(S, +)$  is a bi-automorphism group of  $(S, +)$  then its direct product structure (297) induces a direct product structure for its subset of gyrations

$$\text{gyr}[s_1, s_2] = (\text{lgyr}[s_1, s_2], \text{rgyr}[s_1, s_2]) \quad (301)$$

for all  $s_1, s_2 \in (S, +)$ , where

$$\begin{aligned} \text{gyr}[s_1, s_2] &\in \text{Aut}_0(S, +) \\ \text{lgyr}[s_1, s_2] &\in \text{Aut}_L(S, +) \\ \text{rgyr}[s_1, s_2] &\in \text{Aut}_G(S, +). \end{aligned} \quad (302)$$

The gyrations  $\text{gyr}[s_1, s_2]$  in (301) of the gyrogroup  $(S, +)$  are naturally said to be *bi-gyrations*. The application of a bi-gyration  $\text{gyr}[s_1, s_2]$  to  $s$  is denoted by

$$\text{gyr}[s_1, s_2]s = (\text{lgyr}[s_1, s_2], \text{rgyr}[s_1, s_2])s = \text{lgyr}[s_1, s_2]\text{srgyr}[s_1, s_2]. \quad (303)$$

**Definition 53. (Bi-gyrogroups).** A (gyrocommutative) gyrogroup whose gyrations are bi-gyrations is said to be a (bi-gyrocommutative) bi-gyrogroup.

A detailed study of the abstract bi-gyrogroup is presented in [17].

Our study of special (or, unimodular) pseudo-orthogonal groups  $\text{SO}(m, n)$  can be extended straightforwardly to an analogous study of special (or, unimodular) pseudo-unitary groups  $\text{SU}(m, n)$ . Accordingly, bi-gyrocommutative bi-gyrogroup theory for  $(\mathbb{R}^{n \times m}, \oplus')$ , as developed in this article, can be extended straightforwardly to  $(\mathbb{C}^{n \times m}, \oplus')$  where

1. real  $n \times m$  matrices  $P \in \mathbb{R}^{n \times m}$  are replaced by complex  $n \times m$  matrices  $P \in \mathbb{C}^{n \times m}$
2. the transpose  $P^t$  of  $P \in \mathbb{R}^{n \times m}$  is replaced by the conjugate transpose  $P^* = (\bar{P})^t$  of  $P \in \mathbb{C}^{n \times m}$  and
3. the special orthogonal matrices  $O_k \in \text{SO}(k)$ ,  $k = m, n$ , are replaced by special unitary matrices  $U_k \in \text{SU}(k)$ .

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