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FORCE FREE MÖBIUS MOTIONS OF THE CIRCLE

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Abstract. Let $\mathcal{M}$ be the Lie group of Möbius transformations of the circle. Suppose that the circle has initially a homogeneous distribution of mass and that the particles are allowed to move only in such a way that two configurations differ in an element of $\mathcal{M}$. We describe all force free Möbius motions, that is, those curves in $\mathcal{M}$ which are critical points of the kinetic energy. The main tool is a Riemannian metric on $\mathcal{M}$ which turns out to be not complete (in particular not invariant, as happens with non-rigid motions) given by the kinetic energy.

1. Introduction

In the spirit of the classical description of the force free motions of a rigid body in Euclidean space using an invariant metric on $\mathrm{SO}\,(3)$ [1, Appendix 2], the second author defined in [4] an appropriate metric on the Lorenz group $\mathrm{SO}_o\,(n+1,1)$ to study force free conformal motions of the sphere $\mathbb{S}^n$, obtaining a few explicit ones (only through the identity and those which can be described using the Lie structure of the configuration space). In this note, in the particular case $n=1$, that is, Möbius motions of the circle, we obtain all force free motions.

This is an example of a situation in which using concepts of Physics one can state and solve a problem in Differential Geometry (see for instance [2, 3, 6]).

Notice that the canonical action of $\mathrm{PSL}\,(2,\mathbb{R})$ on $\mathbb{RP}^1 \cong \mathbb{S}^1$ is equivalent to the action of the group of Möbius transformations on the circle. Then, the results presented here, up to a double covering, also extend the case $n=1$ of [5], where force free projective motions of the sphere $\mathbb{S}^n$ were studied.

This note, as well as [4, 5], is weakly related with mass transportation [7]. In our situation, the set of admitted mass distributions is finite dimensional, and also the allowed transport maps are very particular.

1.1. Möbius Motions of the Circle

Let $\mathbb{S}^1$ be the unit circle centered at zero in $\mathbb{C}$ with the usual metric and let $\mathcal{M}$ be the Lie group of Möbius transformations of the circle, that is, the group of Möbius transformations of the extended plane preserving the circle. It consists of maps of the form cT_α, where $c \in \mathbb{S}^1$ and

$$T_\alpha(z) = \frac{z+\alpha}{1+\bar{\alpha}z} \tag{1}$$

for $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and all $z \in \mathbb{S}^1$. Although we are interested in the action of $\mathcal{M}$ on the circle, we recall that if the unit disc $\Delta = \{z \in \mathbb{C}\ ;\ |z| < 1\}$ carries the canonical Poincaré metric of constant negative curvature -1 and $\alpha \neq 0$, then T_α is the transvection translating the geodesic with end points $\pm\alpha/|\alpha|$, which sends 0 to α.

A *Möbius motion* of the circle is by definition a smooth curve in $\mathcal{M}$, thought of as a curve of diffeomorphisms of the circle. (Throughout the paper, smooth means of class C^∞.)

In the next two subsections we recall, specialized for the circle, some definitions and statements given in [4] for conformal motions on the n-dimensional sphere.

1.2. The Energy of Möbius Motions of Circle

Suppose that the circle has initially a homogeneous distribution of mass of constant density one and that the particles are allowed to move only in such a way that two configurations differ in an element of $\mathcal{M}$. The configuration space may be naturally identified with $\mathcal{M}$.

Let $\gamma : [t_0, t_1] \to \mathcal{M}$ be a Möbius motion of $\mathbb{S}^1$. The total kinetic energy $E_\gamma(t)$ of the motion γ at the instant t is given by

$$E_\gamma(t) = \tfrac{1}{2}\int_{\mathbb{S}^1} |v_t(q)|^2 \rho_t(q)\ \mathrm{d}m(q) \tag{2}$$

where integration is taken with respect to the canonical volume form of $\mathbb{S}^1$ and, if $q = \gamma(t)(p)$ for $p \in \mathbb{S}^1$, then

$$v_t(q) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_t \gamma(s)(p) \in T_q\mathbb{S}^1, \qquad \rho_t(q) = 1/\det\left(\mathrm{d}\gamma(t)_p\right)$$

are the velocity of the particle q and the density at q at the instant t, respectively. Applying to (2) the formula for change of variables, one obtains

$$E_\gamma(t) = \tfrac{1}{2} \int_{\mathbb{S}^1} \left| \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_t \gamma(s)(p) \right|^2 \, \mathrm{d}m(p). \tag{3}$$

The kinetic energy of γ is defined by

$$E(\gamma) = \int_{t_0}^{t_1} E_\gamma(t) \, \mathrm{d}t.$$

The following definition is based on the principle of least action.

Definition 1. *A smooth curve γ in $\mathcal{M}$, thought of as a Möbius motion of $\mathbb{S}^1$, is said to be* force free *if it is a critical point of the kinetic energy functional, that is,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 E(\gamma_s) = 0$$

for any proper smooth variation γ_s of γ (here $\gamma_s(t) = \Gamma(s,t)$, where $\Gamma : (-\varepsilon, \varepsilon) \times [t_0, t_1] \to \mathcal{M}$ is a smooth map, with $\varepsilon > 0$, $\Gamma(0,t) = \gamma(t)$ and $\Gamma(s, t_i) = \gamma(t_i)$ for all $s \in (-\varepsilon, \varepsilon)$, $i = 0, 1$).

1.3. A Riemannian Metric on the Configuration Space

Given $g \in \mathcal{M}$ and $X \in T_g\mathcal{M}$, let us define the map $\widetilde{X} : \mathbb{S}^1 \to T\mathbb{S}^1$ by

$$\widetilde{X}(q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma(t)(q) \in T_{g(q)}\mathbb{S}^1 \tag{4}$$

where γ is any smooth curve in $\mathcal{M}$ with $\gamma(0) = g$ and $\dot{\gamma}(0) = X$. The map $\widetilde{X}$ is well-defined and smooth and it is a vector field on $\mathbb{S}^1$ if and only if $X \in T_e\mathcal{M}$. Moreover,

$$X \mapsto \|X\|^2 = \frac{1}{2\pi} \int_{\mathbb{S}^1} \left|\widetilde{X}(q)\right|^2 \, \mathrm{d}m(q) \tag{5}$$

is a quadratic form on $T_g\mathcal{M}$ and gives a Riemannian metric on $\mathcal{M}$.

Remarks 2. a) *The fundamental property of the metric* (5) *on $\mathcal{M}$ is that a curve γ in $\mathcal{M}$ is a geodesic if and only if* (*thought of as a Möbius motion*) *it is force free, since by* (5) *and* (3), *$E_\gamma(t) = \pi \|\dot{\gamma}(t)\|^2$.*

b) *The metric on $\mathcal{M}$ is neither left nor right invariant, since we saw in* [4] *that it is not even complete.*

2. Force Free Möbius Motions of the Circle

The next theorem describes completely the geometry of $\mathcal{M}$ endowed with the metric (5) given by the kinetic energy. Recall from (1) that T_a denotes the transvection associated with α and that Δ is the unit disc centered at zero in $\mathbb{C}$.

Theorem 3. *Let* $\mathrm{d}s^2$ *be the metric on the disc* Δ *given in polar coordinates* (r,θ) *by*

$$\mathrm{d}s^2 = \frac{2\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\right)}{1-r^2} \tag{6}$$

and consider on $\mathbb{S}^1 \times \Delta$ *the Riemannian product metric, where* $\mathbb{S}^1$ *has length* 2π*. Then the map*

$$F : \mathbb{S}^1 \times \Delta \to \mathcal{M}, \qquad F\left(u,\alpha\right) = uT_\alpha$$

is an isometry.

Remarks 4. a) *Note that the metric* (6) *on* Δ *is not the canonical metric of constant negative curvature on* Δ*. Indeed, the curvature function can be easily computed to be* $K(r,\theta) = -1/\left(1-r^2\right)$*, in particular, it tends to* $-\infty$ *as* $r \to 1^-$*. Also, the metric on* Δ *is not complete, since the inextendible ray* $(0,1) \ni r \mapsto T_r$ *has length* $\pi/\sqrt{2}$*, since* $\left\|\frac{\partial}{\partial r}\right\|^2 = \frac{2}{1-r^2}$.

b) *In the higher dimensional situation* [4] *it is proven that the group* $\mathrm{SO}(n)$ (*with the metric induced from the one given by the kinetic energy*) *is totally geodesic in the group of directly conformal transformations of* $\mathbb{S}^n$*, but the author did not know whether this subgroup is a Riemannian factor, as it turned to be for* $n=1$*. In the projective case* [5], $\mathrm{SO}(n)$ *is not even totally geodesic.*

Proof of Theorem 3. Let $\mathbb{S}^1 \subset \mathcal{M}$ be the subgroup of isometries of the circle. The torus $\mathbb{S}^1 \times \mathbb{S}^1$ acts on $\mathcal{M}$ on the left by $(u,v)\cdot g = ug\bar{v}$, where $\left(ug\bar{v}\right)(z) = ug\left(z\bar{v}\right)$ for any $z \in \mathbb{S}^1$. We know from the higher dimensional cases in [4] that this action is by isometries of $\mathcal{M}$, provided that this group is endowed with the metric (5).

We fix $0 < r < 1$. By the torus symmetry just described, it suffices to verify that $\mathrm{d}F_{(1,r)} : T_{(1,r)}\left(\mathbb{S}^1 \times \Delta\right) \to T_{F(1,r)}\mathcal{M}$ is a linear isometry. We put coordinates $t \mapsto \mathrm{e}^{\mathrm{i}t}$ on $\mathbb{S}^1$ and $(\rho,\theta) \mapsto \rho\mathrm{e}^{\mathrm{i}\theta}$ on Δ. We denote $\partial_x = \frac{\mathrm{d}}{\mathrm{d}x}$. Let X,Y,Z be the images under $\mathrm{d}F_{(1,r)}$ of $\partial_t, \partial_\rho, \partial_\theta$, respectively. It suffices to show that $\{X,Y,Z\}$ is an orthogonal basis of $T_{F(1,r)}\mathcal{M}$ with

$$\|X\|^2 = 1, \qquad \|Y\|^2 = \frac{2}{1-r^2}, \qquad \|Z\|^2 = \frac{2r^2}{1-r^2}.$$

First, we compute $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ by their definition (4). In each case, we take the curve γ as the image under F of the coordinate curves in $\mathbb{S}^1 \times \Delta$ through the point $(1, r)$. We have

$$\widetilde{X}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 F(\mathrm{e}^{\mathrm{i}t}, r)\,(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \mathrm{e}^{\mathrm{i}t} T_r(z) = \mathrm{i}\mathrm{e}^{\mathrm{i}t} T_r(z)\big|_{t=0} = \mathrm{i}T_r(z) = \mathrm{i}\frac{z+r}{1+rz}$$

$$\widetilde{Y}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\rho}\right|_r F(1,\rho)\,(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\rho}\right|_r T_\rho(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\rho}\right|_r \frac{z+\rho}{1+\rho z} = \frac{1-z^2}{(1+rz)^2}$$

$$\widetilde{Z}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_0 F(1, r\mathrm{e}^{\mathrm{i}\theta})\,(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_0 T_{r\mathrm{e}^{\mathrm{i}\theta}}(z) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_0 \frac{z+r\mathrm{e}^{\mathrm{i}\theta}}{1+r\mathrm{e}^{-\mathrm{i}\theta}z}$$
$$= \frac{\mathrm{i}r(1+2rz+z^2)}{(1+rz)^2}.$$

Next we compute

$$2\pi\|X\|^2 = \int_{\mathbb{S}^1} \left|\widetilde{X}(z)\right|^2 \mathrm{d}m(z) = \int_{\mathbb{S}^1} |\mathrm{i}T_r(z)|^2 \,\mathrm{d}m(z) = \int_{\mathbb{S}^1} 1\,\mathrm{d}m(z) = 2\pi.$$

We have also

$$2\pi\|Y\|^2 = \int_{\mathbb{S}^1} \left|\widetilde{Y}(z)\right|^2 \mathrm{d}m(z) = \int_{\mathbb{S}^1} \left|\frac{1-z^2}{(1+rz)^2}\right|^2 \mathrm{d}m(z).$$

Setting $z = \mathrm{e}^{\mathrm{i}s}$, we have

$$2\pi\|Y\|^2 = \int_0^{2\pi} \frac{1}{\mathrm{i}\mathrm{e}^{\mathrm{i}s}} \left|\frac{1-\mathrm{e}^{\mathrm{i}2s}}{(1+r\mathrm{e}^{\mathrm{i}s})^2}\right|^2 \mathrm{i}\mathrm{e}^{\mathrm{i}s}\,\mathrm{d}s = \int_{\mathbb{S}^1} \frac{1}{\mathrm{i}z}\left|\frac{1-z^2}{(1+rz)^2}\right|^2 \mathrm{d}z.$$

Now, the integrand is a complex analytic function inside the circle (observe that $\bar{z} = 1/z$ for $|z| = 1$), except for a simple pole at $z = 0$ and a pole of order two at $z = -r$, with residues $\frac{\mathrm{i}}{r^2}$ and $\frac{\mathrm{i}(r^2+1)}{-r^2(1-r^2)}$, respectively. One obtains that $\|Y\|^2 = 2/\left(1-r^2\right)$. In the same way one gets $\|Z\|^2 = 2r^2/\left(1-r^2\right)$.

We claim that the vectors X, Y, Z are pairwise orthogonal. Let $h\,(U,V) = U\bar{V}$ denote the Hermitian inner product on $\mathbb{C}$. We compute

$$\int_{\mathbb{S}^1} h\left(\widetilde{X}(z), \widetilde{Y}(z)\right) \mathrm{d}m(z) = \int_{\mathbb{S}^1} f(z)\,\mathrm{d}z$$

where $f(z) = \frac{z^2-1}{z(1+rz)(z+r)}$ is an complex analytic function inside the circle, except for simple poles at $z = 0$ and $z = -r$, with residues $1/r$ and $-1/r$, respectively. Then,

$$\langle X, Y\rangle = \Re \int_{\mathbb{S}^1} h\left(\widetilde{X}(z), \widetilde{Y}(z)\right) \mathrm{d}m(z) = 0.$$

Analogously, we find that $\langle Y, Z\rangle = \langle X, Z\rangle = 0$. ■

Corollary 5. *The force free Möbius motions of the circle, or equivalently, the geodesics of $\mathcal{M}$, are, via F, of the form $\gamma = (\gamma_1, \gamma_2)$, where γ_1 parametrizes the circle with constant speed and γ_2 is a geodesic in the disc Δ whose trajectory coincides with the images of either $c_1(\rho) = (\rho, \theta_0)$ or $c_2(\theta) = (\rho(\theta), \theta)$, where ρ satisfies the differential equation*

$$(\rho')^2 = \frac{\mu + \rho^2}{(1-\rho^2)\,\rho^2} \tag{7}$$

for some constant $\mu > -1$.

Proof. Clearly, a geodesic of a Riemannian product projects to a geodesic in each factor. Besides, as the coefficients of the first fundamental form of Δ depend only on ρ, the corresponding metric is Clairaut. Then, the trajectories of the geodesics of Δ are, in polar form

$$c_1(\rho) = (\rho, \theta_0) \qquad \text{or} \qquad c_2(\theta) = (\rho(\theta), \theta)$$

for some constant θ_0, where $\rho(\theta)$ satisfies Clairaut's differential equation, for some constant λ

$$\lambda E^2(\rho) = E(\rho) + (\rho')^2 G(\rho).$$

Since in our case $E(\rho) = \left\|\frac{\partial}{\partial r}\right\|^2 = \frac{2}{1-\rho^2}$ and $G(\rho) = \left\|\frac{\partial}{\partial \theta}\right\|^2 = \frac{2\rho^2}{1-\rho^2}$, the differential equation is equivalent to (7) for some constant $\mu > -1$. ■

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