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THE GAUSS MAP OF MINIMAL GRAPHS IN THE HEISENBERG GROUP

CHRISTIAM FIGUEROA

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Abstract. In this paper we study some geometric properties of surfaces in the Heisenberg group, $\mathcal{H}_3$. We obtain, using the Gauss map for Lie groups, a partial classification of minimal graphs in $\mathcal{H}_3$.

1. Introduction

The classical Heisenberg group, $\mathcal{H}_3$, is the group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \qquad r, t, s \in \mathbb{R}. \tag{1}$$

This group is a two-step nilpotent (or quasi-abelian) Lie group, which is the nearest condition to be abelian. Endowed with a left invariant metric g, the isometry group of $(\mathcal{H}_3, g)$ is four-dimensional. It is known that there is no three-dimensional Riemannian manifold with isometry group of dimension five, so $(\mathcal{H}_3, g)$ has isometry group of the largest possible dimension for a non-constant curvature space.

In this paper we will fix a left invariant Riemannian metric in $\mathcal{H}_3$ and study the geometry of surfaces with special emphasis on minimal surfaces and the relationship with their Gauss map.

We have organized the paper as follows. Section 2 we present the basic geometry of the Heisenberg group, $\mathcal{H}_3$ including a basis for left invariant fields.

In Section 3 we study the non parametric surfaces in $\mathcal{H}_3$. We calculate the coefficients of the first and second fundamental form and the Gaussian curvature of this type of surfaces.

In Section 4 we present the Gauss map for hypersurfaces of any Lie group and present a relationship between this map and the second fundamental form and give a direct proof of a non existence of umbilical surfaces in $\mathcal{H}_3$.

In Section 5, we present the classification of minimal graphs in $\mathcal{H}_3$ when the rank of its Gauss map is zero and one and finally we present some conditions in order that a minimal graph is a plane.

2. The Geometry of the Heisenberg Group

The three-dimensional Heisenberg group $\mathcal{H}_3$ is a two-step nilpotent Lie group. It has the standard representation in $\mathrm{GL}(3,\mathbb{R})$ specified in (1).

In order to describe a left-invariant metric on $\mathcal{H}_3$, we note that the Lie algebra $\mathfrak{h}_3$ of $\mathcal{H}_3$ is given by the matrices

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

with x, y, z real. The exponential map $\exp : \mathfrak{h}_3 \to \mathcal{H}_3$ is a global diffeomorphism, and is given by

$$\exp(A) = I + A + \frac{A^2}{2} = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the exponential map as a global parametrization, with the identification of the Lie algebra $\mathfrak{h}_3$ with $\mathbb{R}^3$ given by

$$(x, y, z) \leftrightarrow \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

the group structure of $\mathcal{H}_3$ is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{x_1 y_2 - x_2 y_1}{2}). \qquad (2)$$

From now on, modulo the identification given by $\exp$, we consider $\mathcal{H}_3$ as $\mathbb{R}^3$ with the product given in (2). Notice, in this model, the one-parameter subgroups are straight lines. The Lie algebra bracket, in terms of the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, is given by

$$[e_1, e_2] = e_3, \qquad [e_i, e_3] = 0$$

with $i = 1, 2, 3$. Now, using $\{e_1, e_2, e_3\}$ as the orthonormal frame at the identity, we have the following left-invariant metric $\mathrm{d}s^2$ in $\mathcal{H}_3$

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + (\frac{1}{2} y \mathrm{d}x - \frac{1}{2} x \mathrm{d}y + \mathrm{d}z)^2.$$

And the basis of the orthonormal left-invariant vector fields is given by

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad E_2 = \frac{\partial}{\partial x} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad E_3 = \frac{\partial}{\partial z}\cdot$$

Then the Riemann connection of $\mathrm{d}s^2$, in terms of the basis $\{E_i\}$, is given by

$$\begin{aligned}
\nabla_{E_1} E_2 &= \frac{1}{2} E_3 = -\nabla_{E_2} E_1 \\
\nabla_{E_1} E_3 &= -\frac{1}{2} E_2 = \nabla_{E_3} E_1 \\
\nabla_{E_2} E_3 &= \frac{1}{2} E_1 = \nabla_{E_3} E_2
\end{aligned}$$

and $\nabla_{E_i} E_i = 0$ for $i = 1, 2, 3$.

3. Graphs Over the xy-plane in $\mathcal{H}_3$

Let S be a graph of a smooth function $f : \Omega \to \mathbb{R}$ where Ω is an open set of $\mathbb{R}^2$. We consider the following parametrization of S

$$X\,(x, y) = (x, y, f(x, y)), \qquad (x, y) \in \Omega. \tag{3}$$

A basis of the tangent space $T_p S$ associated to this parametrization is given by

$$\begin{aligned}
X_x &= (1, 0, f_x) = E_1 + \left(f_x + \frac{y}{2}\right) E_3 \\
X_y &= (0, 1, f_y) = E_2 + \left(f_y - \frac{x}{2}\right) E_3
\end{aligned} \tag{4}$$

and its unit normal vector is given by

$$\eta\,(x, y) = -\left(\frac{f_x + \frac{y}{2}}{w}\right) E_1 - \left(\frac{f_y - \frac{x}{2}}{w}\right) E_2 + \frac{1}{w} E_3 \tag{5}$$

where

$$w = \sqrt{1 + \left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2}. \tag{6}$$

Then the coefficients of the first fundamental form of S are given by

$$\begin{aligned}
E &= < X_x, X_x > = 1 + \left(f_x + \frac{y}{2}\right)^2 \\
F &= < X_y, X_x > = \left(f_x + \frac{y}{2}\right)\left(f_y - \frac{x}{2}\right) \\
G &= < X_y, X_y > = 1 + \left(f_y - \frac{x}{2}\right)^2.
\end{aligned} \tag{7}$$

If ∇ is the Riemannian connection of $\left(\mathcal{H}_3, \mathrm{d}s^2\right)$, by Weingarten formula for hypersurfaces, we have that

$$A_\eta v = -\nabla_v \eta, \qquad v \in T_p S$$

and the coefficients of the second fundamental form are given by

$$\begin{aligned} L &= -<\nabla_{X_x}\eta, X_x> = \frac{f_{xx} + (f_y - \frac{x}{2})(f_x + \frac{y}{2})}{w} \\ M &= -<\nabla_{X_x}\eta, X_y> = \frac{f_{xy} + \frac{1}{2}\left(f_y - \frac{x}{2}\right)^2 - \frac{1}{2}\left(f_x + \frac{y}{2}\right)^2}{w} \\ N &= -<\nabla_{X_y}\eta, X_y> = \frac{f_{yy} - \left(f_y - \frac{x}{2}\right)\left(f_x + \frac{y}{2}\right)}{w}. \end{aligned} \tag{8}$$

To end this section we calculate the Gauss curvature of a non-parametric surface, that is, a surface which is a graph over the xy-plane. You can see the same formula in [2].

Theorem 1. *Let S be a non-parametric surface in $\mathcal{H}_3$ given by $(x, y, f(x, y))$ with $(x, y) \in \Omega \subset \mathbb{R}^2$. Then the Gauss curvature of S is given by*

$$\begin{aligned} w^4 K = w^2(f_{xy}^2 - f_{xx}f_{yy} - \frac{1}{4}) - (1 + q^2)(f_{xy} + \frac{1}{2})^2 - f_{xx}f_{yy} \\ - (1 + p^2)(f_{xy} - \frac{1}{2})^2 - f_{xx}f_{yy} + pq(f_{yy} - f_{xx}) \end{aligned}$$

where p, q and w are defined by

$$p = f_x + \frac{y}{2}, \qquad q = f_y - \frac{x}{2}, \qquad w = \sqrt{1 + p^2 + q^2}.$$

Proof: We recall the following formula from the Gauss equation for isometric immersions for this case

$$K(X_x, X_y) - \bar{K}(X_x, X_y) = \det A_\eta$$

where X_x, X_y is the basis of S, associated to the parametrization (3), K and $\bar{K}$ are the sectional curvatures of S and $\mathcal{H}_3$ respectively. Using this basis, we have

$$\nabla_{X_x}(\nabla_{X_x} X_x) = [\frac{1}{2} f_{xx} - \frac{1}{2}(f_y - \frac{x}{2})(f_x + \frac{y}{2})]E_1 - (f_{xy} + \frac{1}{2})E_2 + f_{xxy}E_3.$$

In the same way

$$\begin{aligned}\nabla_{X_x}(\nabla_{X_y}X_x) = [\frac{1}{2}f_{xx} - \frac{1}{4}(f_y - \frac{x}{2})(f_x + \frac{y}{2})]E_1 \\ - [f_{xy} + \frac{1}{4}(f_x + \frac{y}{2})^2 - \frac{1}{4}]E_2 + [f_{xxy} - \frac{1}{4}(f_y - \frac{x}{2})]E_3.\end{aligned}$$

Since $[X_x, X_y] = 0$, the curvature tensor of $\mathcal{H}_3$, is

$$R(X_x, X_y)X_x = -\frac{1}{4}(f_y - \frac{x}{2})(f_x + \frac{y}{2})E_1 + [\frac{1}{4}(f_x + \frac{y}{2})^2 - \frac{3}{4}]E_2 + \frac{1}{4}(f_y - \frac{x}{2})E_3$$

and its sectional curvature is given by

$$\bar{K}(X_x, X_y) = \frac{< R(X_x, X_y)X_x, X_y >}{\|X_x \wedge X_y\|^2} = \frac{1}{4} - \frac{1}{w^2}.$$

On the other hand, using (7) and (8), we have

$$\begin{aligned}\det A_\eta &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{f_{xx}f_{yy} + pq(f_{yy} - f_{xx}) - \frac{1}{4}(p^2 + q^2)^2 - f_{xy}^2 - f_{xy}(q^2 - p^2)}{w^4}.\end{aligned}$$

So the Gauss curvature of S, satisfies

$$w^4 K = f_{xx}f_{yy} - f_{xy}^2 + \frac{1}{4} + pq(f_{yy} - f_{xx}) + p^2(f_{xy} - \frac{1}{2}) - q^2(f_{xy} + \frac{1}{2}) - 1.$$

From this relation follows the formula. ■

Using the above formula, Inoguchi classified flat translation invariant surfaces while Dillen and van der Veken, constructed some examples of semi-parallel surfaces in $\mathcal{H}_3$, see [11] and [5] respectively.

4. The Gauss Map

Recall that the Gauss map is a function from an oriented surface, $S \subset \mathbb{E}^3$, to the unit sphere in the Euclidean space. It associates to every point on the surface its oriented unit normal vector. Considering the Euclidean space as a commutative Lie group, the Gauss map is just the translation of the unit normal vector at any point of the surface to the origin, the identity element of $\mathbb{R}^3$. Reasoning in this way we define a Gauss map in the following form.

Definition 2. *Let $S \subset G$ be an orientable hypersurface of a n-dimensional Lie group G, provided with a left invariant metric. The map*

$$\gamma = S \to S^{n-1} = \{v \in \tilde{g}\ ;\ |v| = 1\}$$

where $\gamma(p) = \mathrm{d}L_p^{-1} \circ \eta(p)$, $\tilde{g}$ is the Lie algebra of G and η is the unitary normal vector field of S, is called the Gauss map of S.

We observe that

$$\mathrm{d}\gamma(T_pS) \subseteq T_{\gamma(p)}S^{n-1} = \{\gamma(p)\}^{\perp} = \mathrm{d}L_p^{-1}(T_pS)$$

and therefore $\mathrm{d}L_p \circ \mathrm{d}\gamma(T_pS) \subseteq T_pS$.

We know also that in the Euclidean case the differential of the Gauss map is just the second fundamental form for surfaces in $\mathbb{R}^3$, this fact can be generalized for hypersurfaces in any Lie group. The following theorem, see [14], states a relationship between the Gauss map and the extrinsic geometry of S.

Theorem 3. *Let S be an orientable hypersurfaces of a Lie group. Then*

$$\mathrm{d}L_p \circ \mathrm{d}\gamma_p(v) = -\left(A_\eta(v) + \alpha_{\bar{\eta}}(v)\right), \qquad v \in T_pS$$

where A_η is the Weingarten operator, $\alpha_{\bar{\eta}}(v) = \nabla_v \bar{\eta}$ and $\bar{\eta}$ is the left invariant vector field such that $\eta(p) = \bar{\eta}(p)$.

In the case of orientable surfaces in $\mathcal{H}_3$ we shall obtain the expressions of the operators $\mathrm{d}L_p \circ \mathrm{d}\gamma_p$ and $a_{\bar{\eta}}$, when such a surface is the graph of a smooth function $f(x, y)$. In fact, using the basis $\{X_x, X_y\}$, given by the parametrization (3), we have

$$\begin{aligned} \mathrm{d}\gamma_p(X_x) &= \sum_{i=1}^{3} \frac{\partial a_i}{\partial x} E_i(e) \\ \mathrm{d}\gamma_p(X_y) &= \sum_{i=1}^{3} \frac{\partial a_i}{\partial y} E_i(e) \end{aligned}$$

where a_i are the components of the normal η, see (5), and $p \in S$. Hence

$$\begin{aligned} \mathrm{d}L_p \circ \mathrm{d}\gamma_p(X_x) &= \sum_{i=1}^{3} \frac{\partial a_i}{\partial x} E_i(p) \\ \mathrm{d}L_p \circ \mathrm{d}\gamma_p(X_y) &= \sum_{i=1}^{3} \frac{\partial a_i}{\partial y} E_i(p). \end{aligned}$$

On the other hand, we have that $\mathrm{d}L_p \circ \mathrm{d}\gamma(T_pS) \subseteq T_pS$, so

$$\begin{aligned} \mathrm{d}L_p \circ \mathrm{d}\gamma_p(X_x) &= aX_x + bX_y \\ \mathrm{d}L_p \circ \mathrm{d}\gamma_p(X_y) &= cX_x + dX_y. \end{aligned}$$

Using (4) and comparing the above two systems, we obtain the matrix of $\mathrm{d}L_p \circ \mathrm{d}\gamma_p$ in the basis $\{X_x, X_y\}$

$$\mathrm{d}L_p \circ \mathrm{d}\gamma_p = \begin{pmatrix} -\left(\dfrac{f_x + \dfrac{y}{2}}{w}\right)_x & -\left(\dfrac{f_x + \dfrac{y}{2}}{w}\right)_y \\ -\left(\dfrac{f_y - \dfrac{x}{2}}{w}\right)_x & -\left(\dfrac{f_y - \dfrac{x}{2}}{w}\right)_y \end{pmatrix}. \tag{9}$$

Notice that

$$\det\left(\mathrm{d}L_p \circ \mathrm{d}\gamma_p\right) = \frac{f_{xx}f_{yy} - f_{xy}^2 + \frac{1}{4}}{w^2} \tag{10}$$

and we will call this expression, the determinant of the Gauss map. Finally the matrix of $\alpha_{\bar{\eta}}$ in the basis $\{X_x, X_y\}$ is

$$\alpha_{\bar{\eta}} = \frac{1}{2w}\begin{pmatrix} -\left(f_x + \frac{y}{2}\right)\left(f_y - \frac{x}{2}\right) & 1 - \left(f_y - \frac{x}{2}\right)^2 \\ \left(f_x + \frac{y}{2}\right)^2 - 1 & \left(f_x + \frac{y}{2}\right)\left(f_y - \frac{x}{2}\right) \end{pmatrix} \tag{11}$$

where w is like (6). Observe that in the case of surface in the Heisenberg group the trace of $\alpha_{\bar{\eta}}$ is zero. Our first result is

Theorem 4. *The vertical plane is the unique connected surface in $\mathcal{H}_3$ with the property that its Gauss map is constant.*

Proof: Let S be a surface in $\mathcal{H}_3$ parameterized as the graph of a smooth function $f(x, y)$. As we have seen, a basis of the tangent space of S is given by

$$\begin{aligned} X_x &= E_1 + \left(f_x + \frac{y}{2}\right) E_3 \\ X_y &= E_2 + \left(f_y - \frac{x}{2}\right) E_3. \end{aligned}$$

Now if there is $p \in S$ such that $\mathrm{d}\gamma_p = 0$, then $\mathrm{d}L_p^{-1}(T_pS)$ is a subalgebra of $\mathfrak{h}_3$, see [14]. But this is a contradiction, because $\left[\mathrm{d}L_p^{-1}(X_x), \mathrm{d}L_p^{-1}(X_y)\right] = e_3 \notin \mathrm{d}L_p^{-1}(T_pS)$. Therefore, there are no graphs in $\mathcal{H}_3$ such that its Gauss map is constant.

Now we consider S as a vertical surface. In this case we can consider such a surface as a ruled surface. We parameterize the surface by

$$X(t, s) = (t, a(t), s), \qquad (t, s) \in U \tag{12}$$

where U is an open set of $\mathbb{R}^2$. So the basis associated to this parametrization is

$$\begin{aligned} X_t &= E_1 + \dot{a}E_2 + (a - t\dot{a})\, E_3 \\ X_s &= E_3 \end{aligned}$$

and the unit normal field to this surface is

$$\eta = \frac{\dot{a}}{\sqrt{1 + (\dot{a})^2}} E_1 - \frac{1}{\sqrt{1 + (\dot{a})^2}} E_2.$$

Notice that η is constant iff $\dot{a}\,(t)$ is constant, that is, $a\,(t)$ is affine. ■

We remark that Piu proved that there are no totally geodesic hypersurfaces in $\mathcal{H}_{2n+1}$, see [13] and Sanini generalized this result, that is, there are no totally umbilical hypersurfaces in this group, see [15]. Finally van der Veken gave a full local classification of totally umbilical surfaces in three-dimensional homogeneous spaces with four-dimensional group, see [17]. Also, R. Souam and E. Toubiana, obtained the same result independently, see [16]. We present here an alternative proof of the following result.

Theorem 5. *There are no totally umbilical surfaces in $\mathcal{H}_3$.*

Proof: Let S be an umbilical surface which is, locally, the graph of a differentiable function f. Then

$$\begin{aligned} A_\eta\,(X_x) &= \lambda X_x \\ A_\eta\,(X_y) &= \lambda X_y \end{aligned}$$

where λ is a differentiable function. The Codazzi equation (in the umbilical case) is given by

$$R\,(X_x, X_y)\,\eta = X_y\,(\lambda)\, X_x - X_x\,(\lambda)\, X_y = \lambda_y X_x - \lambda_x X_y.$$

By replacing (4) and (5) into the last expression, we obtain

$$\begin{aligned} \lambda_x &= -\left(f_x + \tfrac{y}{2}\right)/w \\ \lambda_y &= -\left(f_y - \tfrac{x}{2}\right)/w. \end{aligned}$$

From (9) and the differentiability of λ we can see that $\mathrm{d}L_p \circ \mathrm{d}\gamma_p$ is represented by a symmetric matrix in the basis (4). Then using theorem 3 and (11), we conclude that the matrix representation of $\alpha_{\bar{\eta}}$ is also symmetric. Therefore

$$w = \sqrt{3}$$

where w is like (5). So, using again the fact that $\lambda_{xy} = \lambda_{yx}$, i.e.,

$$\left(f_x + \frac{y}{2}\right)_y = \left(f_y - \frac{x}{2}\right)_x$$

which is a contradiction.

Now assume that S is a vertical surface, we may consider it as a ruled surface, where the vertical lines are the rulings and the directrix, $a\,(t)$, lies in the $xy-plane$. As usual, we parameterize this surface like (12). The coefficients of the first fundamental form in the basis $\{X_t, X_s\}$ are given by

$$\begin{aligned} E &= 1 + \dot{a}^2 + (a - t\dot{a})^2/4 \\ F &= (a - t\dot{a})/2 \\ G &= 1 \end{aligned} \tag{13}$$

and the coefficients of the second fundamental form in the same basis are given by

$$\begin{aligned} L &= ((a - t\dot{a})(1 + \dot{a}^2) - 2\ddot{a})/2\sqrt{1 + \dot{a}^2} \\ M &= \sqrt{1 + \dot{a}^2}/2 \\ N &= 0. \end{aligned} \tag{14}$$

Since we have assumed that the surface was umbilical, we conclude that the Weingarten operator is a diagonal matrix in any basis, in particular in the basis associated to the parametrization (12), then $0 = NF - MG$, but this is a contradiction.

■

5. Minimal Graphs Over the xy-plane in $\mathcal{H}_3$

We recall firstly the mean curvature formula of any surface of $\mathcal{H}_3$ in terms of the coefficients of their first and second fundamental forms

$$H = \frac{1}{2}\left(\frac{EN + GL - 2FM}{EG - F^2}\right). \tag{15}$$

If the surface is the graph of a smooth function f, using (7) and (8) into the above equation, we obtain the equation of the minimal graphs in $\mathcal{H}_3$

$$\left(1 + q^2\right) f_{xx} - 2pq f_{xy} + \left(1 + p^2\right) f_{yy} = 0 \tag{16}$$

where $p = \left(f_x + \frac{y}{2}\right)$ and $q = \left(f_y - \frac{x}{2}\right)$. This equation appears for the first time in [1]. Before presenting some consequences of the above equation, we shall show some examples of minimal graphs and compute the rank of their Gauss map, using formula (10).

Example 6. *As in Euclidean space* $\mathbb{E}^3$, *the plane* $f(x,y) = ax + by + c$ *is a minimal graph of* $\mathcal{H}_3$. *The rank of the Gauss map is two.*

Another minimal graph may be obtained by searching for solutions of Scherk type, i.e., for solutions of the form $f(x,y) = \frac{xy}{2} + u(x) + v(y)$. From this method we find, among others, the following example, see [3].

Example 7. *A surface of saddle type*

$$f(x,y) = \frac{xy}{2} + k\left[\ln\left(y + \sqrt{1+y^2}\right) + y\sqrt{1+y^2}\right]$$

where $k \in \mathbb{R}$. *Notice that this minimal surface is ruled by affine lines, i.e., translations of one-parameter subgroups. The rank of its Gauss map is one.*

The following example was found by Daniel [4], using a Weierstrass representation.

Example 8. *Let* $f(x,y) = xh(y)$, *where* $h(y) = u - \frac{1}{2\coth u}$, u *and* y *are related by the equation* $y = \coth u - 2u$, $u > 0$. *In this case, the determinant of the Gauss map is equal to*

$$-\frac{1}{4}\left(\frac{1}{\coth^4 u} - 1\right), \qquad u > 0.$$

So, the rank of its Gauss map is two.

Unlike the case of Euclidean spaces, where the only complete minimal graphs are linear (Bernstein's theorem), we have several solutions defined on the entire xy-plane.

Let us come back to the minimal graph equation (16). Notice that this is a quasilinear, elliptic PDE. with analytic coefficients and therefore its solutions are analytic and satisfy the following maximum principle, see [10].

Theorem 9. *Consider an elliptic, differential equation of the form*

$$F[u] := F\left(x, y, u, Du, D^2u\right) = 0$$

with $F : S = \Omega \times \mathbb{R} \times \mathbb{R}^2 \times S(2,\mathbb{R}) \to \mathbb{R}$ *where* $S(2,\mathbb{R})$ *is the space of symmetric, real-valued,* 2×2 *matrices. Let* $u_0, u_1 \in C^2(\Omega) \cap C^0(\overline{\Omega})$, *and suppose*

1. $F \in C^1(S)$.
2. F *is elliptic at all functions* $tu_1 + (1-t)u_0$, $\qquad 0 \le t \le 1$.

3. $\dfrac{\partial F}{\partial u} \leq 0$ *in* Ω.

If $u_1 \leq u_0$ *on* $\partial(\Omega)$ *and* $F[u_1] \geq F[u_0]$ *in* Ω*, then either* $u_1 < u_0$ *in* Ω *or* $u_0 \equiv u_1$ *in* Ω*.*

A simple application of these facts imply the following result

Theorem 10. *There are no compact* (*i.e., bounded and closed*) *minimal surfaces in* $\mathcal{H}_3$*.*

Proof: Suppose S is a compact minimal surface (without boundary) in $\mathcal{H}_3$. Take the plane $z = c$, which is a minimal surface, such that the plane is tangent to S and S lies below the plane, so by the maximum principle, S locally coincides with the plane and, by analyticity, S is the plane, which contradicts compactness. ■

Unlike the minimal surface case, for graphs of non-zero constant mean curvature, we have a Bernstein type theorem, see [7].

Theorem 11. *There are no complete graphs of constant mean curvature* $H \neq 0$*.*

We shall now study the stability of minimal graphs. To explain this, we need to characterize the minimal surfaces as solution of a variational problem. Let S be a surface given by $z = f(x,y)$ with $(x,y) \in \Omega \subset \mathbb{R}^2$. Then we consider the following variation of S :

$$S_t(x,y) = (x, y, f(x,y) + th(x,y)), \qquad (x,y) \in \Omega$$

where $h \in C^1$ and $h|_{\partial\Omega} = 0$. Furthermore, the area of S_t over $\bar{\Omega}$ is

$$A(t) = \iint\limits_{\bar{\Omega}} w(t)\, \mathrm{d}x\mathrm{d}y$$

where $w(t) = \sqrt{1 + \left(f_x + th_x + \frac{y}{2}\right)^2 + \left(f_y + th_y - \frac{x}{2}\right)^2}$. Since S has least area among all surfaces of S_t , we have that S must be critical point of $A(t)$ i.e., $A'(0) = 0$. Now we compute the first derivative of $A(t)$

$$A'(t) = \iint\limits_{\bar{\Omega}} w^{-1} \left(\left(f_x + th_x + \frac{y}{2}\right) h_x + \left(f_y + th_y - \frac{x}{2}\right) h_y\right) \mathrm{d}x\mathrm{d}y. \qquad (17)$$

Evaluated at $t = 0$, integrating by parts, and using the fact that $h = 0$ on $\partial\Omega$, we find

$$A'(0) = \iint_{\bar{\Omega}} w^{-1} \left(\left(f_x + \frac{y}{2} \right) h_x + \left(f_y - \frac{x}{2} \right) h_y \right) \mathrm{d}x\mathrm{d}y$$

$$= \iint_{\bar{\Omega}} \left(\left(\frac{f_x + \frac{y}{2}}{w} \right)_x + \left(\frac{f_y - \frac{x}{2}}{w} \right)_y \right) h\mathrm{d}x\mathrm{d}y.$$

It follows that the equation

$$(\frac{f_x + \frac{y}{2}}{w})_x + (\frac{f_y - \frac{x}{2}}{w})_y = 0$$

must hold for all $(x, y) \in \Omega$. Notice, this is nothing but the minimal graph equation (16). Using this equation and the matrix representation of $\mathrm{d}L_p \circ \mathrm{d}\gamma_p$ we conclude the following

Proposition 12. *Let $f : \Omega \to \mathbb{R}$ be a smooth function. A graph of f is a minimal surface in $\mathcal{H}_3$ if and only if the trace of $\mathrm{d}L_p \circ \mathrm{d}\gamma_p$ is equal to zero.*

Now we are ready to prove the following

Proposition 13. *Every minimal graph in $\mathcal{H}_3$ is stable.*

Proof: It is sufficient to consider the second derivative of the area function, $A\,(t)$, evaluated at $t = 0$. From (17) we obtain that

$$A''\,(0) = \iint_{\bar{\Omega}} \frac{h_x^2 + h_y^2 + \left(\left(f_y - \frac{x}{2} \right) h_x - \left(f_x + \frac{y}{2} \right) h_y \right)^2}{w^2} \mathrm{d}x\mathrm{d}y.$$

Notice that $A''\,(0) \geq 0$ and is equal to zero if and only if $h_x = h_y = 0$ that is $h = 0$, because $h|_{\partial\Omega} = 0$. ■

Now we will give a classification of the minimal surfaces in $\mathcal{H}_3$ with Gauss map of rank zero and one. We begin with the rank zero case.

As we have seen in theorem (4) the vertical plane is the unique connected surface in $\mathcal{H}_3$ with the property that its Gauss map is constant. Then it remains to prove that such a surface is minimal. But the only minimal vertical surface in $\mathcal{H}_3$ is the vertical plane. In fact, if we replace, (13) and (14) in the mean curvature formula (15), when $H = 0$, we obtain that $\ddot{a}(t) = 0$, that is, the minimal surface is a vertical plane. So we conclude that the vertical plane is the only minimal surface in $\mathcal{H}_3$ with the property that its Gauss map is constant.

Now we study the minimal graphs of $\mathcal{H}_3$, whose Gauss map have rank one. That is

$$f_{xx}f_{yy} - f_{xy}^2 + \frac{1}{4} = 0. \tag{18}$$

Lemma 14. *Let $(x, y, f(x, y))$, with $(x, y) \in \Omega$, be a minimal graph in $\mathcal{H}_3$, which contains the origin, its normal at the origin is $\eta(0) = \frac{1}{\sqrt{1+4k^2}}(0, -2k, 1)$ and its Gauss map has rank one. Furthermore assume that $f_{yy}(0) = 0$, then*

$$f(x,y) = \begin{cases} \frac{xy}{2} + k\left[\ln\left(y + \sqrt{1+y^2}\right) + y\sqrt{1+y^2}\right] \\ 2ky - \frac{xy}{2}. \end{cases}$$

Proof: Since the unit normal at 0 is $\eta(0) = \frac{1}{\sqrt{1+4k^2}}(0, -2k, 1)$ we have, using (5), that $f_x(0) = 0$ and $f_y(0) = 2k$. On the other hand the Gauss map of such surface has rank one, then, using (18), we have

$$f_{xx}(0)\, f_{yy}(0) - f_{xy}^2(0) + \frac{1}{4} = 0.$$

Since f satisfies the minimal graph equation (16), we obtain

$$(1 + 4k^2) f_{xx}(0) + f_{yy}(0) = 0.$$

From the above two equations and the hypothesis, $f_{yy}(0) = 0$, we conclude that $f_{xy}(0) = \pm\frac{1}{2}$ and $f_{xx}(0) = 0$.

Recalling that f is an analytic function, we can write its Taylor expansion in the form $f(x, y) = 2ky \pm \frac{xy}{2} + \Psi(x, y)$. Substituting into (18) we obtain

$$\Psi_{xx}\Psi_{yy} - \Psi_{xy}^2 = \pm\Psi_{xy}.$$

Let n be the minimal order of $\Psi(x, y)$. We claim that the terms of minimal order of Ψ do not appear mixed. In fact, assuming that this is not the case, the minimal order of Ψ_{xy} and $\Psi_{xx}\Psi_{yy} - \Psi_{xy}^2$ are $(n-2)$ and $2(n-2)$ respectively. This is a contradiction and proves our claim.

We shall now compute the third partial derivatives of f at 0. To do this we differentiate, with respect to x and y, the minimal graph equation (16), the equation (18) and evaluate both at 0. We obtain the following two cases

1. If $f_{xy}(0) = \frac{1}{2}$, we have that $f_{yyy}(0) = 2k$ and the others third derivatives of f are zero. Then the Taylor expansion of f about $(0, 0)$ has the form

$$f(x, y) = 2ky + \frac{xy}{2} + \psi(y) + ax^n + by^n + \tilde{\Psi}(x, y)$$

where $\psi(y)$ is a polynomial such that $3 \leq \deg\psi(y) \leq (n-1)$ with $n \geq 4$. A simple computation shows that

$$\begin{aligned}
f_x &= \tfrac{y}{2} + anx^{n-1} + \tilde{\Psi}_x \\
f_y &= 2k + \tfrac{x}{2} + \psi_y + bny^{n-1} + \tilde{\Psi}_y \\
f_{xx} &= an(n-1)x^{n-2} + \tilde{\Psi}_{xx} \\
f_{xy} &= \tfrac{1}{2} + \tilde{\Psi}_{xy} \\
f_{yy} &= \psi_{yy} + bn(n-1)y^{n-2} + \tilde{\Psi}_{yy}.
\end{aligned}$$

Substituting into (16)

$$\begin{aligned}
&(an(n-1)x^{n-2} + \tilde{\Psi}_{xx})[1 + (2k + \psi_y + bny^{n-1} + \tilde{\Psi}_y)^2] \\
&\quad - 2(\frac{1}{2} + \tilde{\Psi}_{xy})(2k + \psi_y + bny^{n-1} + \tilde{\Psi}_y)(y + anx^{n-1} + \tilde{\Psi}_x) \\
&\quad\quad + (\psi_{yy} + bn(n-1)y^{n-2} + \tilde{\Psi}_{yy})[1 + (y + anx^{n-1} + \tilde{\Psi}_x)^2] = 0.
\end{aligned}$$

If we analyze the coefficient of the term x^{n-2} we obtain that

$$(1 + 4k^2)an(n-1)x^{n-2} = 0.$$

Hence $a = 0$, that is, $f(x, y) = \frac{xy}{2} + g(y)$. We conclude that such a surface is invariant under translation of type $L_{(b,0,0)}$, see [7], and therefore

$$f(x, y) = \frac{xy}{2} + k[\ln(y + \sqrt{1 + y^2}) + y\sqrt{1 + y^2}]$$

for some $k \in \mathbb{R}$.

2. If $f_{xy}(0) = -\frac{1}{2}$, we have that the third partial derivatives of f, evaluated at the origin, are equal to zero, then the Taylor expansion of f about $(0, 0)$ has the form: $f(x, y) = 2ky - \frac{xy}{2} + ax^n + by^n + \tilde{\Psi}(x, y)$, where $n \geq 4$. In this case

$$\begin{aligned}
f_x &= -\tfrac{y}{2} + anx^{n-1} + \tilde{\Psi}_x \\
f_y &= 2k - \tfrac{x}{2} + bny^{n-1} + \tilde{\Psi}_y \\
f_{xx} &= an(n-1)x^{n-2} + \tilde{\Psi}_{xx} \\
f_{xy} &= \tilde{\Psi}_{xy} - \tfrac{1}{2} \\
f_{yy} &= bn(n-1)y^{n-2} + \tilde{\Psi}_{yy}.
\end{aligned}$$

Substituting into (16) we obtain

$$\begin{aligned}(an(n-1)x^{n-2}+\tilde{\Psi}_{xx})[1+(2k-x+bny^{n-1}+\tilde{\Psi}_y)^2]\\ -2(\tilde{\Psi}_{xy}-\frac{1}{2})(2k-x+bny^{n-1}+\tilde{\Psi}_y)(anx^{n-1}+\tilde{\Psi}_x)\\ +(bn(n-1)y^{n-2}+\tilde{\Psi}_{yy})[1+(anx^{n-1}+\tilde{\Psi}_x)^2]=0.\end{aligned}$$

If we analyze the coefficients of x^{n-2} and y^{n-2}, we conclude that $a=b=0$. Therefore

$$f(x,y)=2ky-\frac{xy}{2}\cdot$$

This conclude the proof. ■

Now we shall prove that every minimal graph with Gauss map of rank one must be a ruled surface, that is foliated by straight lines. Let S be such a surface, parameterized as a graph of a differentiable function f, with $f\,(0,0)=0$. Since S has rank one, there exists a curve in S, passing through the origin, such that the unit normal field along this curve is constant. We indicate this curve by $\Gamma(t)$, where

$$\Gamma(t)=(x(t),y(t),z(t))=(t,\alpha(t),f(t,\alpha(t)),\qquad t\in(-\epsilon,\epsilon)$$

and $\alpha(0)=0$. We can assume that the normal field at 0 is given by $\eta(0)=\frac{1}{\sqrt{1+4k^2}}(0,-2k,1)$. Then, along the curve $\Gamma(t)$, and using (5) we obtain

$$f_x(t)+\frac{\alpha(t)}{2}=0,\qquad f_y(t)-\frac{t}{2}=2k,\qquad t\in(-\epsilon,\epsilon).\tag{19}$$

Whence

$$\begin{aligned}f_{xx}(t)+\alpha' f_{xy}(t)+\tfrac{\alpha'}{2}&=0\\ f_{yx}(t)+\alpha' f_{yy}(t)-\tfrac{1}{2}&=0.\end{aligned}\tag{20}$$

We need also the second and third partial derivatives of f evaluated along the curve $\Gamma\,(t)$. From (19)and equations (16) and (18), we obtain the following expressions for the partial derivatives of f

$$f_{xx}(t)=\frac{\sin\theta}{2\sqrt{1+4k^2}},\qquad f_{xy}(t)=\frac{\cos\theta}{2},\qquad f_{yy}(t)=-\frac{\sqrt{1+4k^2}\sin\theta}{2}\cdot$$

Using this partial derivatives and the second equation of (20), we obtain

$$\alpha'(t)=\frac{\cos\theta-1}{\sin\theta\sqrt{1+4k^2}}\cdot\tag{21}$$

Now, by differentiating the equations (16) and (18), with respect to x and y, and evaluating at $(t, \alpha(t))$, we obtain the following system

$$\begin{pmatrix} \frac{-\sqrt{1+4k^2}\sin\theta}{2} & -\cos\theta & \frac{\sin\theta}{2\sqrt{1+4k^2}} & 0 \\ 0 & \frac{-\sqrt{1+4k^2}\sin\theta}{2} & -\cos\theta & \frac{\sin\theta}{2\sqrt{1+4k^2}} \\ (1+4k^2) & 0 & 1 & 0 \\ 0 & (1+4k^2) & 0 & 1 \end{pmatrix}\begin{pmatrix} f_{xxx}(t) \\ f_{xxy}(t) \\ f_{xyy}(t) \\ f_{yyy}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{k\sin\theta}{\sqrt{1+4k^2}} \\ k(1+\cos\theta) \end{pmatrix}.$$

Solving this system, we obtain the third partial derivatives of f along the curve $\Gamma(t)$

$$\begin{aligned} 2f_{xxx}(t) &= \frac{k\sin\theta(1-\cos\theta)}{2(1+4k^2)^{3/2}}, & f_{xxy}(t) &= \frac{k\sin^2\theta}{2(1+4k^2)} \\ f_{yyy}(t) &= \frac{k(1+\cos\theta)^2}{2}, & f_{yyx}(t) &= \frac{k\sin\theta(1+\cos\theta)}{2\sqrt{1+4k^2}}. \end{aligned} \tag{22}$$

We are now ready to prove

Theorem 15. *If $(x, y, f(x, y))$ with $(x, y) \in \Omega \subset \mathbb{R}^2$ is a minimal graph such that its normal at the origin is $\eta(0) = \frac{1}{\sqrt{1+4k^2}}(0, -2k, 1)$ and its Gauss map has rank one, then it is a ruled surface.*

Proof: We shall show that the curve Γ, defined above, must be a straight line. To do this , we differentiate the third component of Γ, that is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_x(t) + \alpha' f_y(t).$$

From this expression and (20), we have that $\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \alpha'' f_y(t)$. By differentiating the second equation of (20) with respect to t, we obtain

$$f_{yxx}(t) + 2\alpha' f_{yyx}(t) + (\alpha')^2 f_{yyy}(t) + \alpha'' f_{yy}(t) = 0. \tag{23}$$

Then we have two cases

If $f_{yy}(0) = 0$, using the proposition (14), we conclude that such a surface is a ruled surface.

If $f_{yy}(0) \neq 0$ then, $f_{yy}(t) \neq 0$, $t \in (-\varepsilon, \varepsilon)$ By replacing (22) and (21) in (23), we obtain that $\alpha'' f_{yy}(t) = 0$, $t \in (-\varepsilon, \varepsilon)$. Then $\alpha(t) = bt$ and therefore, from (19), we conclude that $f(t, bt) = 2kbt$. This completes the proof. ■

The following classification for ruled minimal surfaces in $\mathcal{H}_3$ was proved by Bekkar and Sari, see [3].

Theorem 16. *The ruled minimal surfaces of $\mathcal{H}_3$, up isometry, are*

1. *The plane*

2. *The hyperbolic paraboloid*

3. *The helicoid parameterized by*

$$\begin{cases} x(t,s) &= s\sin t \\ y(t,s) &= s\cos t \\ z(t,s) &= \rho t, \qquad \rho \in \mathbb{R} - \{0\}. \end{cases}$$

4. *The surface given by the equation*

$$z = \frac{xy}{2} - \frac{\lambda}{2}\left[y\sqrt{1+y^2} + \log\left(y + \sqrt{1+y^2}\right)\right], \qquad \lambda \in \mathbb{R} - \{0\}.$$

5. *The surfaces which are locally the graph of the function $z = \frac{y}{2}\left(R\left(x\right) + x\right)$, where R is a solution of the differential equation*

$$R'' \ \left(4 + R^2\right) - 2R\left(R' + 1\right)\left(R' + 2\right) = 0.$$

6. *The surfaces which are locally parameterized by*

$$\begin{cases} x(t,s) &= t + su(t) \\ y(t,s) &= s \\ z(t,s) &= a(t) - \frac{st}{2} \end{cases}$$

where u and a are solutions of the system

$$\begin{aligned} \left(1 + u^2 + t^2\right)u'' - \left(1 + 2u'a'\right)tu' &= 0 \\ \left(1 + u^2 + t^2\right)a'' - \left(1 + 2u'a'\right)\left(ta' - u\right) &= 0. \end{aligned} \tag{24}$$

The above theorem, together with Theorem 15, give us the classification for minimal surfaces in $\mathcal{H}_3$ with Gauss map of rank one. In fact, to do this we determine, among the surfaces given by the above classification, which ones have the rank equal to one.

It is not difficult to compute, using (18) that the rank of the Gauss map of the surfaces of items 1 and 3 of the above theorem have rank different from 1 and the surfaces of items 2 and 4 have rank one.

We shall now study the surface of the item 5, In this case, we have that $f\left(x,y\right) = \frac{y}{2}\left(R\left(x\right) + x\right)$. Hence

$$f_{xx} = R'', \qquad f_{xy} = \frac{1}{2}\left(R' + 1\right), \qquad f_{yy} = 0.$$

Substituting into (18), we obtain $(R'+1)^2 = 1$. Solving this differential equation we obtain the following solution

$$f(x,y) = \frac{y}{2}(a-x).$$

The isometry $L_{(a,0,0)}$ takes this surface to the parabolic hyperboloid $z = \frac{xy}{2}\cdot$
In the case of the surface of item 6, we parameterized this surface as a graph of a differential function f, where

$$f(x,y) = a(t(x,y)) - \frac{1}{2}yt(x,y).$$

First, we compute the following derivatives

$$\begin{aligned}
t_x &= \frac{1}{1+yu'}, \qquad t_y = \frac{-u}{1+yu'} \\
f_{xx} &= a'' t_x^2 + \left(a' + \frac{y}{2}\right) t_{xx} \\
f_{xy} &= \frac{1}{2}t_x + a'' t_x t_y + \left(a' + \frac{y}{2}\right) t_{xy} \\
f_{yy} &= a'' t_y^2 + t_y + \left(a' + \frac{y}{2}\right) t_{yy}.
\end{aligned}$$

Substituting into (18), we obtain that $t_x^2 \left[\frac{1}{2} - \left(a' + \frac{y}{2}\right) u' t_x\right]^2 = \frac{1}{4}\cdot$ Then

$$(u')^2 y^2 + 2u'y + 2u'a' = 0.$$

Notice that this is a polynomial of second degree with respect to y and its coefficients depend only on t, so, there must be zero, that is, $u' = 0$. Hence u is constant. By mean a rotation about the $z-axis$ we may take $u = 0$. By replacing in the second equation of (24), we obtain

$$\left(1+t^2\right) a'' - t a' = 0.$$

The general solution of this equation is

$$a(t) = \frac{\lambda}{2}\left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)\right] + \mu$$

where $\lambda, \mu \in \mathbb{R}$. This minimal surface may be expressed as the graph of the function

$$f(x,y) = \frac{xy}{2} - \frac{\lambda}{2}\left[y\sqrt{1+y^2} + \ln\left(y + \sqrt{1+y^2}\right)\right].$$

Therefore we have the following classification for minimal surfaces in $\mathcal{H}_3$.

Theorem 17. *The minimal graphs in $\mathcal{H}_3$ with Gauss map of rank one, are*

$$f(x,y) = \frac{xy}{2} - \frac{k}{2}\left[y\sqrt{1+y^2} + \ln\left(y + \sqrt{1+y^2}\right)\right]$$

where $k \in \mathbb{R}$.

To conclude this section we shall present some results about complete minimal graphs in $\mathcal{H}_3$. Firstly, we present one directly consequence of the minimal graph equation.

Proposition 18. *If $f(x,y)$ is a function that satisfies (16), then*

$$f_{xx}f_{yy} - f_{xy}^2 \leq 0.$$

Proof: Let $a = 1 + \left(f_y - \frac{x}{2}\right)^2$, $b = \left(f_x + \frac{y}{2}\right)\left(f_y - \frac{x}{2}\right)$ and $c = 1 + \left(f_x + \frac{y}{2}\right)^2$. Then using equation (16), we obtain

$$f_{xx}f_{yy} - f_{xy}^2 = -\frac{1}{a}\left(af_{xy}^2 - 2bf_{yy}f_{xy} + cf_{yy}^2\right).$$

Since $a > 0$ and $ac - b^2 = 1 + \left(f_x + \frac{y}{2}\right)^2 + \left(f_y - \frac{x}{2}\right)^2 > 0$, the result follows. ■

Now we recall the following theorem of Bernstein, see [9] and [12].

Theorem 19. *Let $f(x,y)$ be a real-valued function which satisfies the following conditions*

1. $f(x,y) \in C^2\left(\mathbb{R}^2\right)$.
2. $f_{xx}f_{yy} - f_{xy}^2 \leq 0$, $f_{xx}f_{yy} - f_{xy}^2 \neq 0$.

Then $f(x,y)$ is not bounded.

It follows from the above theorem that a complete minimal graph in $\mathcal{H}_3$ cannot be bounded. More precisely

Proposition 20. *Let $(x, y, f(x,y))$ be a minimal graph in $\mathcal{H}_3$, which is defined in the entire $xy - plane$ and $|f(x,y)| \leq k$, for all (x,y), then f is constant.*

Proof: Since f is bounded we have, using the Bernstein's theorem that

$$f_{xx}f_{yy} - f_{xy}^2 \equiv 0.$$

It follows from the proof of the proposition (18)

$$af_{xy}^2 + 2bf_{yy}f_{xy} + cf_{yy}^2 = 0$$

where a, b and c are as in such proof. Then $f_{yy} = f_{xy} = 0$ and, substituting in minimal graph equation, we obtain that $f_{xx} = 0$. Therefore, f is constant. ■

A consequence of the above proof is that when a rank of the Gauss map of a complete minimal graph is equal to $1/4w^2$ where w is like (6), the surface must be a plane.

Finally, we must mention that Fernadez and Mira gave a classification of the entire minimal graphs in $\mathcal{H}_3$ in terms of the Abresh-Rosenberg holomorphic differential for minimal surfaces in $\mathcal{H}_3$, see [6]. But if we study the images of the Gauss map of the minimal surfaces given in Examples 6 and 8, we observe that they are quite different: the plane covers all the upper hemisphere, whereas the other does not. So we considered of interest to give an explicit classification of all complete minimal graphs whose Gauss map has rank two, using the image of their corresponding Gauss maps.

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Christiam Figueroa
Sección Matemáticas
Departamento de Ciencias
Pontificia Universidad Católica del Perú
Lima, PERÚ
E-mail address: cfiguer@pucp.pe