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COMPLEX REPRESENTATION THEORY OF THE ELECTROMAGNETIC FIELD

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Abstract. A concise discussion of the three-dimensional irreducible (1,0) and (0,1) representations of the restricted Lorentz group and their application to the description of the electromagnetic field is given. It is shown that a mass term is in conflict with relativistic invariance of a formalism using electric and magnetic fields only, contrasting the case of the two-component Majorana field equations. An important difference between the Dirac equation and the Dirac form of Maxwell's equations is highlighted by considering the coupling of the electromagnetic field to the electric current.

1. Introduction

Starting from Lorentz symmetry as the key property of Minkowski space-time in the framework of the special theory of relativity, we may observe that the classical electric and magnetic field can be combined into a single *photon wave function* [3]

$$\mathbf{\Psi} = \frac{1}{\sqrt{2}}(\mathbf{E} + \mathrm{i}\mathbf{B}), \qquad \mathrm{i}^2 = -1 \tag{1}$$

where the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are three-component real fields which, for the sake of convenience, shall be written in column matrix form

$$\mathbf{E}(x) = \begin{pmatrix} E_1(x) \\ E_2(x) \\ E_3(x) \end{pmatrix}, \qquad \mathbf{B}(x) = \begin{pmatrix} B_1(x) \\ B_2(x) \\ B_3(x) \end{pmatrix}, \qquad x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{2}$$

in the following. The column vector x denotes Cartesian space-time coordinates $x = (x^0 = ct, x^1, x^2, x^3)^{\mathrm{T}} = (x^0, \mathbf{x}^{\mathrm{T}})^{\mathrm{T}} = (x_0, -x_1, -x_2, -x_3)^{\mathrm{T}}$ in an orthonormal standard coordinate system in Minkowski space. Throughout the paper, we will choose a system of units where the speed of light is $c = 1$.

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Hence, the Maxwell-Faraday equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\mathrm{curl}\, \mathbf{E} \tag{3}$$

and Ampère's circuital law *in vacuo*

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} \tag{4}$$

can be cast into one single Lorentz-covariant equation of motion

$$\frac{\partial \mathbf{\Psi}}{\partial t} = -\mathrm{i} \cdot \nabla \times \mathbf{\Psi}\,. \tag{5}$$

This was already recognized in lectures by Riemann in the nineteenth century [8]. A short related note can also be found the lecture notes of Sommerfeld [7].

2. Field Equations

Taking the divergence of equation (5)

$$\nabla \cdot \dot{\mathbf{\Psi}} = -\mathrm{i} \cdot \nabla \cdot (\nabla \times \mathbf{\Psi}) = 0 \tag{6}$$

readily shows that the divergence of the electric and magnetic field is conserved. Therefore, if the analytic condition $\mathrm{div}\, \mathbf{E} = \mathrm{div}\, \mathbf{B} = 0$ holds due to the absence of electric or magnetic charges on a space-like slice of space-time, it holds everywhere.

The presence of electric charges and the absence of magnetic charges breaks the gauge symmetry of equation (5)

$$\mathbf{\Psi} \mapsto \mathrm{e}^{\mathrm{i}\alpha} \mathbf{\Psi}\,, \qquad \alpha \in \mathbb{R}\,. \tag{7}$$

Introducing the antisymmetric matrices $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$, and $\tilde{\Sigma}_3$ defined by the help of the totally antisymmetric tensor in three dimensions ε fulfilling $\varepsilon_{123} = 1$, $\varepsilon_{lmn} = -\varepsilon_{mln} = -\varepsilon_{lnm}$

$$(\tilde{\Sigma}_l)_{mn} = \mathrm{i}\varepsilon_{lmn} \tag{8}$$

$$\tilde{\Sigma}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad \tilde{\Sigma}_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad \tilde{\Sigma}_3 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

equation (5) can be written in the form ($\partial_k = \partial / \partial x^k$, $k = 1, 2, 3$)

$$\frac{\partial \mathbf{\Psi}}{\partial t} = \tilde{\Sigma}_k \partial_k \mathbf{\Psi} \tag{10}$$

or, defining matrices Γ^μ by $\Gamma^0 = \mathrm{Id}_3$, where Id_3 denotes the 3×3 identity matrix, and $\Gamma_k = \tilde{\Sigma}_k = -\Gamma^k$ for $k = 1, 2, 3$, equation (5) finally reads

$$\mathrm{i}\Gamma^\mu \partial_\mu \mathbf{\Psi} = 0\,. \tag{11}$$

Obviously, the complex conjugate wave functions $\mathbf{\Psi}^*$ fulfills the equation

$$\mathrm{i}\bar{\Gamma}^\mu \partial_\mu \mathbf{\Psi}^* = 0 \tag{12}$$

where $\bar{\Gamma}^\mu = (\mathrm{Id}_3, \tilde{\Sigma}_1, \tilde{\Sigma}_2, \tilde{\Sigma}_3)$.

Defining by the help of the hermitian conjugate field $\mathbf{\Psi}^+ = \mathbf{\Psi}^{*\mathrm{T}}$ the four density components

$$T^{0\mu} = \mathbf{\Psi}^+ \Gamma^\mu \mathbf{\Psi} \tag{13}$$

we recover after a short calculation the energy density and the Poynting vector of the electromagnetic field

$$\omega = T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \qquad s^k = T^{0k} = (\mathbf{E} \times \mathbf{B})_k\,. \tag{14}$$

Equations (14) signal a crucial difference between the Dirac equation for a spin-$\frac{1}{2}$ (anti-) particle with mass m

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{15}$$

and equation (11), since the (probability) four-current density

$$j^\mu_{\mathrm{Dirac}} = \psi^+ \gamma^0 \gamma^\mu \psi = \bar{\psi}\gamma^\mu \psi \tag{16}$$

transforms a a vector field, whereas $T^{0\mu} = T^{\mu 0}$ is related to the electromagnetic stress-energy tensor $T^{\mu\nu}$.

$\mathbf{\Psi}$ and $\mathbf{\Psi}^*$ transform according to the (1,0) and (0,1) = (1,0)* representation of the proper Lorentz group (see below). We mention as a historical fact that Fredrik Jozef Belinfante coined the expression *undor* when dealing with fields transforming according to some specific representations of the Lorentz group. Here, we term the complex Riemann-Silberstein three-component field $\mathbf{\Psi}$ a *bivector field* [6], in order to allow for a clear distinction from vector or spinor fields. Furthermore, this term was already used by Ludwig Silberstein [6] in 1907.

3. Transformation Properties

$\mathbf{E}$ and $\mathbf{B}$ transform as vectors under spatial rotations $(x^0, \mathbf{x}') \to (x'^0, \mathbf{x}') = (x^0, R\mathbf{x})$ according to

$$C'(x') = RC(x) \tag{17}$$

where R is an orientation preserving rotation matrix in the special orthogonal group

$$\mathrm{SO}(3) = \{R \in \mathrm{Mat}(3,\mathbb{R}) \; ; \; R^{\mathrm{T}} R = \mathrm{Id}_3, \det R = 1\} \,. \tag{18}$$

The definition of $\mathrm{SO}(3)$ is rooted in the preservation of the Euclidean scalar product of real three-vectors $(\mathbf{x},\mathbf{y}) = \mathbf{x}^{\mathrm{T}}\mathbf{y} = (R\mathbf{x}, R\mathbf{y}) = \mathbf{x}^{\mathrm{T}} R^{\mathrm{T}} R\mathbf{y}$. From $R^{\mathrm{T}} R = \mathrm{Id}_3$ follows $\det R = \pm 1$; the additional condition $\det R = 1$ excludes spatial reflections from $\mathrm{SO}(3)$.

However, the electromagnetic field components are not related to spatial components of a four-vector with respect to the proper Lorentz group

$$\mathrm{SO}^+(1,3) = \{\Lambda \in \mathrm{Mat}(4,\mathbb{R}) \; ; \; \Lambda^{\mathrm{T}} g \Lambda = g, \; \Lambda^0{}_0 \geq 1, \; \det \Lambda = 1\} \tag{19}$$

with the metric tensor g defined according to the sign convention given by

$$g = \mathrm{diag}(1,-1,-1,-1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{20}$$

The explicit representation of $\Lambda \in \mathrm{SO}^+(1,3)$ by its matrix elements is given by

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} . \tag{21}$$

Still, the electromagnetic field $\mathbf{\Psi}$ transforms as a vector under the complex special orthogonal group in three dimensions

$$\mathrm{SO}(3,\mathbb{C}) = \{Q \in \mathrm{Mat}(3,\mathbb{C}) \; ; \; Q^{\mathrm{T}} Q = \mathrm{Id}_3, \det Q = 1\} \,. \tag{22}$$

This observation is related to the fact that the proper Lorentz group and the complex rotation group $\mathrm{SO}(3,\mathbb{C})$ are isomorphic indeed

$$\mathrm{SO}^+(1,3) \cong \mathrm{SO}(3,\mathbb{C}) \,. \tag{23}$$

The elegance and the analogy of the considerations presented so far to the Dirac [4], Weyl [9] or *massive* two-component Majorana formalism [5] is obvious. Still, a mass term is absent in equation (11). It is one purpose of this paper to explicitly show that such a term cannot be established, which enforces a new concept like gauge theories when massive spin-one particles are involved in a field theory.

4. Symmetries: Generators of $\mathrm{SO}^+(1,3)$, $\mathrm{SO}(3)$ and $\mathrm{SO}(3,\mathbb{C})$

A pure Lorentz boost in x^1-direction with velocity $\beta = \beta_1$ and Lorentz factor $\gamma = \gamma_1$ is expressed by the matrix

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \gamma^2 - \gamma^2\beta^2 = 1. \tag{24}$$

which can be written to first order in β as

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1 + \beta L_1 \tag{25}$$

where L_1 is a generator for boosts in x^1-direction. The original Lorentz boost is recovered by exponentiating the generator multiplied by the boost parameter ξ_1

$$\exp(\xi_1 L_1) = \begin{pmatrix} \cosh\xi_1 & -\sinh\xi_1 & 0 & 0 \\ -\sinh\xi_1 & \cosh\xi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \cosh(\xi_1) = \gamma_1\,. \tag{26}$$

Additional generators L_2 and L_3 for boost in x^2- and x^3-direction are

$$L_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

and generators for rotations around the x^1-, x^2-, and x^3-axis are

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Altogether, these six generators of the proper Lorentz group span the Lie algebra $\mathfrak{so}^+(1,3)$, satisfying the commutation relations

$$[S_l, S_m] = -\varepsilon_{lmn} S_n, \qquad [L_l, L_m] = +\varepsilon_{lmn} S_n, \qquad [L_l, S_m] = -\varepsilon_{lmn} L_n \tag{28}$$

with the totally antisymmetric tensor or SO(3)-structure constants ε in three dimensions. Note that generators are often multiplied with the imaginary unit i in the physics literature in order to get Hermitian matrices.

Restricting our considerations to the rotation group $\mathrm{SO}(3)$ only, a basis of the Lie Algebra $\mathfrak{so}(3)$ is given by $(\Sigma_l)_{mn} = \varepsilon_{lmn}$, or explicitly

$$\Sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

with

$$[\Sigma_l, \Sigma_m] = -\varepsilon_{lmn} \Sigma_n \,. \tag{30}$$

By decomposing a real rotation matrix according to $R = \mathrm{Id}_3 + \delta R$, we obtain in a straightforward manner

$$R^{\mathrm{T}} R = (\mathrm{Id}_3 + \delta R)^{\mathrm{T}} (\mathrm{Id}_3 + \delta R) = \mathrm{Id}_3 + \delta R^{\mathrm{T}} + \delta R + \delta R^{\mathrm{T}} \delta R = \mathrm{Id}_3 \,. \tag{31}$$

For *small* δR, $\delta R^{\mathrm{T}} \delta R$ is negligible and $\delta R^{\mathrm{T}} + \delta R = 0$ holds approximately, correspondingly the generators in $\mathfrak{so}(3)$ must be antisymmetric. Therefore, the real and antisymmetric $\Sigma-$matrices form a basis of $\mathfrak{so}(3)$.

By definition, the same argument given by equation (31) holds for the complex group $\mathrm{SO}(3, \mathbb{C})$. A complete basis of the Lie algebra $\mathfrak{so}(3, \mathbb{C})$ is thus obtained by adding the antisymmetric matrices $\tilde{\Sigma}_k = \mathrm{i} \Sigma_k$ to the generators Σ_1, Σ_2, and Σ_3 of the real rotation group $\mathrm{SO}(3)$. This complexification leads to

$$[\Sigma_l, \Sigma_m] = -\varepsilon_{lmn} \Sigma_n, \qquad [\tilde{\Sigma}_l, \tilde{\Sigma}_m] = \varepsilon_{lmn} \Sigma_n, \qquad [\tilde{\Sigma}_l, \Sigma_m] = -\varepsilon_{lmn} \tilde{\Sigma}_n \tag{32}$$

i.e., the same abstract Lie algebra is obtained if one identifies the generators of $\mathrm{SO}^+(1,3)$ and $\mathrm{SO}(3, \mathbb{C})$ according to $S_l \leftrightarrow \Sigma_l$ and $L_l \leftrightarrow \tilde{\Sigma}_l$ for $l = 1, 2, 3$.

An arbitrary matrix $\Lambda \in \mathrm{SO}^+(1,3)$ can be written in the form

$$\Lambda = \exp(\xi_1 L_1 + \xi_2 L_2 + \xi_3 L_3 + \alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 S_3) \tag{33}$$

establishing a one-one correspondence to $Q \in \mathrm{SO}(3, \mathbb{C})$ via

$$Q = \exp(\xi_1 \tilde{\Sigma}_1 + \xi_2 \tilde{\Sigma}_2 + \xi_3 \tilde{\Sigma}_3 + \alpha_1 \Sigma_1 + \alpha_2 \Sigma_2 + \alpha_3 \Sigma_3) \,. \tag{34}$$

E.g., one has in correspondence to equation (26) for a bivector boost in x^1-direction an $\mathrm{SO}(3, \mathbb{C})$ transformation matrix

$$\exp(\xi_1 \tilde{\Sigma}_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \xi_1 & \mathrm{i} \sinh \xi_1 \\ 0 & -\mathrm{i} \sinh \xi_1 & \cosh \xi_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_1 & \mathrm{i} \gamma_1 \beta_1 \\ 0 & -\mathrm{i} \gamma_1 \beta_1 & \gamma_1 \end{pmatrix} \tag{35}$$

and a rotation around the x^1-axis is obtained by acting on the bivector with

$$\exp(\alpha_1\Sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & \sin\alpha_1 \\ 0 & -\sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \in \mathrm{SO}(3)\,. \tag{36}$$

A short exercise shows that acting with $\exp(\xi_1\tilde{\Sigma}_1)$ on the bivector field generates the correct Lorentz transformations of the electromagnetic field derived in many standard textbooks for a boost in x^1-direction

$$\begin{pmatrix} E_1'+\mathrm{i}B_1' \\ E_2'+\mathrm{i}B_2' \\ E_3'+\mathrm{i}B_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma_1 & \mathrm{i}\gamma_1\beta_1 \\ 0 & -\mathrm{i}\gamma_1\beta_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} E_1+\mathrm{i}B_1 \\ E_2+\mathrm{i}B_2 \\ E_3+\mathrm{i}B_3 \end{pmatrix} \tag{37}$$

therefore

$$\begin{aligned} E_1' &= E_1\,, & B_1' &= B_1 \\ E_2' &= \gamma_1(E_2-\beta_1 B_3)\,, & B_2' &= \gamma_1(B_2+\beta_1 E_3) \\ E_3' &= \gamma_1(E_3+\beta_1 B_2)\,, & B_3' &= \gamma_1(B_3-\beta_1 E_2). \end{aligned} \tag{38}$$

Generally, one has in the case of a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\mathbf{\Psi}'(x') = Q\mathbf{\Psi}(x) = Q\mathbf{\Psi}(\Lambda^{-1}x')\,. \tag{39}$$

Note that the transformation property of $\mathbf{\Psi}$ implies that the real and imaginary part of

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{\Psi} = \mathbf{\Psi}^{\mathrm{T}}Q^{\mathrm{T}}Q\mathbf{\Psi} = \mathbf{\Psi}^{\mathrm{T}}Q^{-1}Q\mathbf{\Psi} = \frac{1}{2}(\mathbf{E}^2-\mathbf{B}^2)+\mathrm{i}\mathbf{E}\cdot\mathbf{B} \tag{40}$$

are Lorentz-invariant quantities.

5. Mass Terms

5.1. Majorana Approach

Acting with the operator $-\mathrm{i}\bar{\Gamma}^\mu\partial_\mu$ on the field equation (11) leads to

$$\bar{\Gamma}_\mu\Gamma_\nu\partial_\nu\partial_\mu\mathbf{\Psi} = \partial_0^2\mathbf{\Psi} + \nabla\times\nabla\times\mathbf{\Psi} = \partial_0^2\mathbf{\Psi} - \Delta\mathbf{\Psi} + \nabla(\nabla\cdot\mathbf{\Psi}) = 0 \tag{41}$$

therefore the wave function $\mathbf{\Psi}$ fulfills the free wave equation

$$\Box\mathbf{\Psi} = \partial^\mu\partial_\mu\mathbf{\Psi} = 0 \tag{42}$$

in the absence of charges, ensuring the correct relativistic energy-momentum relation. Introducing a naive mass term for the $\mathbf{\Psi}$-field like

$$\mathrm{i}\Gamma^\mu\partial_\mu\mathbf{\Psi} - m\mathbf{\Psi} = 0 \tag{43}$$

would spoil the relativistic invariance of the field equation. As a more general approach one may introduce an (anti-)linear operator S and make the ansatz

$$\mathrm{i}\Gamma^\mu \partial_\mu \mathbf{\Psi} - mS\mathbf{\Psi} = 0\,. \tag{44}$$

Since plane-wave solutions $\sim \mathrm{e}^{\pm \mathrm{i}px}$ must obey $p^2 = p_\mu p^\mu = m^2$

$$\mathrm{i}\bar{\Gamma}^\nu \partial_\nu (\mathrm{i}\Gamma^\mu \partial_\mu \mathbf{\Psi}) = -\Box \mathbf{\Psi} = m^2 \mathbf{\Psi} = \mathrm{i}\bar{\Gamma}^\nu \partial_\nu (mS\mathbf{\Psi}) \tag{45}$$

the transformed bivector fulfills the wave equation

$$\mathrm{i}\bar{\Gamma}^\mu \partial_\mu (S\mathbf{\Psi}) - m\mathbf{\Psi} = 0. \tag{46}$$

Acting on equation (46) with S leads to the requirement

$$S\mathrm{i}\bar{\Gamma}^\mu \partial_\mu (S\mathbf{\Psi}) = mS\mathbf{\Psi} = \mathrm{i}\Gamma^\mu \partial_\mu \mathbf{\Psi} \tag{47}$$

or

$$S\mathrm{i}\bar{\Gamma}^\mu S = \mathrm{i}\Gamma^\mu\,. \tag{48}$$

If S is a linear operator, it must fulfill

$$S^2 = \mathrm{Id}_3, \qquad S\tilde{\Sigma}_k S = -\tilde{\Sigma}_k, \qquad k = 1, 2, 3 \tag{49}$$

or

$$S\tilde{\Sigma}_k S^{-1} = -\tilde{\Sigma}_k, \qquad k = 1, 2, 3. \tag{50}$$

This is impossible, since the structure constants of Lie algebra a stable under similarity transformations.

If S is anti-linear, it can be written as $S = \tilde{S}K$, where K denotes complex conjugation. The complex conjugation of the imaginary unit in equation (48) et cetera then leads to

$$\tilde{S}\tilde{S}^* = -\mathrm{Id}_3, \qquad \tilde{S}\tilde{\Sigma}_k^* \tilde{S}^* = \tilde{\Sigma}_k, \qquad k = 1, 2, 3 \tag{51}$$

or

$$\tilde{S}\tilde{\Sigma}_k^* \tilde{S}^{-1} = -\tilde{\Sigma}_k, \quad k = 1, 2, 3. \tag{52}$$

Again, no such $\tilde{S}$ exists in three dimensions, since $\det(S\tilde{S}^*) = \det(\tilde{S})\det(\tilde{S}^*) = \det(\tilde{S})\det(\tilde{S})^* > 0$ contradicts $\det(-\mathrm{Id}_3) = -1$. This can be contrasted with the case in two dimensions, where the Pauli matrices $\vec{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2, \sigma_3)$ obey

$$\epsilon \vec{\boldsymbol{\sigma}}^* \epsilon^{-1} = -\vec{\boldsymbol{\sigma}} \tag{53}$$

where ϵ is the totally antisymmetric tensor in two dimensions

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon^2 = (\eta\epsilon)(\eta\epsilon)^* = -\mathrm{Id}_2, \qquad \det(\epsilon) = \det(-\mathrm{Id}_2) \tag{54}$$

such that the Majorana equation(s) as a generalization of the Weyl equations [9] and Lorentz invariant two-component field equations describing a massive particle exist [1,5]

$$\mathrm{i}\sigma^{\mu}\partial_{\mu}\mathbf{\Psi} - m\eta\epsilon\mathbf{\Psi}^{*} = 0 \tag{55}$$

where $\sigma^{\mu} = (\mathrm{Id}_{2}, \vec{\boldsymbol{\sigma}})$ and η is a phase.

5.2. Mass Term II: Gauge Formalism

One might try to invoke a mass term by naively introducing an $\mathrm{SO}(3,\mathbb{C})$ bivector "gauge potential"

$$\mathbf{H} = \mathbf{H}^{R} + \mathrm{i}\mathbf{H}^{I} = \begin{pmatrix} H_1^R + \mathrm{i}H_1^I \\ H_2^R + \mathrm{i}H_2^I \\ H_3^R + \mathrm{i}H_3^I \end{pmatrix} \tag{56}$$

related to the massive bivector field via

$$\mathbf{\Psi} = \mathrm{i}\bar{\Gamma}^{\nu}\partial_{\nu}\mathbf{H} = (\mathrm{i}\partial_0 + \nabla\times)(\mathbf{H}^R + \mathrm{i}\mathbf{H}^I) = (-\dot{\mathbf{H}}^I + \nabla\times\mathbf{H}^R) + \mathrm{i}(\dot{\mathbf{H}}^R + \nabla\times\mathbf{H}^I)$$

fulfilling the massive wave equation

$$\Box\mathbf{H} + m^2\mathbf{H} = 0. \tag{57}$$

This would imply

$$\mathrm{i}\Gamma^{\mu}\partial_{\mu}(\mathrm{i}\bar{\Gamma}^{\nu}\partial_{\nu}\mathbf{H}) = (-\partial_0^2 - \nabla\times\nabla\times)\mathbf{H} = (-\partial_0^2 + \Delta - \nabla(\nabla\cdot)\mathbf{H} = -m^2\mathbf{H}$$

and therefore $\nabla(\nabla\cdot\mathbf{H}) = 0$. This condition is, however, not Lorentz invariant.

One has to accept that a mass term is impossible for simple group theoretical reasons. Since a four-gradient transforms according to the $(\frac{1}{2},\frac{1}{2})$-representation of the Lorentz group, it produces quantities transforming according to the representations $(\frac{1}{2},\frac{1}{2})\otimes(1,0) = (\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{1}{2})$ when acting on a bivector under $(1,0)$. In the case of Majorana fermions, one has $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},0) = (1,\frac{1}{2})\oplus(0,\frac{1}{2})$, such that a the derivative of a field field transforming according to $(\frac{1}{2},0)$ can be coupled to its complex conjugate field tranforming according to $(0,\frac{1}{2})$.

6. Coupling to a Current: Transformation Properties

Adding the real electric current to Ampère's circuital law

$$\frac{\partial\mathbf{E}}{\partial t} = \nabla\times\mathbf{B} - \mathbf{j} \tag{58}$$

where a possible factor $\frac{1}{\sqrt{2}}$ according to the normalization chosen in equation (1) and the coupling constant have been absorbed in $\mathbf{j}$, shows that equation (11) cannot be interpreted as the "Dirac form" of Maxwell's equations, since the current $\mathbf{j} = (j^1, j^2, j^3)$ consists of three spatial components of a charge-current four-vector density

$$j = \{j^\mu\} = (j^0, j^1, j^2, j^3) = (\rho, j^1, j^2, j^3) = (\rho, \mathbf{j}). \tag{59}$$

Equation (5) becomes

$$\frac{\partial \mathbf{\Psi}}{\partial t} = -\mathrm{i} \cdot \nabla \times \mathbf{\Psi} - \mathbf{j} \tag{60}$$

and taking the divergence of equation (60) leads to

$$\nabla \cdot \dot{\mathbf{\Psi}} = -\mathrm{i} \cdot \nabla \cdot (\nabla \times \mathbf{\Psi}) - \operatorname{div} \mathbf{j} = \dot{\rho} \tag{61}$$

since the continuity equation $\dot{\rho} + \operatorname{div} \mathbf{j} = 0$ holds. Due to the absence of magnetic charges, equation (61) is equivalent to $\operatorname{div} \dot{E} = \dot{\rho}$.

Therefore, although the bivector field only couples to the spatial components $\mathbf{j}$ of the charge-current four-vector density $(\rho, \mathbf{j})$, the charge distribution is encoded in the divergence of the bivector itself and does not appear as an independent dynamical variable, since the current density j together with the initial conditions for the charge distribution fix the actual charge density or the divergence of the bispinor at any time.

Considering the transformation properties of a Dirac spinor under a Lorentz transformation $x' = \Lambda x$ for a moment

$$\psi'(x') = Q_D(\Lambda)\psi(x), \qquad \Lambda \in \mathrm{SO}^+(1,3), \qquad x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \tag{62}$$

we observe that the Dirac equation

$$\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi = 0 \tag{63}$$

also holds in the primed coordinate system. Namely, requiring

$$\mathrm{i}\gamma^\mu \partial'_\mu \psi'(x') = \mathrm{i}\gamma^\mu \partial'_\mu Q_D \psi(x) = m Q_D \psi(x) \tag{64}$$

implies

$$\mathrm{i}\gamma^\mu \Lambda_\mu^{\ \nu} \partial_\nu Q_D \psi(x) = m Q_D \psi(x) \tag{65}$$

or

$$\Lambda^\alpha_{\ \nu} \Lambda_\mu^{\ \nu} Q_D^{-1} \gamma^\mu Q_D = \Lambda^\alpha_{\ \nu} \gamma^\nu \,. \tag{66}$$

Using $\Lambda^\alpha_{\ \nu} \Lambda_\mu^{\ \nu} = \delta^\alpha_\mu$, we obtain

$$Q_D^{-1} \gamma^\alpha Q_D = \Lambda^\alpha_{\ \nu} \gamma^\nu \,. \tag{67}$$

This important property of the Dirac matrices which expresses the manifest Lorentz covariance of the Dirac equation holds if the transformation of the spinor components is performed by an appropriately chosen matrix Q_D.

However, a simple relation analogous to equation (67) à la

$$Q^{-1}\Gamma^{\alpha}Q = \Lambda^{\alpha}_{\ \nu}\Gamma^{\nu} \tag{68}$$

or

$$Q^{T}\Gamma^{\alpha}Q = \Lambda^{\alpha}_{\ \nu}\Gamma^{\nu}, \qquad Q^{*}\Gamma^{\alpha}Q = \Lambda^{\alpha}_{\ \nu}\Gamma^{\nu} \tag{69}$$

does not hold, as can be shown by a straightforward calculation.

In the bivector case, we have

$$\mathrm{i}\Gamma^{\mu}\partial_{\mu}\mathbf{\Psi} = -\mathrm{i}\mathbf{j} \tag{70}$$

and therefore in a primed coordinate system

$$\mathrm{i}\Gamma^{\mu}_{ab}\partial'_{\mu}\Psi'_{b}(x') = -\mathrm{i}\mathbf{j}'_{a}(x') \tag{71}$$

implying (with latin indices a, b ... = 1,2,3 denoting bivector indices or components of the current density)

$$\mathrm{i}\Gamma^{\mu}_{ab}\Lambda_{\mu}^{\ \nu}\partial_{\nu}Q_{bc}\Psi_{c}(x) = -\mathrm{i}\Lambda^{a}_{\ \nu}j^{\nu}(x) = -\mathrm{i}\Lambda^{a}_{\ 0}\,\mathrm{div}\,\mathbf{E}(x) - \mathrm{i}\Lambda^{a}_{\ c}\,j^{c}(x) \tag{72}$$

since $j^{\mu}(x) = (\mathrm{div}\,\mathbf{E}, j^1, j^2, j^3)$.

The divergence of the electric field can be written by the help of the Kronecker delta as

$$\mathrm{div}\,\mathbf{E} = \delta^{\nu}_{c}\partial_{\nu}\Psi_{c} \tag{73}$$

since the magnetic field is divergence free from the start. Hence equation (72) becomes

$$\mathrm{i}(\Gamma^{\mu}_{ab}\Lambda_{\mu}^{\ \nu}Q_{bc} + \Lambda^{a}_{\ 0}\delta^{\nu}_{c})\partial_{\nu}\Psi_{c}(x) = -\mathrm{i}\Lambda^{a}_{\ c}j^{c}. \tag{74}$$

Defining $\tilde{\Lambda}$ as the inverse of the 3-by-3 submatrix $\Lambda^{a}_{\ b}$ according to $\tilde{\Lambda}^{d}_{\ a}\Lambda^{a}_{\ b} = \delta^{d}_{b}$, one may multiply equation (74) by $\tilde{\Lambda}^{d}_{\ a}$ and obtains the original equation, i.e.,

$$\mathrm{i}(\tilde{\Lambda}^{d}_{\ a}\Gamma^{\mu}_{ab}\Lambda_{\mu}^{\ \nu}Q_{bc} + \tilde{\Lambda}^{d}_{\ a}\Lambda^{a}_{\ 0}\delta^{\nu}_{c})\partial_{\nu}\Psi_{c}(x) = -\mathrm{i}\tilde{\Lambda}^{d}_{\ a}\Lambda^{a}_{\ c}j^{c} = -\mathrm{i}j^{d} \tag{75}$$

and therefore

$$\Gamma^{\nu}_{dc} = \tilde{\Lambda}^{d}_{\ a}\Gamma^{\mu}_{ab}\Lambda_{\mu}^{\ \nu}Q_{bc} + \tilde{\Lambda}^{d}_{\ a}\Lambda^{a}_{\ 0}\delta^{\nu}_{c}\,. \tag{76}$$

Restricting our consideration to spatial rotations, we have $\Lambda^{a}_{\ b} = Q_{ab}$ and $\tilde{\Lambda}^{-1} = Q^{-1}$, furthermore $\Lambda^{a}_{\ 0} = 0$ for $a = 1, 2, 3$. Accordingly

$$\Gamma^{\nu} = \Lambda_{\mu}^{\ \nu}Q^{-1}\Gamma^{\mu}Q \tag{77}$$

in analogy to equation (66), since for pure rotations, spatial derivatives, the bivector and the current density transform as vectors under $\mathrm{SO}(3)$, and the complicated situation in equation (76) arising from coupling a bivector to a four-vector does not arise.

7. Conclusions

In this paper, the main features of the gauge free formalism describing a massless spin-one field coupled to a conserved current based on the representation of the proper (also called *restricted*) Lorentz group $\mathrm{SO}^+(1,3)$ by the complex orthogonal group $\mathrm{SO}(3,\mathbb{C})$ are investigated. It is shown that a mass term analogous to the Majorana or Dirac case is impossible for a bivector field for group theoretical reasons, although the equations of motion in the bivector formalism display some commonalities with the spinor formalism. It is hoped that the paper fills a gap in the literature concerning the representation theory of the Lorentz group in low dimensions relevant for relativistic field theory.

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