



SIGMA FUNCTIONS FOR A SPACE CURVE OF TYPE $(3, 4, 5)$

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Communicated by Vasil V. Tsanov

Abstract. In this article, a generalized Kleinian sigma function for an affine $(3, 4, 5)$ space curve of genus two was constructed as the simplest example of the sigma function for an affine space curve, and in terms of the sigma function, the Jacobi inversion formulae for the curve are obtained. An interesting relation between a space curve with a semigroup generated by $(6, 13, 14, 15, 16)$ and Norton number associated with Monster group is also mentioned in the Appendix by the second author.

1. Introduction

Recently the Kleinian sigma function for hyperelliptic curves, a natural generalization of the Weierstrass sigma function, is re-evaluated because in terms of the sigma functions, it is more convenient to investigate the properties of the abelian functions and their interesting properties are revealed naturally [2, 6, 17].

Further Enolskii, Eilbeck, and Leykin [5] discovered a construction which generalizes the Kleinian sigma function associated with hyperelliptic curves to one for an affine (r, s) plane curve, where r and s ($r < s$) are coprime positive integers $g = (r - 1)(s - 1)/2$. There they have constructed also the fundamental differential of the second kind over an affine (r, s) plane curve and using it, obtained the Legendre relation as the symplectic structure over the curve. Using the Legendre relation, they defined the generalized Kleinian sigma function over the image of the abelian map \mathbb{C}^g . They have found also the natural Jacobi inversion formulae in terms of their sigma function. We call this construction *EEL construction* in this article. Using the EEL construction, we have some interesting results [20, 21].

In this article, we consider a generalized Kleinian sigma function for an affine $(3, 4, 5)$ space curve of genus two, which is the simplest affine space curve. Our purpose of this article is to show that the sigma function is also defined for an affine space curve as we can do for plane curves.

Following the EEL-construction, we define the fundamental differential of the second kind over it and obtain the Legendre relation as the symplectic structure over it.

With the abelian map to \mathbb{C}^2 , we show that the symplectic structure determines the sigma function. Further using the sigma function, we obtain the Jacobi inversion formulae for the curve and the Jacobian following the previous works [20, 21].

It means that the generalization of the sigma functions for the affine plane curves to ones for the space curves is basically possible and is useful. Recently, Korotkin with Shramchenko defined a sigma function for a compact Riemann surface [15] but it is not directly associated with an algebraic curve. Further Ayano introduced sigma functions for space curves of special class [1], which are called telescopic curves, but the class does not include this (3,4,5) curve.

In Remark 21, we also show a problem of a space curve associated with the semi-group generated by (6, 13, 14, 15, 16) with an Appendix by Komeda. The semi-group might be related to Norton number associated with the Monster group, the simple largest sporadic finite group [22].

2. Preliminary

2.1. Numerical Semigroup

Here we give a short overview of recent study of the numerical semigroups as sub-semigroups of non-negative integers \mathbb{N}_0 related to algebraic curves. We call an additive semigroup in \mathbb{N}_0 *numerical semigroup* if its complement in \mathbb{N}_0 is finite. For a numerical semigroup $H = H(M)$ generated by M , the number of elements of $L(H) := \mathbb{N}_0 \setminus H$ is called genus and $L(H)$ is called gap sequence. For example, we have semigroups H_2, H_4, H_{12} generated by $M_2 := \langle 3, 4, 5 \rangle, M_4 := \langle 3, 7, 8 \rangle, M_{12} := \langle 6, 13, 14, 15, 16 \rangle$ respectively whose genera are $g(H_g)$ for $g = 2, 4, 12$ due to $L(H_2) = \{1, 2\}, L(H_4) = \{1, 2, 4, 5\}, L(H_{12}) = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}$.

For a complete non-singular irreducible curve C of genus g over an algebraically closed field k of characteristic 0, the field of its rational functions $k(C)$, and a point $P \in C$, we define

$$H(P) := \{n \in \mathbb{N}_0 ; \text{there exists } f \in k(C) \text{ such that } (f)_\infty = nP\} \quad (1)$$

and refer further on as the Weierstrass semigroup of the point P . If $L(H(P)) := \mathbb{N}_0 \setminus H(P)$ differs from the set $\{1, 2, \dots, g\}$, we call P Weierstrass point of C .

A numerical semigroup H is said to be Weierstrass if there exists a pointed algebraic curve (C, P) such that $H = H(P)$. Hurwitz considered whether every numerical semigroup H is Weierstrass. This was a long-standing problem but Buchweitz finally showed that every H is not Weierstrass. His first counterexample is the semigroup H_B generated by 13, 14, 15, 16, 17, 18, 20, 22 and 23,

whose genus is 16. Thus in general, it is not so trivial whether a given semigroup is Weierstrass or not. Komeda has been investigated this problem with Ohbuchi and Kim [10–12, 14].

2.2. Commutative Algebra

Here we review a normal ring and normalization in commutative ring [16]. We assume that every ring is a commutative ring with unit.

B is a ring and A is a subring of B . B is said to be an *extension* of A . An element b of B is said to be *integral* over A if b satisfies a monic polynomial over A , i.e., there exist n and $\{a_i\}_{i=1,\dots,n} \in A$ such that $b^n + a_1 b^{n-1} + \dots + a_n = 0$.

We say that B is *integral* over A , or B is an *integral ring* over A , or B is an *integral extension* of A if every element b of B is integral over A .

An *integral closure* in B over A is defined by $\tilde{A} := \{b \in B ; b \text{ is integral over } A\}$. If $A = \tilde{A}$, A is *integral closed* in B .

Definition 1. Let A is a ring and $Q(A)$ is a quotient ring of A . We assume that A is an integral domain. A is normal if A is integral closed in $Q(A)$, i.e., for $\tilde{A} := \{q \in Q(A) ; \text{there exist } n \text{ and } a_i \in A \text{ such that } q^n + a_1 q^{n-1} + \dots + a_n = 0\}$, $A = \tilde{A}$.

We define the minimum extension \hat{A} of A in $Q(A)$ so that \hat{A} is integral closed in $Q(A)$. We say that \hat{A} is normalization of A or the normalized ring of A .

Through the correspondence between an algebraic variety and a commutative ring, we have the well-known normalization theorem [9, p.5, p.68]

Theorem 2. For any irreducible algebraic curve $X \subset \mathbb{C}\mathbb{P}^2$, there exists a compact Riemann surface \tilde{X} and a holomorphic mapping $s : \tilde{X} \rightarrow \mathbb{C}\mathbb{P}^2$ such that $s(\tilde{X}) = X$ and s is injective on the inverse image of the set of smooth points of X . Further the Riemann surface is unique up to its isomorphism; if there are two Riemann surfaces \tilde{X} and \tilde{X}' given by normalizations of X , there is a biholomorphic from \tilde{X} to \tilde{X}' .

As illustrations of Theorem 2, we give three examples.

Example 3. $(x^3 - y^2)$, $R := \mathbb{C}[X, Y]/(X^3 - Y^2)$ is not normal because $\frac{Y}{X} \in \tilde{R} \setminus R \subset Q(R)$ due to $(\frac{Y}{X})^2 - X = 0$. Since $R \approx \mathbb{C}[t^2, t^3]$, the normalized ring is $\hat{R} = \mathbb{C}[t]$.

Example 4. ($y^3 = x^5 - 1$ and $w^3 = z - z^6$). Following Theorem 2, we consider the covering of a curve of $f(x, y) := y^3 - x^5 + 1$. Let us consider a homogeneous polynomial $F(X, Y, Z) := Y^3 Z^2 - X^5 + Z^5 \in \mathbb{C}[X, Y, Z]$. Around $Z \neq 0$, we have $F(X, Y, Z) = Z^5 \left(\frac{Y^3}{Z^3} - \frac{X^5}{Z^5} + 1 \right)$ and thus by regarding that $x = X/Z$ and $y = Y/Z$, we have $F(X, Y, Z) = Z^5 f(X/Z, Y/Z)$. $R_0 := \mathbb{C}[x, y]/(f(x, y))$ is a normal ring. On the other hand, around $Z = 0$ and $X \neq 0$, we have $F(X, Y, Z) = X^5 \left(\frac{Y^3 Z^2}{X^5} - 1 + \frac{Z^5}{X^5} \right)$, and then we obtain a polynomial, $g(w, z) = w^3 z^2 - 1 + z^5$ by regarding $w = Y/X$ and $z = Z/X$. However $R_\infty := \mathbb{C}[w, z]/(g(w, z))$ is not a normal ring. As a vector space, R_∞ is $\mathbb{C}1 + \mathbb{C}z + \mathbb{C}z^2 + \cdots + \mathbb{C}w + \mathbb{C}wz + \mathbb{C}wz^2 + \cdots + \mathbb{C}w^2 + \mathbb{C}w^2 z + \mathbb{C}w^2 z^2 + \cdots + \mathbb{C}w^3 + \mathbb{C}w^3 z$. We show that $q \in Q(R_\infty) \setminus R_\infty$ exists such that $q^n + a_1 q^{n-1} + \cdots + a_n = 0$ for certain $a_i \in R_\infty$. Noting $\frac{1}{1-z} g(w, z) = \frac{w^3 z^2}{1-z} + 1 + z + z^2 + z^3 + z^4 = 0 \in Q(R_\infty)$, we consider $q := \frac{w^3}{1-z} + \frac{1+z}{z^2} \in Q(R_\infty) \setminus R_\infty$, which is integral over R_∞ . By normalization, we define $\hat{w} := wz = y/x^2$. $\hat{R}_\infty := \mathbb{C}[\hat{w}, z]/(\hat{g}(w, z))$ is a normal ring, where $\hat{g}(\hat{w}, z) := \hat{w}^3 - z + z^6$. The minimal condition is obvious.

Example 5. (a space curve $y^3 = x^2(x^2 - 1)$ and $w^3 = x(x^2 - 1)^2$). Let us consider a polynomial $f(x, y) = y^3 - x^2(x^2 - 1)$ and show that $R_0 := \mathbb{C}[x, y]/(f(x, y))$ is not a normal ring. As a vector space, R_0 is $\mathbb{C}1 + \mathbb{C}x + \mathbb{C}x^2 + \cdots + \mathbb{C}y + \mathbb{C}yx + \mathbb{C}yx^2 + \cdots + \mathbb{C}y^2 + \mathbb{C}y^2 x + \mathbb{C}y^2 x^2 + \cdots$. We show that $w \in Q(R_0) \setminus R_0$ exists such that $w^n + a_1 w^{n-1} + \cdots + a_n = 0$ for certain a_i 's of R_0 . In other words, noting that $y \sim \sqrt[3]{x^2(x^2 - 1)}$ and $y^2 \sim x \sqrt[3]{x(x^2 - 1)}$, one of w is that $w := \frac{y^2}{x}$ which is integral over R_0 because $w^3 = \frac{y^6}{x^3} = x(x^2 - 1)^2$ or $w^3 - x(x^2 - 1)^2 = 0 \in R_0$. Let $g(x, w) = x(x^2 - 1)^2$. Noting the relations that $w = \frac{y^2}{x}$, $y = \frac{w^2}{x^2 - 1}$, and $wy = (x^2 - 1)x$, we have $\hat{R}_0 := \mathbb{C}[x, y, w]/(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$, as the normalized ring of R_0 , where $f_1(x, y, w) = y^2 - wx$, $f_2(x, y, w) = wy - (x^2 - 1)x$, and $f_3(x, y, w) = w^2 - y(x^2 - 1)$. The minimal condition is also obvious. This example corresponds to the special case of the affine $(3, 4, 5)$ space curve in this article. Due to Theorem 2, the corresponding Riemann surface uniquely exists up to an isomorphism.

3. A Curve (3,4,5)

Since H_2 generated by $\langle 3, 4, 5 \rangle$ is Weierstrass and is the simplest semigroup whose cardinality of the generators is greater than two, we consider a curve $C(H_2)$ explicitly in order to construct the sigma functions for $C(H_2)$ following the EEL construction.

Following Theorem 2, in order to construct a non-singular curve $X_2 = C(H_2)$, we consider two singular curves X_3 and X_4 generated by ∞ points and the zeroes of

$$f_{3,12}(x, y_4) := y_4^3 - k_4(x), \quad f_{4,15}(x, y_5) := y_5^3 - k_5(x)$$

where $k_4(x) := k_2(x)k_1(x)^2$, $k_5(x) := k_2(x)^2k_1(x)$, $k_2(x) := (x - b_1)(x - b_2) = x^2 + \lambda_1^{(2)}x + \lambda_2^{(2)}$, and $k_1(x) := (x - b_0) = x + \lambda_1^{(1)}$ for finite $b_a \in \mathbb{C}$ ($a = 1, 2, 3$) which is distinct from each other. Let us consider commutative rings $R_3 := \mathbb{C}[x, y_4]/(f_{3,12}(x, y_4))$ and $R_4 := \mathbb{C}[x, y_5]/(f_{4,15}(x, y_5))$ related to X_3 and X_4 respectively. These genera of the semigroups associated with their Weierstrass non-gap sequences at ∞ -points are three and four respectively, though the geometric genera are not. Following the normalization in Section 2, we normalize R_3 and R_4 . Since in terms of the language of the commutative algebra [16], $\frac{y_4^2}{(x - b_0)}$ is integral over R_3 in $Q(R_3)$ and $\frac{y_5^2}{(x - b_0)(x - b_2)}$ is integral over R_4 in $Q(R_4)$, R_3 and R_4 are not normal rings. Thus we will normalise them in $\mathbb{C}[x, y_4, y_5]$ in the meaning of the commutative algebra [16] (see Example 5 in §2.2).

For the zeroes of $f_{3,12}(x, y_4)$ and $f_{3,15}(x, y_4)$, we could have the relations,

$$y_4y_5 = k_2(x)k_1(x), \quad y_5 = \frac{y_4^2}{(x - b_0)}, \quad y_4 = \frac{y_5^2}{(x - b_1)(x - b_2)}. \quad (2)$$

Here for the primitive root ζ_3 ($\zeta_3^3 = 1, \zeta_3 \neq 1$), ζ_3 acts on X_3 and X_4 respectively. The first relation is chosen in the possibilities $y_4y_5 = \zeta_3^i k_2(x)k_1(x)$ $i = 0, 1, 2$.

As a normalization of these singular curves, we have the commutative ring,

$$R_2 \equiv R := \mathbb{C}[x, y_4, y_5]/(f_8, f_9, f_{10})$$

and $X_2 := \text{Spec } R$. Here we define $f_8, f_9, f_{10} \in \mathbb{C}[x, y_4, y_5]$ by

$$f_8 = y_4^2 - y_5k_1(x), \quad f_9 = y_4y_5 - k_2(x)k_1(x), \quad f_{10} = y_5^2 - y_4k_2(x)$$

which are also regarded as the 2×2 minors of $\begin{vmatrix} k_2(x) & y_4 & y_5 \\ y_4 & y_5 & k_3(x) \end{vmatrix}$. Here ζ_3 acts on X_2 by $\hat{\zeta}_3(x, y_4, y_5) = (x, \zeta_3y_4, \zeta_3^2y_5)$.

Let X be the Riemann surface which is naturally obtained as an extension of X_2 as mentioned in Theorem 2, i.e., $X = X_2 \cup \{\infty\}$ as a set. It is noted that when x diverges, y_4 and y_5 also diverge vice versa. Thus the infinity point ∞ uniquely exists in X . \mathbb{G}_m acts on R by setting $g_m^{-3}x, g_m^{-a}y_a$ for $x, y_a, g_m \in \mathbb{G}_m$ and $a = 4, 5$. By Nagata's Jacobi-method [16], it can be proved that X is non-singular.

Though they do not explicitly appear, we may also implicitly consider parametrizations of y_4 and y_5 by $y_4 = w_2 w_1^2$, and $y_5 = w_2^2 w_1$, where $w_1^3 = k_1$ and $w_2^3 = k_2$. When we consider $\tilde{R} := \mathbb{C}[x, w_1, w_2]/(w_1^3 - k_1(x), w_2^3 - k_2(x))$, it is related to a natural covering of X .

3.1. The Weierstrass Gap and Holomorphic One Forms

The Weierstrass gap sequences at ∞ are given in Table 1. For the local parameter t_∞ at ∞ , we have

$$x = \frac{1}{t_\infty^3}, \quad y_4 = \frac{1}{t_\infty^4}(1 + d_{\geq}(t_\infty)), \quad y_5 = \frac{1}{t_\infty^5}(1 + d_{\geq}(t_\infty)). \tag{3}$$

Here for a given local parameter t at some P in X , the series of t , whose orders of zero at P are greater than ℓ or equal to ℓ , is denoted by $d_{\geq}(t^\ell)$. $H(\infty)$ in (1) is $H(3, 4, 5)$ as Pinkham considered $(3, 4, 5)$ curve as the simplest example of the numerical semigroup $H(3, 4, 5)$ [23, Section 14]. Its monomial curve is defined by, $Z_4^2 = Z_3 Z_5$, $Z_4 Z_5 = Z_3^5$, $Z_5^2 = Z_3^3 Z_4$, or the 2×2 minor of $\begin{vmatrix} Z_3 & Z_4 & Z_5 \\ Z_4 & Z_5 & Z_3^2 \end{vmatrix} \cdot Z_3$, Z_4 and Z_5 correspond to $\frac{1}{x} \frac{1}{y_4}$ and $\frac{1}{y_5}$ respectively and these relations correspond to (2).

Table 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	8	9	10
X_3	1	-	-	x	y_4	-	x^2	xy_4	y_4^2	x^3	$x^2 y_4$	xy_4^2	x^4	$x^3 y_4$	$x^2 y_4^2$	x^5	$x^4 y_4$
X_4	1	-	-	x	-	y_5	x^2	-	xy_5	x^3	y_5^2	$x^2 y_5$	x^4	$x^2 y_5$	xy_5^2	x^5	$x^2 y_5^2$
X_2	1	-	-	x	y_4	y_5	x^2	xy_4	xy_5	$y_4 y_5$	$x^2 y_4$	$x^2 y_5$	x^4	$x^3 y_4$	$x^3 y_5$	x^5	$x^4 y_4$

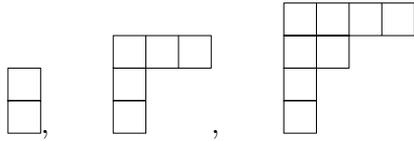
There we define $\phi_i^{(g)}$ as a non-gap monomial in R_g for $g = 2, 3, 4$ and e.g., $\phi_0^{(2)} = 1$, $\phi_1^{(2)} = x$, $\phi_2^{(2)} = y_4$, $\phi_3^{(2)} = y_5$, $\phi_4^{(2)} = x^2, \dots$ and $\phi_0^{(3)} = 1$, $\phi_1^{(3)} = x$, $\phi_2^{(3)} = y_4$, $\phi_3^{(3)} = x^2$, $\phi_4^{(3)} = xy_4, \dots$. We introduce the weight $N^{(g)}(n)$ by letting $N^{(g)}(n) := -\text{wt}(\phi_n^{(g)})$, where $\text{wt}()$ is the degree of divisor at ∞ of each curve X 's. It is noted that H_2 is identical to $\{N^{(2)}(n) ; n = 0, 1, 2, \dots\}$. For later convenience, we also introduce $\phi_{H^1_i} \in R$ ($i = 1, 2, 3, \dots$) by $\phi_{H^1_0} := y_4$, $\phi_{H^1_1} := y_5$

$$\phi_{H^1_2} := xy_4, \phi_{H^1_3} := xy_5, \text{ for } i > 3, \phi_{H^1_i} := \begin{cases} x^{(i-4)/3} y_4 y_5 & i \equiv 1 \pmod{3}, \\ x^{(i+1)/3} y_4 & i \equiv 2 \pmod{3}, \\ x^{i/3} y_5 & i \equiv 0 \pmod{3}. \end{cases}$$

We also define the weight $N_{H^1}(n)$ by $N_{H^1}(n) := -\text{wt}(\phi_{H^1 n})$, $N_{H^1}(0) = 4$, $N_{H^1}(1) = 5$, and $N_{H^1}(n) = n + 5$ for $n \geq 2$. By letting

$$\begin{aligned} \Lambda_i^{(2)} &:= N_{H^1}(2) - N_{H^1}(i-1) + i - 3, \\ \Lambda_i^{(g)} &:= N^{(g)}(g) - N^{(g)}(i-1) - g + i - 1, \quad (g = 3, 4) \end{aligned}$$

the corresponding Young diagrams, $\Lambda \equiv \Lambda^{(2)} := (\Lambda_1, \Lambda_2) = (1, 1)$, $\Lambda^{(3)} := (\Lambda_1^{(3)}, \Lambda_2^{(3)}, \Lambda_3^{(3)}) = (3, 1, 1)$ and $\Lambda^{(4)} := (\Lambda_1^{(4)}, \Lambda_2^{(4)}, \Lambda_3^{(4)}, \Lambda_4^{(4)}) = (4, 2, 1, 1)$ are given respectively as



The Young diagram Λ is not symmetric, whereas ${}^t\Lambda^{(3)} = \Lambda^{(3)}$ and ${}^t\Lambda^{(4)} = \Lambda^{(4)}$. Then the following propositions are obvious

Proposition 6. *The b -small bases of the holomorphic one forms over X are expressed by $\nu_1^I = \frac{dx}{3y_5}$ and $\nu_2^I = \frac{dx}{3y_4}$ or $\nu_i^I := \frac{\phi_{H^1 i-1} dx}{3y_4 y_5}$, $i = 1, 2$.*

We note their divisors and linear equivalences for $B_a := (b_a, 0, 0)$, $a = 0, 1, 2$, $(\nu_1^I) = \infty + B_0 \sim (dx/y_5^2) = 2(3\infty - B_1 - B_2)$ and $(\nu_1^I) \sim (\nu_2^I) = B_1 + B_2 \sim (dx/y_4^2) = 2(2\infty - B_0) = 2(\infty + (\infty - B_0))$.

Proposition 7. $\sum_{i=0}^n a_i \tilde{\nu}_i$ belongs to $H^1(X \setminus \infty, \mathcal{O}_X)$, where $\tilde{\nu}_i := \frac{\phi_{H^1 i} dx}{3y_4 y_5}$ and the order of the singularity of $(\tilde{\nu}_i)$ at ∞ is given by $N_{H^1}(n) - 5$.

Lemma 8. $a_0 \frac{dx}{y_4 y_5} + a_1 \frac{x dx}{y_4 y_5} + a_2 \frac{x^2 dx}{y_4 y_5}$ is not holomorphic one form over X if a_i does not vanish.

Proof: For $n < 3$, every $\sum_{i=0}^n a_i \frac{x^i dx}{y_4 y_5}$ has singularities at points in $X \setminus \infty$. ■

We choose the bases α_i, β_j ($1 \leq i, j \leq 2$) of $H_1(X, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$, and we denote the period matrices by $[\omega' \ \omega''] = \frac{1}{2} \left[\int_{\alpha_i} \nu_j^I \quad \int_{\beta_i} \nu_j^I \right]_{i,j=1,2}$. Let Π_2 be a lattice generated by ω' and ω'' . For a point $P \in X$, the abelian map $\hat{u}_o : X \rightarrow \mathbb{C}^2$ is defined by

$$\hat{u}_o(P) = \int_{\infty}^P \nu^I \in \mathbb{C}^2$$

and for a point $(P_1, \dots, P_k) \in S^k X$, i.e., the k -th symmetric product of X , the shifted abelian map $\hat{u} : S^k X \rightarrow \mathbb{C}^2$ by

$$\hat{u}(P_1, \dots, P_k) := \hat{u}_o(P_1, \dots, P_k) + \hat{u}_o(B_0)$$

where $\hat{u}_o(P_1, \dots, P_k) := \sum_{i=1}^k \hat{u}_o(P_i)$. Then we define the Jacobian \mathcal{J}_2 and its subvariety \mathcal{W}^k , $k = 0, 1, 2$ by

$$\kappa : \mathbb{C}^2 \rightarrow \mathcal{J}_2 = \mathbb{C}^2 / \Pi_2 = \mathcal{W}^2, \quad \mathcal{W}^k := \kappa \hat{u}(S^k X)$$

respectively. Further the singular locus of $S^2 X$ is denoted by $S_1^2 X$ as in [20].

For a point $(P_1, P_2) \in S^2 X$ around the infinity point, by letting their local parameters $t_{\infty,1}$ and $t_{\infty,2}$, $u \equiv {}^t(u_1, u_2) := \hat{u}_o(P_1, P_2)$ is given by $u_1 = \frac{1}{2}(t_{\infty,1}^2 + t_{\infty,2}^2)(1 + d_{>0}(t_{\infty,1}, t_{\infty,2}))$, $u_2 = (t_{\infty,1} + t_{\infty,2})(1 + d_{>0}(t_{\infty,1}, t_{\infty,2}))$, where $d_{\geq}(t_1, t_2)$ is a natural extension of $d_{\geq}(t)$.

3.2. Differentials of the Second and the Third Kinds

Following the EEL-construction [5] for a (n, s) curve, we give here an algebraic representation of a differential form which is equal to the fundamental normalized differential of the second kind in [7, Corollary 2.6], up to a tensor of holomorphic one forms

Definition 9. A two-form $\Omega(P_1, P_2)$ on $X \times X$ is called a fundamental differential of the second kind if it is symmetric, $\Omega(P_1, P_2) = \Omega(P_2, P_1)$, it has its only pole (of second order) along the diagonal of $X \times X$, and in the vicinity of each point (P_1, P_2) is expanded in power series as

$$\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t_{P_2})^2} + d_{\geq}(1) \right) dt_{P_1} \otimes dt_{P_2}, \quad \text{as } P_1 \rightarrow P_2 \quad (4)$$

where t_P is a local coordinate at a point $P \in X$.

Here we use the convention that for $P_a \in X$, P_a is represented by $(x_a, y_{4,a}, y_{5,a})$ or $(x_{P_a}, y_{4,P_a}, y_{5,P_a})$ and for $P \in X$, P is expressed by (x, y_4, y_5) . Then the following propositions holds.

Proposition 10. By letting

$$\Sigma(P, Q) := \frac{y_{4,P}y_{5,P} + y_{4,P}y_{5,Q} + y_{4,Q}y_{5,P}}{(x_P - x_Q)^3 y_{4,P}y_{5,P}} dx_P$$

$\Sigma(P, Q)$ has the properties

- 1) $\Sigma(P, Q)$ as a function of P is singular at $Q = (x_Q, y_{4,Q}, y_{5,Q})$ and ∞ , and vanishes at $\hat{\zeta}_3^\ell(Q) = (x_Q, \zeta_3^\ell y_{4,Q}, \zeta_3^{2\ell} y_{5,Q})$, $\ell = 1, 2$, and
- 2) $\Sigma(P, Q)$ as a function of Q is singular at P and at ∞ .

Proof: Direct computations provide the results. ■

Proposition 11. *There exist differentials $\nu_j^{II} = \nu_j^{II}(x, y_4, y_5)$, $j = 1, 2$ of the second kind such that they have their only pole at ∞ and satisfy the relation*

$$d_Q \Sigma(P, Q) - d_P \Sigma(Q, P) = \sum_{i=1}^2 \left(\nu_i^I(Q) \otimes \nu_i^{II}(P) - \nu_i^I(P) \otimes \nu_i^{II}(Q) \right) \quad (5)$$

where $d_Q \Sigma(P, Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_{4,P} y_{5,P} + y_{4,P} y_{5,Q} + y_{4,Q} y_{5,P}}{(x_P - x_Q) 3y_{4,P} y_{5,P}}$.

The differentials $\{\nu_1^{II}, \nu_2^{II}\}$ are determined modulo the \mathbb{C} -linear space spanned by $\langle \nu_j^I \rangle_{j=1,2}$ and we fix

$$\{\nu_1^{II}, \nu_2^{II}\} = \left\{ \frac{-\left(2x + \lambda_1^{(2)}\right) dx}{3y_4}, \frac{-x dx}{3y_5} \right\}$$

as their representative.

Proof: $\frac{\partial}{\partial x_Q} \frac{y_{4,P} y_{5,P} + y_{4,P} y_{5,Q} + y_{4,Q} y_{5,P}}{(x_P - x_Q) 3y_{4,P} y_{5,P}} dx_P$ is equal to

$$\frac{1}{(x_P - x_Q) 9y_{4,P} y_{5,P} y_{4,Q} y_{5,Q}} \left[\frac{3(y_{4,P} y_{5,P} + y_{4,P} y_{5,Q} + y_{4,Q} y_{5,P}) y_{4,Q} y_{5,Q}}{(x_P - x_Q)} + \left(y_{4,P} \frac{y_{4,Q}}{y_{5,Q}} (2k_{2,Q} k'_{2,Q} k_{1,Q} + k_{2,Q}^2 k'_{1,Q}) + y_{5,P} \frac{y_{5,Q}}{y_{4,Q}} (2k_{2,Q} k_{1,Q} k'_{1,Q} + k_{2,Q}^2 k'_{1,Q}) \right) \right].$$

Here $k_{a,P} = k_a(x_P)$ and $k'_{a,P} = dk_a(x_P)/dx_P$. We have

$$\begin{aligned} & \frac{\partial}{\partial x_Q} \frac{y_{4,P} y_{5,P} + y_{4,P} y_{5,Q} + y_{4,Q} y_{5,P}}{(x_P - x_Q) 3y_{4,P} y_{5,P}} - \frac{\partial}{\partial x_P} \frac{y_{4,Q} y_{5,Q} + y_{4,Q} y_{5,P} + y_{4,P} y_{5,Q}}{(x_Q - x_P) 3y_{4,Q} y_{5,Q}} \\ &= \frac{1}{(x_P - x_Q) 9y_{4,P} y_{5,P} y_{4,Q} y_{5,Q}} (B_2(P, Q) - B_2(Q, P)) \end{aligned}$$

where $B_2(P, Q) = y_{4,P} y_{5,Q} (2x_Q + \lambda_1^{(2)} - x_P)$. Then we obtain the statements. ■

Corollary 12. 1) The one form, $\Pi_{P_1}^{P_2}(P) := \Sigma(P, P_1) - \Sigma(P, P_2)$, is a differential of the third kind, whose only (first-order) poles are $P = P_1$ and $P = P_2$, and residues $+1$ and -1 respectively.

2) $\Omega(P_1, P_2)$ is defined by $d_{P_2}\Sigma(P_1, P_2) + \sum_{i=1}^2 \nu_i^I(P_1) \otimes \nu_i^{II}(P_2)$

$$\Omega(P_1, P_2) = \frac{F(P_1, P_2)dx_1 \otimes dx_2}{(x_{P_1} - x_{P_2})^2 g_{4,P_1} y_{5,P_1} y_{4,P_2} y_{5,P_2}}$$

where F is an element of $R \otimes R$.

Proof: Direct computations give the claims. ■

Lemma 13. We have $\lim_{P_1 \rightarrow \infty} \frac{F(P_1, P_2)}{\phi_{H^1_1}(P_1)(x_{P_1} - x_{P_2})^2} = \phi_{H^1_2}(P_2) = x_{P_2} y_{4,P_2}$.

Proof: B_2 which appears in the proof of Proposition 11 ensures the result. ■

For later convenience we introduce the quantity, $\Omega_{Q_1, Q_2}^{P_1, P_2} := \int_{P_2}^{P_1} \int_{Q_2}^{Q_1} \Omega(P, Q)$,

$$\Omega_{Q_1, Q_2}^{P_1, P_2} = \int_{P_2}^{P_1} (\Sigma(P, Q_1) - \Sigma(P, Q_2)) + \sum_{i=1}^4 \int_{P_2}^{P_1} \nu_i^I(P) \int_{Q_2}^{Q_1} \nu_i^{II}(P). \quad (6)$$

4. The sigma Function for (3, 4, 5) Curve

4.1. Generalized Legendre Relation

Corresponding to the complete integral of the first kind, we define the complete integral of the second kind

$$[\eta' \ \eta''] := \frac{1}{2} \left[\int_{\alpha_i} \nu_j^{II} \quad \int_{\beta_i} \nu_j^{II} \right]_{i,j=1,2}.$$

Let τ_{Q_1, Q_2} be the normalized differential of the third kind such that τ_{Q_1, Q_2} has residues $+1$ and -1 at Q_1 and Q_2 respectively, is regular everywhere else, and is normalized, $\int_{\alpha_i} \tau_{P, Q} = 0$ for $i = 1, 2$ [7, p.4]. The following Lemma corresponding to Corollary 2.6 (ii) in [7] holds

Lemma 14. *By letting $\gamma = \omega'^{-1}\eta'$, we have*

$$\Omega_{Q_1, Q_2}^{P_1, P_2} = \int_{P_2}^{P_1} \tau_{Q_1, Q_2} + \sum_{i,j=1}^2 \gamma_{ij} \int_{P_2}^{P_1} \nu_i^I \int_{Q_2}^{Q_1} \nu_j^I.$$

Proof: The same as [20, I: Lemma 4.1]. ■

The following Proposition provides a symplectic structure in the Jacobian \mathcal{J}_2 , known as *generalized Legendre relation* [3, 4, 20]

Proposition 15. $M \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} {}^t M = 2\pi\sqrt{-1} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ for $M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}$.

Proof: The same as [20, I: Proposition 4.2]. ■

4.2. The σ Function

Due to the Riemann relations [7], $\text{Im}(\omega'^{-1}\omega'')$ is positive definite. Theorem 1.1 in [7] gives $\delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \in \left(\frac{\mathbb{Z}}{2}\right)^4$ be the theta characteristic which is equal to the Riemann constant ξ_R and the period matrix $[2\omega' \ 2\omega'']$. We note that $\xi_R = \hat{u}(P_R)$ for a point $P_R \in X$ satisfying $2P_R + 2B_0 - 4\infty \sim 0$. We define an entire function of (a column-vector) $u = {}^t(u_1, u_2) \in \mathbb{C}^2$

$$\sigma(u) = ce^{-\frac{1}{2} {}^t u \eta' \omega'^{-1} u} \sum_{n \in \mathbb{Z}^2} e^{\left[\pi\sqrt{-1} \left\{ {}^t(n+\delta'') \omega'^{-1} \omega''(n+\delta'') + {}^t(n+\delta'')(\omega'^{-1}u+\delta') \right\} \right]}$$

where c is a certain constant as in (7).

For a given $u \in \mathbb{C}^2$, we introduce u' and u'' in \mathbb{R}^2 so that $u = 2\omega'u' + 2\omega''u''$.

Proposition 16. *For $u, v \in \mathbb{C}^2$, $\ell (= 2\omega'\ell' + 2\omega''\ell'') \in \Pi_2$, and introducing $L(u, v) := 2 {}^t u(\eta'v' + \eta''v'')$, $\chi(\ell) := \exp[\pi\sqrt{-1}(2({}^t\ell''\delta' - {}^t\ell'\delta'') + {}^t\ell'\ell'')]$, we have the translational relation*

$$\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2}\ell, \ell))\chi(\ell).$$

Proof: The same as [20, I: Proposition 4.3]. ■

The vanishing locus of σ is simply given by $\Theta^1 := (\mathcal{W}^1 \cup [-1]\mathcal{W}^1) = \mathcal{W}^1$.

4.3. The Riemann Fundamental Relation

As in [20, I: Proposition 4.4], we have the Riemann fundamental relation

Proposition 17. For $(P, Q, P_i, P'_i) \in X^2 \times (S^2(X) \setminus S_1^2(X)) \times (S^2(X) \setminus S_1^2(X))$

$$\exp \left(\sum_{i,j=1}^2 \Omega_{P_i, P'_j}^{P, Q} \right) = \frac{\sigma(\hat{u}_o(P) - \hat{u}(P_1, P_2))\sigma(\hat{u}_o(Q) - \hat{u}(P'_1, P'_2))}{\sigma((\hat{u}_o(Q) - \hat{u}(P_1, P_2))\sigma(\hat{u}_o(P) - \hat{u}(P'_1, P'_2)))}.$$

Using the differential identity, $\sum_{i,j=1}^2 \phi_{H^1 i-1}(P'_i)\phi_{H^1 j-1}(P'_j) \frac{\partial^2}{\partial \hat{u}_i(P'_1)\partial \hat{u}_j(P'_2)}$
 $= 9y_{4,P'_1}y_{5,P'_1}y_{4,P'_2}y_{5,P'_2} \frac{\partial^2}{\partial x'_1 \partial x'_2}$, taking logarithm of both sides of the relation and differentiating them along $P'_1 = P$ and $P'_2 = P_a$, we have the differential expressions of the relation, as mentioned in [20, I: Proposition 4.5]

Proposition 18. For $(P, P_1, P_2) \in X \times S^2(X) \setminus S_1^2(X)$ and $u := \hat{u}(P_1, P_2)$, the equality

$$\sum_{i,j=1}^2 \wp_{i,j}(\hat{u}_o(P) - u) \phi_{H^1 i-1}(P)\phi_{H^1 j-1}(P_a) = \frac{F(P, P_a)}{(x - x_a)^2}$$

holds for every $a = 1, 2$, where we set

$$\wp_{ij}(u) := -\frac{\sigma_i(u)\sigma_j(u) - \sigma(u)\sigma_{ij}(u)}{\sigma(u)^2} \equiv -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u).$$

4.4. Jacobi Inversion Formulae

As in [20], we introduce meromorphic functions on the curve X

Definition 19. For $P, P_1, \dots, P_n \in (X \setminus \infty) \times S^n(X \setminus \infty)$, $n = 1, 2$, we define

$$\begin{aligned} \mu_1(P; P_1) &:= y_5 - \frac{y_{5,1}}{y_{4,1}} y_4 \\ \mu_2(P; P_1, P_2) &:= xy_4 - \frac{y_{4,1}x_2y_{4,2} - y_{4,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}} y_5 + \frac{y_{5,1}x_2y_{4,2} - y_{5,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}} y_4. \end{aligned}$$

We note that μ_n for X is characterized by the condition on the polynomial $\mu_n = \sum_{i=0}^n a_i \phi_{H^1 i}(P)$, $a_i \in \mathbb{C}$ and $a_n = 1$, which has a zero at each point P_i and

has the smallest possible order such that it multiplied by $dx/3y_4y_5$ belongs to $H^1(X \setminus \infty, \mathcal{O}_X)$. For given P_1 , the solution of $\mu_1(P; P_1) = 0$ corresponds to a point $Q_1 = [-1]P_1$ with B_a , $a = 0, 1, 2$, and for given P_1 and P_2 , the solution of $\mu_2(P; P_1, P_2) = 0$ gives two points Q_1, Q_2 with B_a , $a = 0, 1, 2$ such that $Q_1 + Q_2 = [-1](P_1 + P_2)$. Here we use $B_0 + B_1 + B_2 - 3\infty \sim 2B_0 - 2\infty$.

Using μ_n , we have our main theorem in this article

Theorem 20. 1) For $(P, P_1, P_2) \in X \times (S^2(X) \setminus S_1^2(X))$, we have

$$1-1) \mu_2(P; P_1, P_2) = xy_4 - \wp_{22}(\hat{u}(P_1, P_2))y_4 + \wp_{21}(\hat{u}(P_1, P_2))y_5.$$

$$1-2) \wp_{22}(\hat{u}(P_1, P_2)) = \frac{y_{4,1}x_2y_{4,2} - y_{4,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}$$

$$\wp_{21}(\hat{u}(P_1, P_2)) = \frac{y_{5,1}x_2y_{4,2} - y_{5,2}x_1y_{4,1}}{y_{4,1}y_{4,2} - y_{4,2}y_{4,1}}.$$

2) For $(P, P_1) \in X \times (X \setminus S_1^1(X))$ and $u = \hat{u}(P_1) \in \kappa^{-1}(\mathcal{W}^1)$

$$\mu_1(P; P_1) = y_5 - \frac{\sigma_1(u)}{\sigma_2(u)}y_4, \quad \text{and} \quad \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{y_5}{y_4}.$$

Proof: 1) is the same as [20, I: Proposition 4.6]. As in [20, I: Theorem 5.1], by considering $\lim_{P_2 \rightarrow \infty} \frac{\wp_{21}(\hat{u}(P_1, P_2))}{\wp_{22}(\hat{u}(P_1, P_2))}$, we have the second result. ■

Following the statement by Buchstaber, Leykin and Enolskii, Nakayashiki showed that the leading of the sigma function for (r, s) curve is expressed by Schur function [21]. Noting (3) and degrees of u , the above Jacobi inversion formulae gives

$$\sigma(u) = \frac{1}{2}u_2^2 - u_1 + \sum_{|\alpha|>2} a_\alpha u^\alpha$$

where $a_\alpha \in \mathbb{Q}[b_1, \dots, b_5]$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $u^\alpha = u_1^{\alpha_1}u_2^{\alpha_2}$. The prefactor c is determined by this relation. Since for a Young diagram Λ , S_Λ and s_Λ are the Schur functions defined by

$$S_\Lambda(T_1, T_2) = t_1t_2 = \frac{1}{2}T_1^2 - T_2 \tag{7}$$

where $T_1 := t_1 + t_2$ and $T_2 := \frac{1}{2}(t_1^2 + t_2^2)$, we have

$$\sigma(u) = S_\Lambda(u_1, u_2) + \sum_{|\alpha|>2} a_\alpha u^\alpha.$$

Remark 21. We showed that the EEL construction works well even for a space curve, and the sigma function associated with the curve is naturally defined. Since this construction is very natural, this study sheds a new light on the way to construction of the sigma functions for space curves. We conjectured that the EEL construction could be applied to every space curve if it is Weierstrass.

As an interesting example of a space curve, we will give a comment on a problem as follows, for which we started to study sigma functions for affine space curves.

McKay considers a relation between dispersionless KP hierarchy and the replicable functions in order to obtain a further profound interpretation of the moonshine phenomena of Monster group [22]. He conjectured that it might be related to the quantised elastica [18, 19]. By studying a relation between a replicable function and an algebraic curve associated with elastica, Matsutani found that a semigroup H_{12} generated by $M_{12} := \langle 6, 13, 14, 15, 16 \rangle$ has gap sequence, $L(H_{12}) = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}$, which is identical to the Norton number, $N_{12} := \{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}$ by exchanging 10 and 19. The Norton number plays the essential role in the moonshine phenomena for the Monster group [22]. The replicable function is given as an element of $\mathbb{Q}[a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{11}, a_{17}, a_{19}, a_{23}][[t]]$. The replicable function is a generalization of the elliptic J -function, which causes the moonshine phenomena of the Monster group.

After then, Komeda proved that H_{12} is the Weierstrass semigroup and gave the fundamental relations Propositions 23 as given in Appendix, which is reported more precisely in [13]. Then we applied the EEL-construction to the curve and obtain a sigma function for a Jacobi variety \mathcal{J}_{12} for $C(H_{12})$ [13]. Since the Jacobi variety \mathcal{J}_{12} is given as 12-dimensional complex torus whose real dimension is 24, it might remind us of Witten conjecture associated with Monster group problem [8]; Witten conjectured that a 24 dimensional manifold exists such that the Monster group acts on it via Weierstrass sigma function.

Appendix (by J. Komeda). Weierstrass Properties of $\langle 6, 13, 14, 15, 16 \rangle$

The proofs of these propositions are given in the article [13]. Here we show only the sketch of the first one because the second one is not difficult.

Proposition 22. *The numerical semigroup $\langle 6, 13, 14, 15, 16 \rangle$ is Weierstrass.*

Proof: Let (C, P) be a pointed curve with $H(P) = \langle 3, 7, 8 \rangle$. Then

$$2 = h^0(4P) = 4 + 1 - 4 + h^0(K - 4P) = 1 + h^0(K - 4P)$$

which implies that $K - 4P \sim P_1 + P_2$ for some points P_1 and $P_2 \in C$. Here K is a canonical divisor on C . Moreover,

$$2 = h^0(5P) = 5 + 1 - 4 + h^0(K - 5P) = 2 + h^0(K - 5P)$$

which implies that $h^0(K - 5P) = 0$. Hence, we get $P_i \neq P$ for $i = 1, 2$. Thus, $K \sim 4P + P_1 + P_2$ with $P_i \neq P$ for $i = 1, 2$. We set $D = 7P - P_1 - P_2$. Then $\deg(2D - P) = 9 = 2 \times 4 + 1$, which implies that the complete linear system $|2D - P|$ is very ample, hence base-point free. Therefore, $2D \sim P + Q_1 + \dots + Q_9$ (= a reduced divisor). Let \mathcal{L} be the invertible sheaf $\mathcal{O}_C(-D)$ on C and ϕ an isomorphism $\mathcal{L}^{\otimes 2} \approx \mathcal{O}_C(-P - Q_1 - \dots - Q_9) \subset \mathcal{O}_C$. Then the vector bundle $\mathcal{O}_C \oplus \mathcal{L}$ has an \mathcal{O}_C -algebra structure through ϕ . The canonical morphism $\pi : \tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \rightarrow C$, is a double covering. Its branch locus of π is $\{P, Q_1, \dots, Q_9\}$. Let \tilde{P} be the ramification point of π over P . Then it can be showed that $H(\tilde{P}) = \langle 6, 13, 14, 15, 16 \rangle$ using the formula, $h^0(2n\tilde{P}) = h^0(nP) + h^0(nP - D)$ for any non-negative integer n .

By considering $h^0(2n\tilde{P})$ for $n = 3, 4, 5, 6, 7, 8, 9$, we show $H(\tilde{P}) = \langle 6, 13, 14, 15, 16 \rangle$. ■

Proposition 23. *Let B_{12} a monomial ring which is given by $k[t^a]_{a \in M_{12}}$ for the numerical semigroup H_{12} . For a k -algebra homomorphism*

$$\varphi_{12} : k[Z] := k[Z_6, Z_{13}, Z_{14}, Z_{15}, Z_{16}] \rightarrow k[t^a]_{a \in M_{12}}$$

where Z_a is the weight of $a = 6, 13, 14, 15, 16$, the kernel of φ_{12} is generated by the following relations $f_{12,b}^{(Z)}$ ($b = 1, \dots, 9$)

$$\begin{aligned} f_{12,1}^{(Z)} &= Z_{13}^2 - Z_6^2 Z_{14}, & f_{12,2}^{(Z)} &= Z_{13} Z_{14} - Z_6^2 Z_{15}, & f_{12,3}^{(Z)} &= Z_{14}^2 - Z_{13} Z_{15} \\ f_{12,4}^{(Z)} &= Z_{14}^2 - Z_6^2 Z_{16}, & f_{12,5}^{(Z)} &= Z_{13} Z_{16} - Z_{14} Z_{15}, & f_{12,6}^{(Z)} &= Z_{15}^2 - Z_6^5 \\ f_{12,7}^{(Z)} &= Z_{14} Z_{16} - Z_6^5, & f_{12,8}^{(Z)} &= Z_{15} Z_{16} - Z_6^3 Z_{13}, & f_{12,9}^{(Z)} &= Z_{16}^2 - Z_6^3 Z_{14}. \end{aligned}$$

Acknowledgements

One of the authors (S.M.) would like to thank John McKay for posing his attention to the Norton problem and the encouragement. This work started in a seminar at Yokohama National University 2008, and was stimulated by the International Conference at HWK 2011. S. M. is grateful also to Kenichi Tamano, Norio Konno, Claus Lämmerzahl, Jutta Kunz, and Victor Enolskii. He is also most grateful to Emma Previato, Yoshihiro Ōnishi and Yuji Kodama for crucial discussions.

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