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THE CLASSICAL MAGNETIZED KEPLER PROBLEMS IN HIGHER ODD DIMENSIONS

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Abstract. The Kepler problem for planetary motion is a two-body dynamic model with an attractive force obeying the inverse square law, and has a direct analogue in any dimension. While the magnetized Kepler problems were discovered in the late 1960s, it is not clear until recently that their higher dimensional analogues can exist at all. Here we present a possible route leading to the discovery of these high dimensional magnetized models.

1. The Kepler Problem and its High Dimensional Analogues

The Kepler problem is the mathematical model for a solar system with a single planet or an atom with a single electron, depending on whether it is considered classically or quantum mechanically. At the classical level, this is a dynamic problem with configuration space $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$ and equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} \tag{1}$$

where $\mathbf{r}$ is a function of time t taking value in $\mathbb{R}^3_*$, $r = |\mathbf{r}|$ and $\mathbf{r}''$ is the second time-derivative of $\mathbf{r}$. Since the force on the left hand side of equation (1) is a central force, the *angular momentum* $\mathbf{L} := \mathbf{r} \times \mathbf{r}'$ is a constant of motion. A hidden fact is that the *Runge-Lenz vector* $\mathbf{A} := \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$ is a constant of motion. Although the Runge-Lenz vector has been re-discovered several times [3], ironically, neither Carl Runge nor Wilhelm Lenz discovered it.

The intrinsic version of the angular momentum is the two-vector

$$L := \mathbf{r} \wedge \mathbf{r}'$$

in term of which, the Runge-Lenz vector can be written as

$$\mathbf{A} := \mathbf{r}' \lrcorner L + \frac{\mathbf{r}}{r}$$

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where $\lrcorner$ is the interior product. Note that L and $\mathbf{A}$ are still constants of motion even if we replace the configuration space $\mathbb{R}^3_*$ by $\mathbb{R}^n_*$ for a generic positive integer n. This implies that a dynamic problem with equation (1) as its equation of motion in $\mathbb{R}^n_*$ behaviors like the Kepler problem. In fact, a simple computation shows that $L \wedge \mathbf{A} = 0$ and

$$L \wedge \mathbf{r} = 0, \qquad r - \mathbf{A}.\mathbf{r} = |L|^2. \tag{2}$$

Therefore, if $L \neq 0$, equation (2) implies that $\mathbf{r}$ lies in a plane determined by L, moreover, the orbit is a conic with the Runge-Lenz vector $\mathbf{A}$ as its eccentricity vector. It is not hard to see that the total energy for a motion with this orbit is

$$E = -\frac{1 - |\mathbf{A}|^2}{2|L|^2}. \tag{3}$$

Hereafter, this analogous dynamical problem in dimension n shall be referred to as the *Kepler problem in dimension* n, and its non-colliding orbits (i.e., the ones with $L \neq 0$) shall be referred to as the *Kepler orbits*.

To continue the discussion in this article, we need to review the notions of poly-vectors in the Euclidean space $\mathbb{R}^n$ or the Lorentz space $\mathbb{R}^{1,n}$, plus the wedge product, interior product, and inner product involving the poly-vectors.

1.1. Notations and Conventions

Boldface Latin letters are reserved for vectors in the Euclidean space $\mathbb{R}^n$ only, and the inner product (i.e., the dot product) of the vectors $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u}.\mathbf{v}$. Vectors in the Lorentz space $\mathbb{R}^{1,n}$ are referred to as *Lorentz vectors*. For the Lorentz vectors $a = (a_0, \mathbf{a})$ and $b = (b_0, \mathbf{b})$, the (Lorentz) inner product of a and b, written as $a.b$, is defined as

$$a_0 b_0 - \mathbf{a}.\mathbf{b}.$$

The vector $\mathbf{r}$ is reserved for a point in $\mathbb{R}^n$, and the Lorentz vector x is reserved for a point in $\mathbb{R}^{1,n}$. We often write $x = (x_0, \mathbf{r})$ rather than $(x_0, \mathbf{x})$. For the standard basis vectors $e_0, e_1, \ldots, e_n$ in $\mathbb{R}^{1,n}$, we have $e_0.e_0 = 1$ and $e_i.e_i = -1$ for $i > 0$. When we view the Lorentz e_i $(i > 0)$ as a vector inside the subspace $\mathbb{R}^n$, we write it as $\mathbf{e}_i$.

Let V be either $\mathbb{R}^n$ or $\mathbb{R}^{1,n}$, and let $k > 0$ be an integer. A k-vector in V is just an element of $\wedge^k V$. A one-vector is just a vector. A k-vector is called *decomposable* if it is the wedge product of k vectors.

The inner product extends from vectors to poly-vectors and is denoted by $\langle\, , \rangle$. By definition, for vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in V, let $[u_i.v_j]$ be the square matrix whose (i,j)-entry is $u_i.u_j$. Then

$$\langle u_1 \wedge \ldots \wedge u_k, v_1 \wedge \ldots \wedge v_k \rangle = \det[u_i.v_j].$$

We define the interior product $\lrcorner$ as the adjoint of the wedge product with respect to the inner product for poly-vectors: for poly-vectors X, u, and v in V with $\deg X + \deg u = \deg v$, we have

$$\langle X \wedge u, v \rangle = \langle u, X \lrcorner v \rangle.$$

For any poly-vector X, we write X^2 for $\langle X, X\rangle$. When $\langle X, X\rangle \geq 0$, we write $|X|$ for $\sqrt{\langle X, X\rangle}$. We always write r for $|\mathbf{r}|$. Finally, we remark that a poly-vector X in $\mathbb{R}^n$ is also viewed as a poly-vector X in $\mathbb{R}^{1,n}$ in a natural way.

By an orthochronous Lorentz transformation of $\mathbb{R}^{1,n}$ we mean a Lorentz transformation which leaves invariant the future light cone. The group of orthochronous Lorentz transformations of $\mathbb{R}^{1,n}$ shall be denoted by $\mathrm{O}^+(1,n)$, a Lie subgroup of $\mathrm{O}(1,n)$ of index two. We shall let Str_n be the subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ which leaves the future light cone invariant, so $\mathrm{Str}_n = \mathbb{R}_+ \times \mathrm{O}^+(1,n)$.

2. The Light Cone Formulation

The Kepler problem in dimension n has a mathematically appealing formulation in the Lorentz space $\mathbb{R}^{1,n}$, as explicitly pointed out in reference [10]. To see this, we observe that $\mathbb{R}^n_*$ is diffeomorphic to the future light cone

$$\{x \in \mathbb{R}^{1,n}\ ;\ x^2 = 0, x_0 > 0\}$$

so the Kepler problem in dimension n can be reformulated as a dynamic problem on the future light cone. As a result, an oriented Kepler orbit in $\mathbb{R}^n_*$ can be reformulated as an oriented curve inside the future light cone, i.e., the intersection of the cylinder over the oriented Kepler orbit with the future light cone. This intersection turns out to be a conic section, a reason why a Kepler orbit must be a conic. To see the intersection plane for this conic section, we first write A for $(1, \mathbf{A})$ and observe that $L \wedge A = L \wedge e_0 \neq 0$. Then, for x on the future light cone, since equation (2) can be recast as $x \wedge (L \wedge A) = 0$, $A.x = |L \wedge A|^2$ or equivalently

$$m \wedge x = 0, \qquad a.x = 1 \tag{4}$$

where

$$m = \frac{L \wedge A}{|L \wedge A|}, \qquad a = \frac{A}{|L \wedge A|^2} \tag{5}$$

the intersection plane is the affine plane defined by equation (4).

In summary, in the light cone formulation, a Kepler orbit (i.e., non-colliding orbit) is the intersection of the future light cone with an affine plane defined by equation (4), where m is a decomposable three-vector in $\mathbb{R}^{1,n}$ with $m^2 = 1$ and $e_0 \wedge m = 0$, a is a vector with $a_0 > 0$ and $m \wedge a = 0$. Note that, in this formulation, formula (3) becomes

$$E = -\frac{a^2}{2a_0}\cdot \tag{6}$$

2.1. A Potential Connection with Lorentz Transformations

The linear action of Str_n on $\mathbb{R}^{1,n}$ leaves invariant the future light cone and turns an affine plane into another affine plane, so it seems that a Kepler orbit would be transformed into another Kepler orbit. However, this cannot be true.

To see this, we first introduce the space $\mathcal{M}_+$ consisting of pairs (m, a) where m is a decomposable three-vector in $\mathbb{R}^{1,n}$ with $m^2 = 1$, and a is a vector with $m \wedge a = 0$, $a^2 > 0$ and $a_0 > 0$. Next, we observe that the set of oriented elliptic Kepler orbits can be parametrized by the subspace

$$\mathcal{M}_+^0 := \{(m, a) \in \mathcal{M}_+ \; ; \; m \wedge e_0 = 0\}$$

of $\mathcal{M}_+$. Since $\dim \mathcal{M}_+ = \dim \mathcal{M}_+^0 + (n-2)$, for $n \geq 3$, $\mathcal{M}_+^0$ is a proper subspace of $\mathcal{M}_+$. Finally we observe that the corresponding action of Str_n on $\mathcal{M}_+$

$$\mathrm{Str}_n \times \mathcal{M}_+ \to \mathcal{M}_+, \qquad ((\alpha, \Lambda), (m, a)) \mapsto (\Lambda.m, \alpha(\Lambda.a))$$

is transitive, so the action of Str_n can take a point in $\mathcal{M}_+^0$ to a point outside of $\mathcal{M}_+^0$. Consequently the linear action of Str_n on $\mathbb{R}^{1,n}$ can take a Kepler orbit to a conic section which is not a Kepler orbit.

Could these extra conic sections be orbits of some additional dynamic models? The answer is yes provided that n is an odd integer. When $n = 3$, these additional dynamic models are Kepler problem's magnetized companions [6, 15], under the name MIC-Kepler problems or MICZ-Kepler problems. When $n = 5$, they are the Iwai's $\mathrm{SU}(2)$-Kepler problems [4]. For $n \geq 7$, they are the models discovered recently by present author [11].

3. MICZ Kepler Problems

An MICZ Kepler problem is the mathematical model for a hypothetical hydrogen atom where the nucleus is dyon. Here the equation of motion is

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \mu^2\frac{\mathbf{r}}{r^4} - \mathbf{r}' \times \mu\frac{\mathbf{r}}{r^3} \tag{7}$$

with the parameter μ being the magnetic charge of the nucleus. These models are so similar to the Kepler problem so that the solution of them are nearly direct repetition of the solution for the Kepler problem. For example, the constants of motions are[1]

$$L = \mathbf{r}\wedge\mathbf{r}' + \mu\frac{*\mathbf{r}}{r}, \qquad \mathbf{A} = \mathbf{r}' \lrcorner L + \frac{\mathbf{r}}{r}$$

and equations (4) and (5) stay the same, except that we no longer have $e_0 \wedge m = 0$. Equation (6) stays the same, too. It was demonstrated in reference [9] that the set of bounded oriented orbits of MICZ Kepler problems is parametrized by $\mathcal{M}_+$, hence admits a transitive action of Str_3.

4. High Dimensional Analogues of MICZ Kepler Problems

The high-dimensional analogue of equation (7) is far from straightforward. The reason is that, if $k > 1$, instead of governing motions on $\mathbb{R}^{2k+1}_*$, the equation of motion governs motions on a manifold P_μ which fibers over $\mathbb{R}^{2k+1}_*$.

To describe the fiber bundle $P_\mu \to \mathbb{R}^{2k+1}_*$, we let $G = \mathrm{SO}(2k)$ and consider the canonical principal G-bundle over S^{2k}

$$\begin{array}{c}\mathrm{SO}(2k+1)\\ \downarrow\\ \mathrm{S}^{2k}.\end{array}$$

This bundle comes with a natural connection

$$\omega(g) := \mathrm{Pr}_{\mathfrak{so}(2k)}\left(g^{-1}\mathrm{d}g\right)$$

where $g^{-1}\mathrm{d}g$ is the Maurer-Cartan form for $\mathrm{SO}(2k+1)$, so it is an $\mathfrak{so}(2k+1)$-valued differential one-form on $\mathrm{SO}(2k+1)$, and $\mathrm{Pr}_{\mathfrak{so}(2k)}$ denotes the orthogonal projection of $\mathfrak{so}(2k+1)$ onto $\mathfrak{g} := \mathfrak{so}(2k)$.

Under the map

$$\pi : \mathbb{R}^{2k+1}_* \to \mathrm{S}^{2k}$$

[1]here $*$ is the Hodge-star operator.

$$\mathbf{r} \mapsto \frac{\mathbf{r}}{r}$$

the above bundle and connection are pulled back to a principal G-bundle

$$\begin{array}{c} P \\ \big\downarrow \\ X := \mathbb{R}^{2k+1}_* \end{array}$$

with a connection which is usually referred to as the *generalized Dirac monopole* [2, 7]. Now

$$P_\mu \to \mathbb{R}^{2k+1}_*$$

is the associated fiber bundle with fiber being a certain co-adjoint orbit $\mathcal{O}_\mu$ of G, the so-called *magnetic orbit* [11] with magnetic charge $\mu \in \mathbb{R}$.

To describe $\mathcal{O}_\mu$, let us use $M_{a,b}$ ($1 \le a, b \le 2k$) to denote the element of $\mathfrak{g}$ such that, in the defining representation of $\mathfrak{g}$, $M_{a,b}$ is represented by the skew-symmetric real matrix whose (a, b)-entry is -1, (b, a)-entry is 1, and all other entries are 0. For the invariant metric $(\,,\,)$ on $\mathfrak{g}$, we take the one such that $M_{a,b}$ ($1 \le a < b \le 2k$) form an orthonormal basis for $\mathfrak{g}$. Via this invariant metric, one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, hence co-adjoint orbits with adjoint orbits. By definition, for any $\mu \in \mathbb{R}$

$$\mathcal{O}_\mu := \mathrm{SO}(2k).\frac{1}{\sqrt{k}}(|\mu| M_{1,2} + \ldots + |\mu| M_{2k-3,2k-2} + \mu M_{2k-1,2k}).$$

It is easy to see that $\mathcal{O}_\mu = \{0\}$ if $\mu = 0$ and is diffeomorphic to $\mathrm{SO}(2k)/\mathrm{U}(k)$ if $\mu \neq 0$.

We are now ready to describe the equation of motion for the magnetized Kepler problem in dimension $2k + 1$. Let $\mathbf{r}$: $\mathbb{R} \to X$ be a smooth map, and ξ be a smooth lifting of $\mathbf{r}$

$$\begin{array}{ccc} & & P_\mu \\ & \xi \nearrow & \big\downarrow \\ \mathbb{R} & \xrightarrow{\mathbf{r}} & X \end{array}$$

Let Ad_P be the adjoint bundle $P \times_G \mathfrak{g} \to X$, d_∇ be the canonical connection, i.e., the generalized Dirac monopole on $\mathbb{R}^{2k+1}_* = X$. Then the curvature $\Omega := \mathrm{d}^2_\nabla$ is a smooth section of the vector bundle $\wedge^2 T^* X \otimes \mathrm{Ad}_P$. The equation of motion is

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \frac{\mu^2}{k}\frac{\mathbf{r}}{r^4} + (\xi, \mathbf{r}' \lrcorner \Omega), \qquad \frac{D\xi}{\mathrm{d}t} = 0. \tag{8}$$

Here $\dfrac{D\xi}{\mathrm{d}t}$ is the covariant derivative of ξ, $(\,,)$ refers to the inner product on the fiber of the adjoint bundle coming from the invariant inner product on $\mathfrak{g}$, and two-forms are identified with two-vectors via the standard euclidean structure of $\mathbb{R}^{2k+1}$. equation (8) defines a super integrable model, referred to as the *classical Kepler problem with magnetic charge* μ *in dimension* $2k+1$, which generalize the classical MICZ-Kepler problem. Indeed, in dimension three, the bundle is topological trivial, $\xi = \mu M_{12}$, and $\Omega = \frac{*(\sum_{i=1}^{3} x^i\,\mathrm{d}x^i)}{r^3} M_{12}$, then equation (8) reduces to equation (7), i.e., the equation of motion for the MICZ-Kepler problem with magnetic charge μ. In dimension 5, it is essentially Iwai's $\mathrm{SU}(2)$-Kepler problem, cf. reference [4].

The equation of motion appears to be mysterious, but it does not. As demonstrated in [11], with a key input from the work of Sternberg, Weinstein, and Montgomery [14], it emerges naturally from the notion of universal Kepler problem in reference [12]. As a side remark, we would like to point out that the quantum magnetized Kepler problems were obtained much earlier in [8].

4.1. Orbits

While the orbits for the magnetized Kepler problems in dimension three have been thoroughly studied from the very beginning [6], in higher dimensions, in view of the fact that the equation of motion is a bit more sophisticated, one might expect that the orbits are a bit hard to find. That is probably the reason why the orbits for Iwai's $\mathrm{SU}(2)$-Kepler problems were never investigated in [4] and the subsequent papers [5].

The second of equations in (8) implies that an orbit inside P_μ is the horizontal lifting of its projection onto $\mathbb{R}^{2k+1}_*$. So it suffices to understand the projection of the orbit onto $\mathbb{R}^{2k+1}_*$. This projection curve was found [1] to be either a part of straight line (colliding orbit) or a conic (non-colliding orbit).

4.2. Outlook

An interesting further study is to work out the geometric quantization of the classical models introduced here so that one can reproduce the quantum models introduced in reference [8]. It is expected also that the earlier work carried out by I. Mladenov and V. Tsanov [13] for the Kepler problems in higher dimensions or the MICZ Kepler problems shall serve as a good guidance in such studies.

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