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ON BOHR-SOMMERFELD-HEISENBERG QUANTIZATION

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Abstract. This paper presents the theory of Bohr-Sommerfeld-Heisenberg quantization of a completely integrable Hamiltonian system in the context of geometric quantization. The theory is illustrated with several examples.

1. Introduction

Most texts on quantum mechanics have a short section on the old quantum theory. They discuss Bohr's quantization of the harmonic oscillator and Sommerfeld's results on the energy spectrum of the hydrogen atom. Usually they mention of Heisenberg's quantum mechanics and give a description of Schrödinger's wave mechanics. Schrödinger's theory is further discussed in the framework of modern quantum mechanics. Heisenberg's theory is relegated to a criptic remark that Dirac proved that the theories of Heisenberg and of Schrödinger are equivalent. In [9], Dirac showed that Heisenberg's matrices can be also obtained in the Schrödinger theory, but he did not state that these theories give the same physical results.

Geometric quantization provides an explanation of Dirac's theory in the framework of modern differential geometry. Within geometric quantization, it is easy to understand Bohr-Sommerfeld quantization rules for completely integrable Hamiltonia systems, see [13]. If a Hamiltonian system with n-degrees of freedom has globally defined action-angle variables (A_i, φ_i), then the Bohr-Sommerfeld conditions define the structure of an n-dimensional lattice on the corresponding basis of the space of quantum states. Moreover, the action functions A_i are quantizable and the quantum operators $\boldsymbol{Q}_{A_k}$, $k = 1, 2, ...n$, corresponding to the action variables $A_1, ...A_n$, are diagonal in this basis.

The lattice structure of the basis defined by the Bohr-Sommerfeld conditions enables us to define n shifting operators that move the basic vectors along the vectors of the lattice. In the case of a regular infinite lattice the shifting operators may be interpreted as the quantization of the functions $\exp(-\mathrm{i}\varphi_1), ..., \exp(-\mathrm{i}\varphi_n)$ and their complex conjugates. In most examples, the lattice is not sufficiently regular

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and some of the angles $\varphi_1, ..., \varphi_n$ might not be defined at some points. In this case the shifting operators have to be redefined.

In this paper, we describe our understanding of Heisenberg's quantum mechanics within the framework of geometric quantization. We do not know if our approach has any relation to Heisenberg's ideas. However, we hope to convince the reader that we obtain a well defined quantum theory consistent with the principles of geometric quantization. More precisely, the theory we obtain generalizes geometric quantization, as formulated by Kostant [1,11], to the case of a singular polarization.

The theory, as formulated here, is limited by the requirement of the existence of global actions. If actions exist only locally, then one has to deal with quantum monodromy [5].

2. Completely Integrable Systems

Let (P, ω) be a symplectic manifold of dimension $2n$. We consider a completely integrable system on (P, ω) with action angle coordinates (A_i, φ_i) defined on an open dense subset U of P. The symplectic form ω restricted to U is $\omega_{|U} = \mathrm{d}\theta$, where $\theta = \sum_{i=1}^{n}(A_i \mathrm{d}\varphi_i)$.

Assumption 1. *We assume that the action coordinates A_i are globally defined on P.*

This implies that we have a symplectic action of the torus group $\mathbb{T}^n$ with the momentum map $J : P \to \mathbb{R}^n : p \mapsto J(p) = (A_1(p), ..., A_n(p))$, where we have identified the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$.

3. Bohr-Sommerfeld Quantization

The Hamiltonian vector field X_f of a function $f \in C^\infty(P)$ is defined by the equation $X_f \lrcorner \omega = -\mathrm{d}f$, where $\lrcorner$ is the left interior product (contraction on the left).

For each $i = 1, ..., n$, the Hamiltonian vector field X_{A_i} generates the action on P of the i^{th} component $\mathbb{T}_i$ of the torus group $\mathbb{T}^n = \mathbb{T} \times \mathbb{T} \times ... \times \mathbb{T}$. We denote by $\mathcal{O}_i(p)$ the orbit of $\mathbb{T}_i$ through $p \in P$. Clearly, A_i is constant on each orbit $\mathcal{O}_i(p)$.

BOHR-SOMMERFELD QUANTIZATION RULE. For each $i = 1, ..., n$, the quantum spectrum of A_i consists of the values $A_i(p)$ on orbits $\mathcal{O}_i(p)$ satisfying the condition

$$\int_{\mathcal{O}_i(p)} A_i \mathrm{d}\varphi_i = m_i h \tag{1}$$

where m_i is an integer and h denotes Planck's constant.

Integrating equation (1), we conclude that the quantum spectrum of the $i^{\rm th}$ action is given by

$$A_i = m_i \hbar \tag{2}$$

where $\hbar$ is Planck's constant divided by 2π.

Assumption 2. *For each n-tuple* $\mathbf{m} = (m_1, ..., m_n)$ *of integers, the set*

$$\mathbb{T}_{\mathbf{m}} = \{p \in P \ ; \ A_i(p) = m_i \hbar, \ i = 1, ..., n\} \tag{3}$$

is connected.

Under this assumption, $\mathbb{T}_{\mathbf{m}}$ is a torus. Otherwise, it would be the union of disjoint tori, and we would have to introduce an additional index to label connected components. In the following, we shall refer to sets $\mathbb{T}_{\mathbf{m}}$ defined by equation (3) as Bohr-Sommerfeld tori.

4. Link to Geometric Quantization

Suppose that we want to perform geometric quantization of our completely integrable system in the real polarization D spanned by the Hamiltonian vector fields X_{A_i} of the momenta $A_1, ..., A_n$.

Let L be a prequantization line bundle of (P, ω). Thus, L is a complex line bundle over P, with a connection ∇ such that

$$(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\sigma = -\frac{\mathrm{i}}{\hbar}\,\omega(X_1, X_2)$$

for each section σ of L and every pair X_1, X_2 of vector fields on P.

The quantum states of the system are given by sections σ of L that are covariantly constant along the polarization D. If Λ is a leaf of D, it is a torus, and the restriction $\sigma_{|\Lambda}$ of a section σ of L that is covariantly constant along D vanishes unless the holonomy group of the restriction of ∇ to Λ vanishes.

Proposition 3. *The holonomy group of the restriction of* ∇ *to a leaf* Λ *of* D *vanishes if and only if* Λ *satisfies the Bohr-Sommerfeld conditions, that is* $\Lambda = \mathbb{T}_{\mathbf{m}}$ *for some* n*-tuple* $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{Z}^n$.

Proof: The proof of this proposition can be found in [13]. ■

For a completely integrable system, tori $\mathbb{T}_{\mathbf{m}}$ are submanifolds of P of codimension at least n, and the only smooth section of L that is covariantly constant along D is the zero section. However, we may identify quantum states with distribution sections of L that are smooth and covariantly constant along leaves of D. Under this interpretation of quantum states, to every non-empty Bohr-Sommerfdeld torus $\mathbb{T}_{\mathbf{m}}$ in P, we can associate a non-zero distribution section $\sigma_{\mathbf{m}}$ of L with support in $\mathbb{T}_{\mathbf{m}}$, and such that the restriction $\sigma_{\mathbf{m}|\mathbb{T}_{\mathbf{m}}}$ of $\sigma_{\mathbf{m}}$ to $\mathbb{T}_{\mathbf{m}}$ is a smooth covariantly constant section of the restriction of L to $\mathbb{T}_{\mathbf{m}}$. On the space $\mathfrak{S}$ of distribution sections of L that are spanned by the sections $\sigma_{\mathbf{m}}$ we introduce a scalar product $\langle \cdot \mid \cdot \rangle$ such that

$$\langle \sigma_{\mathbf{m}} \mid \sigma_{\mathbf{m}'} \rangle = \delta_{\mathbf{m}\mathbf{m}'} = \delta_{m_1 m_1'} .... \delta_{m_n m_n'} . \tag{4}$$

For each Bohr-Sommerfeld torus $\mathbb{T}_{\mathbf{m}}$, the section $\sigma_{\mathbf{m}}$ introduced above is defined by $\mathbb{T}_{\mathbf{m}}$ up to an arbitrary non-zero complex factor. Therefore, the collection $\{\mathbb{T}_{\mathbf{m}}\}$ of all Bohr-Sommerfeld tori in P determines only the orthogonality property of basic vectors $\sigma_{\mathbf{m}}$. For a covariantly constant section section σ with support in $\mathbb{T}_{\mathbf{m}}$, the norm $\|\sigma\|$ depends on the choice of the basic section $\sigma_{\mathbf{m}}$.

We denote by $\mathfrak{H}$ the Hilbert space obtained by the completion of $\mathfrak{S}$ in the norm given by $\langle \cdot \mid \cdot \rangle$. $\mathfrak{H}$ is our space of quantum states. To each function $f \in C^\infty(P)$, such that $f = F(A_1, ..., A_n)$ for some $F \in C^\infty(\mathbb{R}^n)$, the Bohr-Sommerfeld quantization associates the quantum operator $\mathbf{Q}_f$ on $\mathfrak{H}$ such that, for every basic section $\sigma_{\mathbf{m}}$

$$\mathbf{Q}_f \sigma_{\mathbf{m}} = F(\mathbf{m}\hbar)\sigma_{\mathbf{m}}.$$

It follows from Assumption 2 that the spectrum of the action operators $\mathbf{Q}_{A_i}$ is simple.

A shortcoming of the Bohr-Sommerfeld quantization is that it is defined only on the commutative algebra consisting of smooth functions of the actions. In particular, Bohr-Sommerfeld quantization does not allow for quantization of any function of the angles. Moreover, it leads only to diagonal operators in $\mathfrak{H}$.

5. Shifting Operators

Bohr-Sommerfeld conditions together with Assumption 1 and Assumption 2 imply that the basis $\{\sigma_{\mathbf{m}}\}$ is a lattice. Therefore, there are well defined operators corresponding to shifting along the generators of the lattice.

For each $i = 1, ..., n$, let

$$\mathbf{m}_i = \{m_1, ..., m_{i-1}, m_i - 1, m_{i+1}, ..., m_n\}$$

and

$$\mathbf{m}^i = \{m_1, ..., m_{i-1}, m_i + 1, m_{i+1}, ..., m_n\}.$$

We define shifting operators $\mathbf{a}_i$ on $\mathfrak{H}$ by

$$\mathbf{a}_i \sigma_{\mathbf{m}} = \begin{cases} \sigma_{\mathbf{m}_i} & if \quad \mathbb{T}_{\mathbf{m}_i} \neq \emptyset \\ 0 & if \quad \mathbb{T}_{\mathbf{m}_i} = \emptyset. \end{cases} \tag{5}$$

The adjoint operators $\mathbf{a}_i^\dagger$ are given by

$$\mathbf{a}_i^\dagger \sigma_{\mathbf{m}} = \begin{cases} \sigma_{\mathbf{m}^i} & if \quad \mathbb{T}_{\mathbf{m}^i} \neq \emptyset \\ 0 & if \quad \mathbb{T}_{\mathbf{m}^i} = \emptyset. \end{cases} \tag{6}$$

Proposition 4. *The shifting operators satisfy the following commutation relations*

$$[\mathbf{a}_k, \mathbf{Q}_{A_j}] \;=\; \delta_{kj} \hbar \, \mathbf{a}_k \tag{7a}$$

$$[\mathbf{a}_k^\dagger, \mathbf{Q}_{A_j}] \;=\; -\delta_{kj} \hbar \, \mathbf{a}_k^\dagger. \tag{7b}$$

The Poisson bracket relations between actions and angles are

$$\{\mathrm{e}^{-\mathrm{i}\varphi_k}, A_j\} = -\mathrm{i}\delta_{kj} \, \mathrm{e}^{-\mathrm{i}\varphi_k}.$$

Hence, Dirac's quantization conditions

$$[\mathbf{Q}_{f_1}, \mathbf{Q}_{f_2}] = \mathrm{i}\hbar \, \mathbf{Q}_{\{f_1, f_2\}} \tag{8}$$

suggest the identification $\mathbf{a}_k = \mathbf{Q}_{\mathrm{e}^{-\mathrm{i}\varphi_k}}$ and $\mathbf{a}_k^\dagger = \mathbf{Q}_{\mathrm{e}^{\mathrm{i}\varphi_k}}$, where φ_k is the angle coordinate corresponding to the action A_k, provided the functions $\mathrm{e}^{-\mathrm{i}\varphi_k}$ and $\mathrm{e}^{\mathrm{i}\varphi_k}$ are globally defined on P.

6. Heisenberg Quantization

Since not all sets $\mathbb{T}_{\mathbf{m}}$ are n-tori, we cannot expect that all exponential functions $\mathrm{e}^{-\mathrm{i}\varphi_k}$ are globally defined. We can try to replace $\mathrm{e}^{-\mathrm{i}\varphi_k}$ by a globally defined smooth function f_k of the form $\chi_k = r_k \mathrm{e}^{-\mathrm{i}\varphi_k}$, where the coefficient r_k depends only on the actions and vanishes at the points at which $\mathrm{e}^{\mathrm{i}\varphi_k}$ is not defined. In the following we shall refer to functions χ_k as Heisenberg functions.

We have the following Poisson bracket relations

$$\{\chi_k, A_j\} = -\mathrm{i}\delta_{kj} \, \chi_k \qquad \text{and} \qquad \{\bar{\chi}_k, A_j\} = \mathrm{i}\delta_{kj} \, \bar{\chi}_k. \tag{9}$$

By Dirac's quantization conditions, we get

$$[\mathbf{Q}_{\chi_k}, \mathbf{Q}_{A_j}] \;=\; \delta_{kj} \hbar \, \mathbf{Q}_{\chi_k} \tag{10a}$$

$$[\mathbf{Q}_{\bar{\chi}_k}, \mathbf{Q}_{A_j}] \;=\; -\delta_{kj} \hbar \, \mathbf{Q}_{\bar{\chi}_k}. \tag{10b}$$

For each basic vector $\sigma_{\mathbf{m}}$ of $\mathfrak{H}$

$$\begin{aligned} \mathbf{Q}_{A_j}(\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}) &= \mathbf{Q}_{\chi_j}(\mathbf{Q}_{A_j}\sigma_{\mathbf{m}}) - [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{A_j}]\sigma_{\mathbf{m}} \\ &= \mathbf{Q}_{\chi_j}(\hbar m_j \sigma_{\mathbf{m}}) - \hbar \mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} \\ &= \hbar(m_j - 1)\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}. \end{aligned} \tag{11}$$

Thus, $\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}_j}$. A similar argument shows that $\mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}}$ is proportional to $\sigma_{\mathbf{m}^j}$. Hence, $\mathbf{Q}_{\chi_j}$ and $\mathbf{Q}_{\bar{\chi}_j}$ act as shifting operators, namely

$$\mathbf{Q}_{\chi_j}\sigma_{\mathbf{m}} = b_{\mathbf{m},j}\sigma_{\mathbf{m}_j} \qquad \text{and} \qquad \mathbf{Q}_{\bar{\chi}_j}\sigma_{\mathbf{m}} = c_{\mathbf{m},j}\sigma_{\mathbf{m}^j} \tag{12}$$

for some coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$.

We can use Dirac's quantization conditions

$$[\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\chi_k}] = \mathrm{i}\hbar\, \mathbf{Q}_{\{\chi_j,\chi_k\}} \qquad \text{and} \qquad [\mathbf{Q}_{\chi_j}, \mathbf{Q}_{\bar{\chi}_k}] = \mathrm{i}\hbar\, \mathbf{Q}_{\{\chi_j,\bar{\chi}_k\}} \tag{13}$$

and the identification

$$\mathbf{Q}^{\dagger}_{\chi_j} = \mathbf{Q}_{\bar{\chi}_j} \tag{14}$$

to determine the coefficients $b_{\mathbf{m},j}$ and $c_{\mathbf{m},j}$, which must satisfy the consistency conditions

$$b_{\mathbf{m},j} = 0 \quad \text{if} \quad \mathbb{T}_{\mathbf{m}_j} = \emptyset \qquad \text{and} \qquad c_{\mathbf{m},j} = 0 \quad \text{if} \quad \mathbb{T}_{\mathbf{m}^j} = \emptyset. \tag{15}$$

The Bohr-Sommerfeld-Heisenberg quantization described here is an extension of the Bohr-Sommerfeld theory. In the Bohr-Sommerfeld-Heisenberg quantization, the Hilbert space $\mathfrak{H}$ of quantum states is the same as in the Bohr-Sommerfeld theory. However, in the Bohr-Sommerfeld-Heisenberg theory, we can quantize functions that are first degree polynomials in χ_k and $\bar{\chi}_k$ with coefficients given by smooth functions of the actions

$$F(A_1, ..., A_n) + \sum_{k=1}^{n}[F_k(A_1, ..., A_n)\chi_k + \tilde{F}_k(A_1, ..., A_n)\bar{\chi}_k].$$

The resulting operators on $\mathfrak{H}$ are first degree polynomials in shifting operators. Higher powers of shifting operators are well defined on $\mathfrak{H}$, but they need not be quantizations of the corresponding powers of the functions f_k or $\bar{f}_k$ (the usual factor ordering problem).

7. Examples

7.1. The One-Dimensional Harmonic Oscillator

The phase space of the one-dimensional harmonic oscillator is $P = \mathbb{R}^2$ with coordinates (p, q) and the symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$. The Hamiltonian is $H =$

$\frac{1}{2}(p^2+q^2)$. In polar coordinates $(p,q)=(r\cos\varphi, r\sin\varphi)$, where $r=\sqrt{p^2+q^2}$ and $\varphi=\tan\frac{q}{p}$, we have $\omega=\mathrm{d}H\wedge\mathrm{d}\varphi$. Here $H=\frac{1}{2}r^2$ is the action variable, while φ is the corresponding angle. The Heisenberg function $\chi=p-\mathrm{i}q=r\mathrm{e}^{-\mathrm{i}\varphi}$ leads to quantization equivalent to the Bargmann quantization [2]. It should be noted that $r=\sqrt{2H}$ is not a smooth function of H, but χ is in $C^\infty(P)$. For full details see [7].

7.2. Coadjoint Orbits of SO(3)

Following Souriau [15] we use the presentation of coadjoint orbits of SO (3) as spheres $S_r^2=\{(x^1,x^2,x^3)\in\mathbb{R}^3\ ;\ (x^1)^2+(x^2)^2+(x^3)^2=r^2\}$ endowed with a symplectic form $\omega=\frac{1}{r}\mathrm{vol}_{S_r^2}$, where $\mathrm{vol}_{S_r^2}$ is the standard area form on S_r^2 with $\int_{S_r^2}\mathrm{vol}_{S_r^2}=4\pi r^2$. A coadjoint orbit S_r^2 is quantizable if $r=\frac{n}{2}\hbar$, where n is an integer.

For each $i=1,2,3$, we denote by J^i the restriction of x^i to the sphere S_r^2. The functions J^1, J^2 and J^3 are components of the momentum map of the co-adjoint action. They satisfy the Poisson bracket relations $\{J^i,J^j\}=\sum_{k=1}^3\varepsilon_{ijk}J^k$. In spherical polar coordinates

$$J^1=r\sin\theta\cos\varphi,\qquad J^2=r\sin\theta\sin\varphi,\qquad J^3=r\cos\theta$$

and

$$\omega=r\sin\theta\,\mathrm{d}\varphi\wedge\mathrm{d}\theta=-\mathrm{d}(r\cos\theta\,\mathrm{d}\varphi)=\mathrm{d}(J^3\mathrm{d}(-\varphi)).$$

Thus, $(J^3,-\varphi)$ are action-angle coordinates for an integrable system (J^3,S_r^2,ω). In this case, a Heisenberg function is $\chi=J_+=\sqrt{r^2-(J^3)^2}\,\mathrm{e}^{\mathrm{i}\varphi}$. The resulting Bohr-Sommerfeld-Heisenberg quantization leads to the irreducible unitary representation of $\mathrm{SO}(3)$ corresponding to the co-adjoint orbit S_r^2. For more details, see [7]. The presented treatment closely resembles the approach of Schwinger [12].

7.3. The Two-Dimensional Harmonic Oscillator

The configuration space of the two-dimensional harmonic oscillator is $\mathbb{R}^2$ with coordinates $x=(x^1,x^2)$. The phase space is $T^*\mathbb{R}^2=\mathbb{R}^4$ with coordinates $(x,y)=(x^1,x^2,y^1,y^2)$ and canonical symplectic form $\omega=\mathrm{d}(y^1\mathrm{d}x^1+y^2\mathrm{d}x^2)$. The Hamiltonian function of the two-dimensional harmonic oscillator is

$$H(x,y)=\tfrac{1}{2}\big((x^1)^2+(x^2)^2\big)+\tfrac{1}{2}\big((y^1)^2+(y^2)^2\big).$$

Orbits of the Hamiltonian vector field X_H of H are periodic of period 2π. The function $L(x,y)=x^1y^2-x^2y^1$ generates an action of S^1 on $T^*\mathbb{R}^2$ that preserves the Hamiltonian H. Hence, $(H, L, T^*\mathbb{R}^2, \omega)$ is a completely integrable system. Let

$$\begin{aligned} x_1 &= \tfrac{-1}{\sqrt{2}}(r_1 \cos\vartheta_1 + r_2 \cos\vartheta_2), & y_1 &= \tfrac{1}{\sqrt{2}}(r_1 \sin\vartheta_1 + r_2 \sin\vartheta_2) \\ x_2 &= \tfrac{1}{\sqrt{2}}(-r_1 \sin\vartheta_1 + r_2 \sin\vartheta_2), & y_2 &= \tfrac{1}{\sqrt{2}}(-r_1 \cos\vartheta_1 + r_2 \cos\vartheta_2) \end{aligned} \tag{16}$$

be a change of coordinates from rectangular (x, y) variables to polar variables $(r_1, r_2, \vartheta_1, \vartheta_2)$. A computation shows that $H(r, \vartheta) = \frac{1}{2}(r_1^2 + r_2^2)$ and $L(r, \vartheta) = \frac{1}{2}(r_1^2 - r_2^2)$ and that the change of coordinates (16) pulls back the symplectic form $\omega = \mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2$ to the symplectic form $\Omega = \mathrm{d}(\frac{1}{2}r_1^2) \wedge \mathrm{d}\vartheta_1 + \mathrm{d}(\frac{1}{2}r_2^2) \wedge \mathrm{d}\vartheta_2$. Let $A_1 = \frac{1}{2}r_1^2 = \frac{1}{2}\big(E(r,\vartheta) + L(r,\vartheta)\big) \geq 0$ and $A_2 = \frac{1}{2}r_2^2 = \frac{1}{2}\big(E(r,\vartheta) - L(r,\vartheta)\big) \geq 0$. Then $(A_1, A_2, \vartheta_1, \vartheta_2)$ with $A_1 > 0$, and $A_2 > 0$ and symplectic form $\Omega = \mathrm{d}A_1 \wedge \mathrm{d}\vartheta_1 + \mathrm{d}A_2 \wedge \mathrm{d}\vartheta_2$ are real analytic action-angle coordinates for the two-dimensional harmonic oscillator. These coordinates extend real analytically to the closed domain $A_1 \geq 0$ and $A_2 \geq 0$. The Heisenberg functions $\chi_1 = r_1 \mathrm{e}^{\mathrm{i}\vartheta_1}$ and $\chi_2 = r_2 \mathrm{e}^{\mathrm{i}\vartheta_2}$ give rise to the Bohr-Sommerfeld-Heisenberg quantization of the two-dimensional harmonic oscillator. For more details see [6].

7.4. The Mathematical Pendulum

The phase space of the mathematical pendulum is T^*S^1 with the canonical coordinates (p, α) and symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}\alpha$. The Hamiltonian of the system is $H = \frac{1}{2}p^2 - \cos\alpha + 1$. The Hamiltonian system (H, T^*S^1, ω) violates Assumption 2, because for $H > 2$, level sets of the Hamiltonian H have two connected components. We are investigating how to extend to this case the theory presented here.

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