Abstract. We show that if \( \psi \) is an \( f \)-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then \( \psi \) is an \( f \)-harmonic map. We prove that if the \( f \)-tension field \( \tau_f(\psi) \) of a map \( \psi \) of Riemannian manifolds is a Jacobi field and \( \phi \) is a totally geodesic map of Riemannian manifolds, then \( \tau_f(\phi \circ \psi) \) is a Jacobi field. We finally investigate the stress \( f \)-bienergy tensor, and relate the divergence of the stress \( f \)-bienergy of a map \( \psi \) of Riemannian manifolds with the Jacobi field of the \( \tau_f(\psi) \) of the map.

1. Introduction

Harmonic maps between Riemannian manifolds were first established by Eells and Sampson in 1964. Afterwards, there are two reports and one survey paper by Eells and Lemaire [15–17] about the developments of harmonic maps up to 1988. Chiang, Ratto, Sun and Wolak also studied harmonic and biharmonic maps in [4–9]. \( f \)-harmonic maps which generalize harmonic maps, were first introduced by Lichnerowicz [25] in 1970, and were studied by Course [12, 13] recently. The \( f \)-harmonic maps relate to the equation of the motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction in mathematical physics. Moreover, \( F \)-harmonic maps between Riemannian manifolds were first introduced by Ara [1, 2] in 1999, which could be considered as the special cases of \( f \)-harmonic maps.

Let \( f : (M_1, g) \to (0, \infty) \) be a smooth function. By definition the \( f \)-biharmonic maps between Riemannian manifolds are the critical points of \( f \)-bienergy

\[
E_f^2(\psi) = \frac{1}{2} \int_{M_1} f |\tau_f(\psi)|^2 dv
\]

where \( dv \) the volume form determined by the metric \( g \). The \( f \)-biharmonic maps between Riemannian manifolds which generalized biharmonic maps by Jiang [20, 21] in 1986, were first studied by Ouakkas, Nasri and Djaa [27] in 2010.
f-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then ψ is an f-harmonic map. It is well-known from [18] that if ψ is a harmonic map of Riemannian manifolds and ϕ is a totally geodesic map of Riemannian manifolds, then ϕ ◦ ψ is harmonic. However, if ψ is f-biharmonic and ϕ is totally geodesic, then ϕ ◦ ψ is not necessarily f-biharmonic. Instead, we prove in Theorem 3.3 that if the f-tension field τf(ψ) of a smooth map ψ of Riemannian manifolds is a Jacobi field and ϕ is totally geodesic, then τf(ϕ ◦ ψ) is a Jacobi field. It implies Corollary 3.4 [8] that if ψ is a biharmonic map between Riemannian manifolds and ϕ is totally geodesic, then ϕ ◦ ψ is a biharmonic map. We finally investigate the stress f-bienergy tensors. If ψ is an f-biharmonic of Riemannian manifolds, then it usually does not satisfy the conservation law for the stress f-bienergy tensor Sf2(ψ). However, we obtain in Theorem 4.2 that if ψ : (M1, g) → (M2, h) be a smooth map between two Riemannian manifolds, then
\[
\text{div } S_f^2(Y) = \pm \langle J_{τ_f}(ψ), dψ(Y) \rangle \quad \text{for all } Y ∈ Γ(TM_1)
\]
(1)
where div Sf2 is the divergence of Sf2 and Jτf(ψ) is the Jacobi field of τf(ψ) (there is a - or + sign convention in the formula). Hence, if τf(ϕ) is a Jacobi field, then ψ satisfies the conservation law for Sf2. It implies Corollary 4.4 [22] that if ψ is a biharmonic map between Riemannian manifolds, then ψ satisfies the conservation law for the stress bi-energy tensor S2(ψ). We also discuss a few applications concerning the vanishing of the stress f-bienergy tensor.

2. Preliminaries

2.1. Motivation

In mathematical physics, the equation of the motion of a continuous system of spins with inhomogeneous neighborhood Heisenberg interaction is
\[
\frac{∂ψ}{∂t} = f(x)(ψ × Δψ) + \nabla f · (ψ × ∇ψ)
\]
(2)
where Ω ⊂ Rm is a smooth domain in the Euclidean space, f is a real-valued function defined on Ω, ψ(x, t) ∈ S2, and × is the cross product in R3 and Δ is the Laplace operator in Rm. Such a model is called the inhomogeneous Heisenberg ferromagnet [10,11,14]. Physically, the function f is called the coupling function, and is the continuum of the coupling constant between the neighboring spins. It is known from [18] that the tension field of a map ψ into S2 is τ(ψ) = Δψ + |∇ψ|^2ψ.
Observe that the right hand side of (2) can be expressed as
\[ \psi \times (f \tau(\psi) + \nabla f \cdot \nabla \psi). \] (3)
Hence, \( \psi \) is a smooth stationary solution (i.e., \( \frac{\partial \psi}{\partial t} = 0 \)) of (2) if and only if
\[ f \tau(\psi) + \nabla f \cdot \nabla \psi = 0 \] (4)
i.e., \( \psi \) is an \( f \)-harmonic map. Consequently, there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (2) on the domain \( \Omega \) and the set of \( f \)-harmonic maps from \( \Omega \) into \( S^2 \). The inhomogeneous Heisenberg spin system (2) is also called inhomogeneous Landau-Lifshitz system (cf. [19, 23, 24]).

2.2. \( f \)-harmonic Maps

Let \( f : (M_1, g) \to (0, \infty) \) be a smooth function. Many aspects of the \( f \)-harmonic maps which generalize harmonic maps, were studied in [12,13,19,24] recently. Let \( \psi : (M_1, g) \to (M_2, h) \) be a smooth map from an \( m \)-dimensional Riemannian manifold \( (M_1, g) \) into an \( n \)-dimensional Riemannian manifold \( (M_2, h) \). A map \( \psi : (M_1, g) \to (M_2, h) \) is \( f \)-harmonic if and only if \( \psi \) is a critical point of the \( f \)-energy
\[ E_f(\psi) = \frac{1}{2} \int_{M_1} f|d\psi|^2 dv. \] (6)
In terms of the Euler-Lagrange equation, \( \psi \) is \( f \)-harmonic if and only if the \( f \)-tension field
\[ \tau_f(\psi) = f \tau(\psi) + d\psi(\text{grad } f) = 0 \] (5)
where \( \tau(\psi) = \text{Tr}_g Dd\psi \) is the tension field of \( \psi \). In particular, when \( f = 1 \), \( \tau_f(\psi) = \tau(\psi) \).

Let \( F : [0, \infty) \to [0, \infty) \) be a \( C^2 \) function such that \( F' > 0 \) on \( (0, \infty) \). \( F \)-harmonic maps between Riemannian manifolds were introduced in [1, 2]. For a smooth map \( \psi : (M_1, g) \to (M_2, h) \) of Riemannian manifolds, the \( F \)-energy of \( \psi \) is defined by
\[ E_F(\psi) = \int_{M_1} F(|d\psi|^2) dv. \] (6)
When $F(t) = t^{\frac{2(p^2-2)}{p}} (p \geq 4), (1 + 2t)^{\alpha} (\alpha > 1, \dim M_1 = 2)$, and $e^t$, they are the energy, the $p$-energy, the $\alpha$-energy of Sacks-Uhlenbeck [28], and the exponential energy, respectively. A map $\psi$ is $F$-harmonic if and only if $\psi$ is a critical point of the $F$-energy functional. In terms of the Euler-Lagrange equation, $\psi : M_1 \to M_2$ is an $F$-harmonic map if and only if the $F$-tension field

$$\tau_F(\psi) = F'(|d\psi|^2)\tau(\psi) + \psi (\text{grad}(F'(|d\psi|^2))) = 0.$$  

(7)

Proposition 1. 1) If $\psi : (M_1, g) \to (M_2, h)$ an $F$-harmonic map without critical points (i.e., $|d\psi| \neq 0$ for all $x \in M_1$), then it is an $f$-harmonic map with $f = F(\frac{2|d\psi|^2}{2})$. In particular, a $p$-harmonic map without critical points is an $f$-harmonic map with $f = \frac{|d\psi|^2}{2}$.

2) [15, 25]. A map $\psi : (M^n_1, g) \to (M^n_2, h)$ is $f$-harmonic if and only if $\psi : (M^n_1, \frac{f \nabla \nabla g}{f}) \to (M^n_2, h)$ is a harmonic map.

Proof: 1) It follows from (5) and (7) immediately (cf. Corollary 1.1 in [26]).

2) See [15].

3. $f$-biharmonic maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. $f$-biharmonic maps between Riemannian manifolds which generalized biharmonic maps [20, 21], were first studied by Ouakkas, Nasri and Djaa [27] recently. An $f$-biharmonic map $\psi : (M_1, g) \to (M_2, h)$ between Riemannian manifolds is the critical point of the $f$-bienergy functional

$$(E_f)(\psi) = \frac{1}{2} \int_{M_1} ||\tau_f(\psi)||^2 dv$$

(8)

where the $f$-tension field $\tau_f(\psi) = f\tau(\psi) + d\psi(\text{grad} f)$. In terms of Euler-Lagrange equation, $\psi$ is $f$-biharmonic if and only if the $f$-biharmonic field $\psi$

$$(\tau)^*_f(\psi) = \pm \Delta^*_f(\tau)(\psi) = 0$$

(9)

where

$$\Delta^*_f(\tau)(\psi) = DfD\tau(\psi) - fD\tau(\psi) = \sum_{i=1}^{m} \left( D_{\epsilon_i} f \epsilon_i \tau(\psi) - f D_{\epsilon_i} \tau(\psi) \right).$$

Here, $D, \hat{D}$ are the connections on $TM_1, \psi^{-1}TM_2$, respectively, $\{\epsilon_i\}_{1 \leq i \leq m}$ is a local orthonormal frame at any point in $M_1$ and $R'$ is the Riemannian curvature of
Suppose that the compact supports of the maps $\partial \psi$.

It follows from Eells-Lemaire [15] results that $\partial \psi$.

Corollary 3 ([20]) If $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map from a compact Riemannian manifold $M_1$ into a Riemannian manifold $M_2$ with non-positive curvature, then $\psi$ is harmonic.

Proof: Since $\psi : M_1 \to M_2$ is $f$-biharmonic, it follows from (9) that

$$\frac{1}{2} f \Delta ||\psi||^2 = f \{ D_\psi \tau_f, D_\psi \tau_f \} + f \{ D^{\star} D \tau_f, \tau_f \}$$

where $D^{\star} = D \circ D - D D$ by (10), $f > 0$ and $R' \leq 0$. It implies that

$$\frac{1}{2} \Delta ||\psi||^2 \geq 0.$$

By applying the Bochner’s technique, we know that $||\psi||^2$ is constant and that

$$D_i \psi_i = 0$$

for all $i = 1, 2, ..., m$.

It follows from Eells-Lemaire [15] results that $\tau_f(\psi) = 0$, i.e., $\psi$ is $f$-harmonic on $M_1$. □

Corollary 3 ([20]) If $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map from a compact Riemannian manifold $M_1$ into a Riemannian manifold $M_2$ with non-positive curvature, then $\psi$ is harmonic.
two Riemannian manifolds and then \( \varphi \) is harmonic. However, if \( \psi: (M_1, g) \to (M_2, h) \) is an \( f \)-biharmonic map, and \( \phi: (M_2, h) \to (M_3, k) \) is totally geodesic, then \( \phi \circ \psi: (M_1, g) \to (M_3, k) \) is not necessarily an \( f \)-biharmonic map. We obtain the following theorem instead.

**Theorem 4.** If \( \tau_\psi(\psi) \) is a Jacobi field for a smooth map \( \psi: (M_1, g) \to (M_2, h) \) of two Riemannian manifolds, and \( \phi: (M_2, h) \to (M_1, k) \) is a totally geodesic map of two Riemannian manifolds, then \( \tau_\psi(\phi \circ \psi) \) is a Jacobi field.

**Proof:** When \( f = 1 \) and \( \psi: M_1 \to M_2 \) is a biharmonic map from a compact Riemannian manifold into a Riemannian manifold \( M_2 \) with non-positive curvature, (11) becomes

\[
\tau_\psi(\psi) = D^* D_\psi(\psi) + R(\tau(\psi), \psi) d\psi = 0.
\]

The identity (13) reduces to

\[
\frac{1}{2} \| \tau(\psi) \|^2 = \langle D_\psi \tau(\psi), D_\psi \tau(\psi) \rangle + \langle D^* D_\psi(\psi), \tau(\psi) \rangle
\]

\[
= \langle D_\psi \tau(\psi), D_\psi \tau(\psi) \rangle - \langle R(d\psi, d\psi) \tau(\psi), \tau(\psi) \rangle \geq 0
\]

since \( \psi \) is biharmonic, and \( M_2 \) is a Riemannian manifold with non-positive curvature \( R' \). Note that (10) is automatically satisfied. It follows from the similar arguments as Theorem 3.1 that \( \psi \) is harmonic.

It is well-known from [18] that if \( \psi: (M_1, g) \to (M_2, h) \) is a harmonic map of two Riemannian manifolds and \( \phi: (M_2, h) \to (M_3, k) \) is totally geodesic of two Riemannian manifolds, then \( \phi \circ \psi: (M_1, g) \to (M_3, k) \) is harmonic. However, if \( \psi: (M_1, g) \to (M_2, h) \) is an \( f \)-biharmonic map, and \( \phi: (M_2, h) \to (M_3, k) \) is totally geodesic, then \( \phi \circ \psi: (M_1, g) \to (M_3, k) \) is not necessarily an \( f \)-biharmonic map. We obtain the following theorem instead.

**Theorem 4.** If \( \tau_\psi(\psi) \) is a Jacobi field for a smooth map \( \psi: (M_1, g) \to (M_2, h) \) of two Riemannian manifolds, and \( \phi: (M_2, h) \to (M_1, k) \) is a totally geodesic map of two Riemannian manifolds, then \( \tau_\psi(\phi \circ \psi) \) is a Jacobi field.

**Proof:** Let \( D, D', D^*, D^*, D' \) and \( \bar{D}, \bar{D}^* \) are the respective connections on \( TM_1, TM_2, \phi^{-1} TM_2, \phi^{-1} TM_3, \phi^{-1} TM_1, T^* M_1 \otimes \phi^{-1} TM_2, T^* M_2 \otimes \phi^{-1} TM_1 \) and \( T^* M_2 \otimes (\phi \circ \psi)^{-1} TM_3 \). Then we have

\[
D^*_\xi d(\phi \circ \psi)(Y) = (\bar{D}^*_{d\phi(X)} d\phi)(Y) + d\phi \circ \bar{D}X d\psi(Y)
\]

(13)

for all \( X, Y \in \Gamma(TM_1) \). We have also

\[
R^M_{\xi}(d\phi(X'), d\phi(Y')) d\phi(Z') = R^{\phi^{-1} TM_3}(X', Y') d\phi(Z')
\]

(14)

for all \( X', Y', Z' \in \Gamma(TM_2) \).

It is well-known from [18] that the tension field of the composition \( \phi \circ \psi \) is given by

\[
\tau(\phi \circ \psi) = d\phi(\tau(\psi)) + Tr_\psi Dd\phi(d\psi, d\psi) = d\phi(\tau(\psi))
\]
since φ is totally geodesic. Then the f-tension field of the composition of φ ◦ ψ is
\[ \tau_f(\phi \circ \psi) = d\phi(\tau_f(\psi)) + f\text{Tr}_gD\phi(d\psi, d\psi) = d\phi(\tau_f(\psi)) \]
since φ is totally geodesic. Recall that \{e_i\}_{i=1}^m is a local orthonormal frame at any point in \( M \), and let \( D^*D = D_{e_i}D_{e_k} - D_{D_{e_i}e_k} \) and \( D^{**}D^* = D_{e_i}^{**}D_{e_k}^* - D_{D_{e_i}e_k}^{**}D_{e_k}^* \).

Then we have
\[ D^{**}D^* \tau_f(\phi \circ \psi) = D^{**}D^*(d\phi \circ \tau_f(\psi)) \]
\[ = D_{e_i}^{**}D_{e_k}^*(d\phi \circ \tau_f(\psi)) - D_{D_{e_i}e_k}^{**}D_{e_k}^*(d\phi \circ \tau_f(\psi)). \]  
(15)

We derive from (13) that
\[ D_{e_i}^*(d\phi \circ \tau_f(\psi)) = (D_{D_{e_i}e_k}^*(d\phi)(\tau_f(\psi)) + d\phi \circ D_{e_k}(\tau_f(\psi)) \]
\[ = d\phi \circ D_{e_k} \tau_f(\psi) \]
(16)

since φ is totally geodesic. Therefore, we arrive at
\[ D_{e_i}^*(d\phi \circ \tau_f(\psi)) = D_{e_k}^*(d\phi \circ D_{e_k} \tau_f(\psi)) = d\phi \circ D_{e_k} \tau_f(\psi) \]
(17)

and
\[ D_{D_{e_i}e_k}^*(d\phi \circ \tau(\psi)) = d\phi \circ D_{D_{e_i}e_k} \tau_f(\psi). \]
(18)

On the other hand, it follows from (14) that
\[ R^{M_1}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i) \]
\[ = R^{M_1}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i) \]
\[ = d\phi \circ R^{M_1}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i). \]  
(19)

By (18) and (19), we obtain
\[ D^{**}D^* \tau_f(\phi \circ \psi) + R^{M_1}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i) \]
\[ = d\phi \circ [D^* \tau_f(\phi \circ \psi) + R^{M_1}(d(\phi \circ \psi)(e_i), \tau_f(\phi \circ \psi))d(\phi \circ \psi)(e_i)]. \]  
(20)

Consequently, if \( \tau_f(\phi \circ \psi) \) is a Jacobi field, then \( \tau_f(\phi \circ \psi) \) is a Jacobi field.
Corollary 5 ([8]) If \( \psi : (M_1, g) \to (M_2, h) \) is a biharmonic map between two Riemannian manifolds and \( \phi : (M_2, h) \to (M_3, k) \) is totally geodesic, then \( \phi \circ \psi : (M_1, g) \to (M_3, k) \) is a biharmonic map.

Proof: If \( f = 1 \) and \( \psi : (M_1, g) \to (M_2, h) \) is a biharmonic map of two Riemannian manifolds, then \( \tau_f(\psi) = \tau(\psi) \) is a Jacobi field. We can apply the analogous arguments as Theorem 3.3, and (20) becomes

\[
\bar{D}'\bar{D}'\tau(\phi \circ \psi) + R^M(\phi \circ \psi)(\epsilon_i, \tau(\phi \circ \psi))d(\phi \circ \psi)(\epsilon_i) = d\phi \circ [\bar{D}'\bar{D}\tau(\psi) + R^M(\phi(\epsilon_i), \tau(\psi))d\psi(\epsilon_i)]
\]

i.e., \( \tau(\phi \circ \psi) = d\phi \circ (\tau_2(\psi)) \), where \( \tau_2(\psi) \) is the bi-tension field of \( \psi \). Hence, we can conclude the result. ■

4. Stress \( f \)-bienergy Tensors

Let \( \psi : (M_1, g) \to (M_2, h) \) be a smooth map between two Riemannian manifolds. The stress energy tensor is defined by Baird and Eells [3] as

\[
S(\psi) = e(\psi)g - \psi^*h
\]

where \( e(\psi) = \frac{\|d\psi\|^2}{2} \). Thus we have \( \text{div} S(\psi) = -\langle \tau(\psi), d\psi \rangle \). Hence, if \( \psi \) is harmonic, then \( \psi \) satisfies the conservation law for \( S \) (i.e., \( \text{div} S(\psi) = 0 \)). In [27], the stress \( f \)-energy tensor of the smooth map \( \psi : M_1 \to M_2 \) was similarly defined as

\[
S^f(\psi) = fe(\psi)g - f\psi^*h
\]

and they obtained

\[
\text{div} S^f(\psi) = -\langle \tau^f(\psi), d\psi \rangle + e(\psi)d\Phi.
\]

In this case, an \( f \)-harmonic map usually does not satisfy the conservation law for \( S^f \). In particular, by letting \( f = F(\frac{\|d\psi\|^2}{2}) \), then \( S^f(\psi) = F(\frac{\|d\psi\|^2}{2})\psi^*g - F'(\frac{\|d\psi\|^2}{2})\psi^*h \). It is different than following Ara’s idea [1] to define \( S^F(\psi) \)

\[
= F(\frac{\|d\psi\|^2}{2})g - F'(\frac{\|d\psi\|^2}{2})\psi^*h,
\]

and we have

\[
\text{div} S^F(\psi) = -\langle \tau_F(\psi), d\psi \rangle.
\]

It implies that if \( \psi : M_1 \to M_2 \) is an \( F \)-harmonic map between Riemannian manifolds, then it satisfies the conservation law for \( S^F \).
The stress bienergy tensors and the conservation laws of biharmonic maps between Riemannian manifolds were first studied by Jiang [22] in 1987. Following his notions, we define the stress \( f \)-bienergy tensor of a smooth map as follows.

**Definition 6.** Let \( \psi : (M_1, g) \to (M_2, h) \) be a smooth map between two Riemannian manifolds. The stress \( f \)-bienergy tensor of \( \psi \) is defined by

\[
S_f^2(X, Y) = \frac{1}{2} |\tau_f(\psi)|^2 \langle X, Y \rangle + \langle d\psi, \bar{D}\tau_f(\psi) \rangle \langle X, Y \rangle
- \langle d\psi(X), D_Y \tau_f(\psi) \rangle - \langle d\psi(Y), D_X \tau_f(\psi) \rangle
\]

for all \( X, Y \in \Gamma(TM_1) \).

Remark that if \( \psi : (M_1, g) \to (M_2, h) \) is an \( f \)-biharmonic map between two Riemannian manifolds, then \( \psi \) does not necessarily satisfy the conservation law for the stress \( f \)-bienergy tensor \( S_f^2 \). Instead, we obtain the following theorem.

**Theorem 7.** If \( \psi : (M_1, g) \to (M_2, h) \) be a smooth map between two Riemannian manifolds, then we have

\[
\text{div} S_f^2(Y) = \pm \langle J_{\tau_f(\psi)}(Y), d\psi(Y) \rangle \quad \text{for all} \quad Y \in \Gamma(TM_1)
\]

where \( J_{\tau_f(\psi)} \) is the Jacobi field of \( \tau_f(\psi) \).

**Proof:** For the map \( \psi : M_1 \to M_2 \) between two Riemannian manifolds, set \( S_f^2 = K_1 + K_2 \), where \( K_1 \) and \( K_2 \) are \((0, 2)\)-tensors defined by

\[
K_1(X, Y) = \frac{1}{2} |\tau_f(\psi)|^2 \langle X, Y \rangle + \langle d\psi, \bar{D}\tau_f(\psi) \rangle \langle X, Y \rangle
- \langle d\psi(X), D_Y \tau_f(\psi) \rangle - \langle d\psi(Y), D_X \tau_f(\psi) \rangle
\]

\[
K_2(X, Y) = - \langle d\psi(X), D_Y \tau_f(\psi) \rangle - \langle d\psi(Y), D_X \tau_f(\psi) \rangle.
\]

Let \( \{e_i\} \) be the geodesic frame at a point \( a \in M_1 \), and write \( Y = Y^i e_i \) at the point \( a \). We first compute

\[
\text{div} K_1(Y) = \sum_i (D_{e_i} K_1)(e_i, Y) = \sum_i (e_i (K_1(e_i, Y) - K_1(e_i, D_{e_i} Y)))
= \sum_i (e_i (\frac{1}{2} |\tau_f(\psi)|^{2} Y^{i} + \sum_k (d\psi(e_k), \bar{D}_{e_k} \tau_f(\psi)) Y^{i}))
= \frac{1}{2} |\tau_f(\psi)|^{2} Y^{i} e_i - \sum_k (d\psi(e_k), \bar{D}_{e_k} \tau_f(\psi)) Y^{i} e_i
\]
where

Proof:

Adding (24) and (25), we arrive at

If

Theorem 9 ([22]): If \( f : (M_1, g) \rightarrow (M_2, h) \) is biharmonic between two Riemannian manifolds, then it satisfies the conservation law for stress bienergy tensor \( S_f \).

Corollary 8: If \( \tau_f (\psi) \) is a Jacobi field \( (i.e., J_{\tau_f (\psi)} = 0) \) for a map \( \psi : M_1 \rightarrow M_2 \), then it satisfies the conservation law \( (i.e., \text{div } S_f^2 = 0) \) for the stress \( f \)-bienergy tensor \( S_f^2 \).

Proof: If \( f = 1 \) and \( \psi : (M_1, g) \rightarrow (M_2, h) \) is biharmonic, then (26) yields to

\[
\text{div } S_f^2(Y) = \pm \langle d\psi(Y), \triangle \tau(Y) \rangle + \sum_i \langle d\psi(X_i), R'(Y, X_i) \tau(Y) \rangle
\]

Adding (24) and (25), we arrive at

\[
\text{div } S_f^2(Y) = \pm \langle d\psi(Y), \triangle \tau(Y) \rangle + \sum_i \langle d\psi(X_i), R'(Y, X_i) \tau(Y) \rangle = \pm \langle J_{\tau_f (\psi)}(Y), d\psi(Y) \rangle
\]

where \( J_{\tau_f (\psi)} \) is the Jacobi field of \( \tau_f (\psi) \).

\[
\]
where $\tau_2(\psi)$ is the bi-tension field of $\psi$ (i.e., $\tau(\psi)$ is a Jacobi field). Hence, we can conclude the result.

Proposition 10. Let $\psi : (M_1, g) \rightarrow (M_2, h)$ be a submersion such that $\tau_f(\psi)$ is basic, i.e., $\tau_f(\psi) = W \circ \psi$ for $W \in \Gamma(TM_2)$. Suppose that $W$ is Killing and $|W|^2 = c^2$ is non-zero constant. If $M_1$ is non-compact, then $\tau_f(\psi)$ is a non-trivial Jacobi field.

Proof: Since $\tau_f(\psi)$ is basic

$$S_f^2(X, Y) = \frac{c^2}{2} + \langle d\psi, D\tau_f(\psi) \rangle(X, Y) - \langle d\psi(X), D\tau_f(\psi) \rangle - \langle d\psi(Y), D\tau_f(\psi) \rangle$$

(26)

where $X, Y \in \Gamma(TM_1)$. Let $a$ be a point in $M_1$ with the orthonormal frame $\{e_i\}_{i=1}^m$ such that $\{e_j\}_{j=1}^n$ are in $T_a^\perp M_1$ and $\{e_k\}_{k=n+1}^m$ are in $T_a^\perp M_1 = \text{Ker} \, d\psi(a)$. Because $W$ is Killing, we have

$$\langle d\psi, D\tau_f(\psi) (a) \rangle = \sum_j (d\psi_a(e_j), D_{\partial/e_j} \tau_f(\psi)) + \sum_k (d\psi_a(e_k), D_{\partial/e_k} \tau_f(\psi))$$

$$= \sum_j (d\psi_a(e_j), D_{\partial/e_j} W) = 0.$$  

(27)

Therefore,

$$S_f^2(a)(X, Y) = \frac{c^2}{2} (X, Y) + \langle d\psi_a(X), D_{\partial/e_j} W \rangle$$

$$- \langle d\psi_a(Y), D_{\partial/e_j} W \rangle = \frac{c^2}{2} (X, Y).$$

If $M_1$ is not compact, $S_f^2 = \frac{c^2}{2} g$ is divergence free and $\tau_f(\psi)$ is a non-trivial Jacobi field due to $c \neq 0$.

Proposition 11. If $\psi : (M_1^1, g) \rightarrow (M_2, h)$ is a map from a surface with $S_f^2 = 0$, then $\psi$ is $f$-harmonic.

Proof: Since $S_f^2 = 0$, it implies

$$0 = \text{Tr} S_f^2 = |\tau_f(\psi)|^2 + 2 \langle d\tau_f(\psi), d\psi \rangle - 2 \langle D\tau_f(\psi), d\psi \rangle = |\tau_f(\psi)|^2.$$  


Proposition 12. If $\psi : (M^n, g) \rightarrow (M_2, h)$ ($m \neq 2$) with $S^f_2 = 0$, then
\[
\frac{1}{m - 2}|\tau_f(\psi)|^2(X, Y) + \langle D_X\tau_f(\psi), d\psi(Y) \rangle + \langle D_Y\tau_f(\psi), d\psi(X) \rangle = 0 \tag{28}
\]
for $X, Y \in \Gamma(T(M_1))$.

Proof: Suppose that $S^f_2 = 0$, it implies $\text{Tr} S^f_2 = 0$. Therefore
\[
\langle \bar{D}\tau_f(\psi), d\psi \rangle = -\frac{m}{2(m - 2)}|\tau_f(\psi)|^2 \quad m \neq 2. \tag{29}
\]
Substituting it into the definition of $S^f_2$, we arrive at
\[
0 = S^f_2(X, Y) = -\frac{1}{m - 2}|\tau_f(\psi)|^2(X, Y)
-\langle D_X\tau_f(\psi), d\psi(Y) \rangle - \langle D_Y\tau_f(\psi), d\psi(X) \rangle. \tag{30}
\]
\[
\blacksquare
\]

Corollary 13. If $\psi : (M_1, g) \rightarrow (M_2, h)$ ($m > 2$) with $S^f_2 = 0$ and rank $\psi \leq m - 1$, then $\psi$ is $f$-harmonic.

Proof: Since rank $\psi(a) \leq m - 1$, for a point $a \in M_1$ there exists a unit vector $X_a \in \text{Ker} d\psi_a$. Letting $X = Y = X_a$, (28) gives to $\tau_f(\psi) = 0$.

\[
\blacksquare
\]

Corollary 14. If $\psi : (M_1, g) \rightarrow (M_2, h)$ is a submersion ($m > n$) with $S^f_2 = 0$, then $\psi$ is $f$-harmonic.

References


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