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QUASICLASSICAL AND QUANTUM SYSTEMS OF ANGULAR MOMENTUM. PART III. GROUP ALGEBRA $\mathfrak{su}(2)$, QUANTUM ANGULAR MOMENTUM AND QUASSICLASSICAL ASYMPTOTICS

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Abstract. This is the third part of our series "Quasiclassical and Quantum Systems of Angular Momentum". In two previous parts we have discussed the methods of group algebras in formulation of quantum mechanics and certain quasiclassical problems. Below we specify to the special case of the group $\mathrm{SU}(2)$ and its quotient $\mathrm{SO}(3,\mathbb{R})$, and discuss just our main subject in this series, i.e., angular momentum problems. To be more precise, this is the purely $\mathrm{SU}(2)$-treatment, so formally this might also apply to isospin. However. it is rather hard to imagine realistic quasiclassical isospin problems.

Contents

1. Introduction

In the previous parts of this paper we have investigated some general problems of the formulation of quantum mechanics based on the H^+-algebras. In particular, we reviewed their important subclass, namely, the associative convolution algebras of functions on locally compact topological groups, first of all, Lie groups. This was mainly the subject of Parts I and II [27,28]. In this final Part III below we concentrate on the main subject of the series, namely, on the theory of systems with quantum angular momentum. It is not decided what is the nature of this angular momentum; it may be either orbital or spin or some their superposition. The nature of the system as that of angular momentum is specified by using the convolution algebra of once integrable functions on the group $\mathrm{SU}(2)$. We begin with the usual expressions involving the Pauli matrices and canonical coordinates of the first kind on $\mathrm{SU}(2)$, i.e., components of the rotation vector, admitting the doubled range of the angle of rotation. We also mention about the projective parametrization based on the so-called vector of finite rotations, when during the multiplication of matrices some purely algebraic rule of computation of parameters is used. Operators $\mathcal{L}_a$, $\mathcal{R}_a$, $\mathcal{A}_a$, introduced previously in Part II [28], i.e., generators of the left and right multiplicative argument-wise action of $\mathrm{SU}(2)$ and its quotient $\mathrm{SO}(3,\mathbb{R})$, are below explicitly expressed in terms of partial differentiation with respect to the group coordinates on $\mathrm{SU}(2)$. Some algebraic formulas for the action of those operators on our configuration space functions and the resulting differential equations satisfied by the unitary irreducible matrix elements $D(j)_{mn}$ and magnetic multipoles $Q^p{}_{kl}$ are below discussed. As it is well known, there are various objections against the using the $\hbar \to 0$ limit transition from the quantum to classical mechanics. In our $\mathrm{SU}(2)$-approach to the theory of angular momentum this problem is particularly essential because, as a matter of fact, apparently there is no use of the Planck constant at all. The method of quasi-classical analysis we suggest below is based on some other limit transition. Namely, instead of the $\hbar \to 0$ procedure, we perform the procedure of eliminating the low quantum numbers in the group algebra on $\mathrm{SU}(2)$. So, the problem is to fix some value of L and to investigate only the sub-algebra of $L(\mathrm{SU}(2))$ obtained as the direct sum of all two-sided ideals with the value of the angular momentum $j > L$. And then we perform the limit transition with $L \to \infty$. After some manipulations one obtains in this "classical limit" a Poisson system in the Lie co-algebra of $\mathrm{SU}(2)$. In the special case of the usual dipole model, one obtains the traditional classical equations of motion. If the model is more complicated, one obtains models with Hamiltonians containing, e.g., higher-order multipole magnetic moments. It is interesting that when we perform the quasi-classical analysis in this sense, then some classically strange maxima/minima of functions on $\mathrm{SU}(2)$ for $k = 2\pi$ approximately cancel each other for the neighbouring values of large scalar angular moments $j, j+1$.

2. Lie Algebra of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$

Theory of angular momentum is based on the group $\mathrm{SU}(2)$ and its quotient, i.e., $\mathrm{SO}(3,\mathbb{R}) = \mathrm{SU}(2)/\mathbb{Z}_2$ (see, e.g. [14, 15]).

Theory of angular momentum is based on the group $\mathrm{SU}(2)$ and its quotient, i.e., $\mathrm{SO}(3,\mathbb{R}) = \mathrm{SU}(2)/\mathbb{Z}_2$. The two-element center and maximal normal divisor $\mathbb{Z}_2$ of the simply-connected group $\mathrm{SU}(2)$ is given by

$$\mathbb{Z}_2 = \{I_2, -I_2\}$$

where I_2 is the 2×2 unit matrix.

Let σ_a, $a = 1, 2, 3$ denote Pauli matrices in the following convention

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They are basic traceless Hermitian 2×2-matrices. The Lie algebra of $\mathrm{SU}(2)$, $\mathfrak{su}(2)$, consists of anti-Hermitian traceless matrices. The basic ones are chosen as

$$e_a := \frac{1}{2\mathrm{i}}\sigma_a. \tag{1}$$

The corresponding structure constants are given by the Ricci symbol, more precisely

$$[e_a, e_b] = \varepsilon^c{}_{ab} e_c$$

where ε_{abc} is just the totally antisymmetric Ricci symbol, $\varepsilon_{123} = 1$, and the raising/lowering of indices is meant here in the sense of the "Kronecker delta" δ_{ab} as the standard metric of $\mathbb{R}^3$. So, this shift of indices is here analytically a purely "cosmetic" procedure, however we use it to follow the standard convention.

We know that $\mathrm{SU}(2)$ is the universal $2:1$ covering group of $\mathrm{SO}(3,\mathbb{R})$, the proper orthogonal group in $\mathbb{R}^3$. The projection epimorphism

$$\mathrm{SU}(2) \ni u \mapsto R = \mathrm{p}(u) \in \mathrm{SO}(3,\mathbb{R})$$

is given by

$$u e_b u^{-1} = u e_b u^{+} = e_a R^a{}_b. \tag{2}$$

With respect to the basis (1) the Killing metric γ has the components

$$\gamma_{ab} = -2\delta_{ab}.$$

Its negative definiteness is due to the compactness of the simple algebra/group $\mathfrak{su}(2)/\mathrm{SU}(2)$. For practical purposes one eliminates the factor (-2) and takes the metric

$$\Gamma_{ab} = -\frac{1}{2}\gamma_{ab} = \delta_{ab}. \tag{3}$$

In terms of the canonical coordinates of the first kind

$$u(\overline{k}) = \mathrm{e}\left(k^a e_a\right) = \cos\frac{k}{2}I_2 - \frac{\mathrm{i}}{k}\sin\frac{k}{2}k^a\sigma_a \tag{4}$$

where k denotes the Euclidean length of the vector $\overline{k} \in \mathbb{R}^3$

$$k = \sqrt{\overline{k}\cdot\overline{k}} = \sqrt{\delta_{ab}k^a k^b}.$$

Its range is $[0, 2\pi]$ and the range of the unit vector (versor) $\overline{n} := \overline{k}/k$ is the total unit sphere $S^2(0,1) \subset \mathbb{R}^3$. This coordinate system is singular at $k = 0$, $k = 2\pi$, where

$$u(0\overline{n}) = I_2, \qquad u(2\pi\overline{n}) = -I_2$$

for any $\overline{n} \in S^2(0,1)$. Of course, the formula (4) remains meaningful for $k > 2\pi$, however, the "former" elements of $\mathrm{SU}(2)$ are then repeated.

Sometimes one denotes

$$\sigma_0 = I_2, \qquad e_0 = \frac{1}{2}I_2.$$

Then (4) may be written down as follows

$$u = \xi^\mu(\overline{k})\,(2e_\mu) \tag{5}$$

where the summation convention is meant over $\mu = 0, 1, 2, 3$

$$\left(\xi^0\right)^2 + \left(\xi^1\right)^2 + \left(\xi^2\right)^2 + \left(\xi^3\right)^2 = 1 \tag{6}$$

and this formula together with the structure of parametrization (4), (5) tells us that $\mathrm{SU}(2)$ is the unit sphere $S^3(0,1)$ in $\mathbb{R}^4$. Roughly speaking, $k = 0$ is the "north pole" and $k = 2\pi$ is the corresponding "south pole".

This "pseudo-relativistic" notation is rather misleading. The point is that the matrices σ_μ, e_μ above are used to represent linear mappings in $\mathbb{C}^2$, i.e., mixed tensors in $\mathbb{C}^2$. In the relativistic theory of spinors, e.g., in Lagrangians for (anti)neutrino fields, σ_μ are used as matrices of sesqulinear Hermitian forms, thus, twice covariant tensors on $\mathbb{C}^2$. The space of such forms carries an intrinsic conformal-Minkowskian structure (Minkowskian up to the normalization of the scalar product). And then σ_μ form a Lorentz-ruled multiplet. This is seen in the standard procedure of using $\mathrm{SL}(2,\mathbb{C})$ as the universal covering of the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$, namely

$$a\sigma_{\mu}a^{+} = \sigma_{\nu}\Lambda^{\nu}{}_{\mu}$$

describes the covering assignment

$$\mathrm{SL}(2,\mathbb{C}) \ni a \mapsto \Lambda \in \mathrm{SO}^{\uparrow}(1,3).$$

The four-dimensional quantity $\left(\xi^{0},\xi^{1},\xi^{2},\xi^{3}\right)$ in (5), (6) may be also interpreted in terms of the group $\mathrm{SO}(4,\mathbb{R})$ and its covering group, however, this interpretation is relatively complicated and must not be confused with the relativistic aspect of the quadruplet of σ_{μ}-matrices as analytical representants of sesquilinear forms.

The Lie algebra of $\mathrm{SO}(3,\mathbb{R})$, $\mathfrak{so}(3,\mathbb{R})$, consists of 3×3 skew-symmetric matrices with real entries. The standard choice of basis of $\mathfrak{so}(3,\mathbb{R})$, adapted to (1) and to the procedure (2), is given by matrices E_a, $a=1,2,3$, with entries

$$\left(E_a\right)^{b}{}_{c} := -\varepsilon_{a}{}^{b}{}_{c}$$

where again ε_{abc} is the totally antisymmetric Ricci symbol, and indices are "cosmetically" shifted with the help of the Kronecker symbol. Then, of course

$$[E_a, E_b] = \varepsilon^{c}{}_{ab}E_c.$$

In spite of having isomorphic Lie algebras, the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R}) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$ are globally different. The main topological distinction is that $\mathrm{SU}(2)$ is simply connected and $\mathrm{SO}(3,\mathbb{R})$ is doubly connected.

Using canonical coordinates of the first kind, we have in analogy to (4) the formula

$$R(\overline{k}) = \mathrm{e}\left(k^{a}E_a\right).$$

Because of the obvious reasons, known from elementary geometry and mechanics, $\overline{k}$ is referred to as the rotation vector, $k=\sqrt{\overline{k}\cdot\overline{k}}$ is the rotation angle, and the unit vector (versor)

$$\overline{n} = \frac{\overline{k}}{k}$$

is the oriented rotation axis. We use the all standard concepts and symbols of the vector calculus in $\mathbb{R}^3$, in particular, scalar products $\overline{a}\cdot\overline{b}$ and vector products $\overline{a}\times\overline{b}$. The rotation angle k runs over the range $[0,\pi]$ and the antipodal points on the sphere $S^2(0,\pi)\subset\mathbb{R}^3$ are identified, they describe the same rotation

$$R\left(\pi\overline{n}\right) = R\left(-\pi\overline{n}\right). \tag{7}$$

Therefore, this sphere, taken modulo the antipodal identification, is the manifold of non-trivial square roots of the identity I_3 in $\mathrm{SO}(3,\mathbb{R})$. It is seen in this picture that $\mathrm{SO}(3,\mathbb{R})$ is doubly connected, because any curve in the ball $K^2(0,1)\subset\mathbb{R}^3$ joining two antipodal points on the boundary $S^2(0,1)$ is closed under this identification, i.e., it is a loop, but it cannot be continuously contracted into a single point.

It is worth to note that formally the values $k>\pi$ are admitted, however, they correspond to rotations by $k<\pi$, taken earlier into account. By abuse of language, in $\mathrm{SU}(2)$ the quantities $\overline{k}$, k are also referred to as the rotation vector and rotation angle. But one must "rotate" by 4π to go back to the same situation, not by 2π. The matrix of $R(\overline{k})$ is given by

$$R\left(\overline{k}\right)^a{}_b=\cos k\delta^a{}_b+\frac{1}{k^2}\left(1-\cos k\right)k^ak_b+\frac{1}{k}\sin k\varepsilon^a{}_{bc}k^c$$

i.e.,

$$R\left(\overline{k}\right)\overline{x}=\cos k\overline{x}+\frac{1-\cos k}{k^2}\left(\overline{k}\cdot\overline{x}\right)\overline{k}+\frac{\sin k}{k}\overline{k}\times\overline{x}$$

or, symbolically

$$R(\overline{k})\cdot\overline{x}=\overline{x}+\overline{k}\times\overline{x}+\frac{1}{2!}\overline{k}\times\left(\overline{k}\times\overline{x}\right)+\cdots+\frac{1}{n!}\overline{k}\times\left(\overline{k}\times\cdots\times\left(\overline{k}\times\overline{x}\right)\cdots\right)+\ldots$$

Let us distinguish between two ways of viewing, representing geometry of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ in terms of some subsets in $\mathbb{R}^3$ as the space of rotation vectors $\overline{k}$ or, alternatively, in terms of closed submanifolds and their quotients in $\mathbb{R}^4$.

As seen from (6), $\mathrm{SU}(2)$ is a unit sphere $S^3(0,1)\subset\mathbb{R}^4$, $\mathrm{SO}(3,\mathbb{R})$ is obtained by the antipodal identification. Then $\mathrm{SO}(3,\mathbb{R})$ is doubly connected because the curves on $S^3(0,1)$ joining antipodal points project to the quotient manifold onto closed loops non-contractible to points in a continuous way. In $\mathbb{R}^3$ the group $\mathrm{SU}(2)$ is represented by the ball $K^2(0,2\pi)$ and the whole shell $S^2(0,2\pi)$ represents the single point $-I_2\in\mathrm{SU}(2)$. Then $\mathrm{SO}(3,\mathbb{R})$ is pictured as the ball $K^2(0,\pi)\subset\mathbb{R}^3$ with the antipodal identification of points on the shell $S^2(0,\pi)$, cf. (7). This exhibits the identification of $\mathrm{SO}(3,\mathbb{R})$ with the projective space $\mathbb{RP}^3$. The antipodally identified points on $S^2(0,\pi)$ represent the improper points at infinity in $\mathbb{R}^3$.

For certain reasons, both practical and deeply geometrical, it is convenient to use also another parametrization of $\mathrm{SO}(3,\mathbb{R})$, using so-called vector of finite rotation

$$\overline{\varkappa}=\frac{2}{k}\,\mathrm{tg}\,\frac{k}{2}\,\overline{k}. \tag{8}$$

One can note that in the neighbourhood of group identity, when $\overline{k}\approx 0$, $\overline{\varkappa}$ differs from $\overline{k}$ by higher-order quantity. The practical advantage of $\overline{\varkappa}$ is that the composition rule and the action of rotations are described by very simple and purely algebraic expressions

$$R\left(\overline{\varkappa}_1\right) R\left(\overline{\varkappa}_2\right) = R\left(\overline{\varkappa}\right)$$

where

$$\overline{\varkappa} = \left(1 - \frac{1}{4}\overline{\varkappa}_1 \cdot \overline{\varkappa}_2\right)^{-1} \left(\overline{\varkappa}_1 + \overline{\varkappa}_2 + \frac{1}{2}\overline{\varkappa}_1 \times \overline{\varkappa}_2\right)$$

$$R\left[\overline{\varkappa}\right]\overline{x} = \overline{x} + \left(1 + \frac{1}{4}\overline{\varkappa}^2\right)^{-1} \overline{\varkappa} \times \left(\overline{x} + \frac{1}{2}\overline{\varkappa} \times \overline{x}\right).$$

An important property of this parametrization is that it describes the projective mapping of $\mathrm{SO}(3,\mathbb{R})$ onto the projective space $\mathbb{RP}^3$. The one-parameter subgroups and their cosets in $\mathrm{SO}(3,\mathbb{R})$ are mapped onto straight-lines in $\mathbb{R}^3$. The manifold of π-rotations (non-trivial square roots of identity) is mapped onto the set of improper points in $\mathbb{RP}^3$, i.e., it “blows up” to infinity.

The homomorphism (2) of $\mathrm{SU}(2)$ onto $\mathrm{SO}(3,\mathbb{R})$, $u \mapsto R(u)$, may be alternatively described in terms of inner automorphisms of $\mathrm{SU}(2)$ and the rotation-vector parametrization

$$uv(\overline{k})u^{-1} = v\left(R(u)\overline{k}\right), \qquad u \in \mathrm{SU}(2). \tag{9}$$

Roughly speaking, inner automorphisms in $\mathrm{SU}(2)$ result in rotation of the rotation vector. The same holds in $\mathrm{SO}(3,\mathbb{R})$

$$OR(\overline{k})O^{-1} = R\left(O\overline{k}\right), \qquad O \in \mathrm{SO}(3,\mathbb{R}).$$

Therefore, inner automorphisms preserve the length k of the rotation vector $\overline{k}$, and the classes of conjugate elements are characterized by the fixed values of the rotation angle (but all possible oriented rotation axes $\overline{n}$). This means that in the above description they are represented by spheres $S^2(0,k) \subset \mathbb{R}^3$ in the space of rotation vectors. There are two one-element singular equivalence classes in $\mathrm{SU}(2)$, namely $\{I_2\}$, $\{-I_2\}$ corresponding respectively to $k = 0$, $k = 2\pi$. Of course, in $\mathrm{SO}(3,\mathbb{R})$ there is only one singular class $\{I_3\}$. More precisely, in $\mathrm{SO}(3,\mathbb{R})$ the class $k = \pi$ is not the sphere, but rather its antipodal quotient, so-called elliptic space. The idempotents $\varepsilon(\alpha)$/characters $\chi(\alpha) = \varepsilon(\alpha)/n(\alpha)$ and all central functions of the group algebras of $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ are constant on the spheres $S^2(0,k)$, i.e., depend on $\overline{k}$ only through the rotation angle k. In many problems it is convenient to parametrize $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$ with the help of spherical variables k, θ, φ in the space $\mathbb{R}^3$ of the rotation vector $\overline{k}$. Historically the most popular parametrization is that based on the Euler angles $(\varphi, \vartheta, \psi)$. It is given by

$$u[\varphi,\vartheta,\psi] = u(0,0,\varphi)u(0,\vartheta,0)u(0,0,\psi) \tag{10}$$

$$R[\varphi,\vartheta,\psi] = R(0,0,\varphi)R(0,\vartheta,0)R(0,0,\psi). \tag{11}$$

Historically φ, ϑ, ψ are referred to respectively as the precession angle, nutation angle and the rotation angle.

Sometimes one uses $u(\vartheta, 0, 0)$, $R(\vartheta, 0, 0)$ instead $u(0, \vartheta, 0)$, $R(0, \vartheta, 0)$ in (10), (11). The only thing which matters here is that one uses the product of three elements which belong to two one-parameter subgroups. The Euler angles are practically important in gyroscopic problems. Canonical parametrization of the second kind

$$u(\alpha, \beta, \gamma) = u(\alpha, 0, 0)u(0, \beta, 0)u(0, 0, \gamma)$$

are not very popular. One must say, however, that many formulas have the same form in variables $(\varphi, \vartheta, \psi)$ and (α, β, γ).

3. Irreducible Unitary Representations

It is well known that in $\mathrm{SU}(2)$ irreducible unitary representations, or rather their equivalence classes, are labelled by non-negative integers and half-integers,

$$\alpha = j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

i.e.,

$$\Omega = \{0\} \bigcup \frac{\mathbb{N}}{2}$$

where $\mathbb{N}$ denotes the set of naturals (positive integers). And

$$n(\alpha) = n(j) = 2j + 1.$$

On $\mathrm{SO}(3, \mathbb{R})$ one uses integers only

$$\alpha = j = 0, 1, 2, \ldots, \qquad \Omega = \{0\} \bigcup \mathbb{N}.$$

For any $\alpha = j$, there is only one irreducible representation of dimension

$$n(\alpha) = n(j) = 2j + 1$$

i.e., only one up to equivalence. It is not the case for many practically important groups, e.g., for $\mathrm{SU}(3)$ or for the non-compact group $\mathrm{SL}(2, \mathbb{C})$.

Historically, the irreducible representations of $\mathrm{SU}(2)$, $\mathrm{SO}(3, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$, and $\mathrm{SO}^{\uparrow}(1, 3)$ were found in two alternative ways

i) algebraic one, based on taking the tensor products of fundamental representation (by itself),

ii) differential one, based on solving differential equations like (39)–(44), (46), (47) in [28].

The left and right generators $\mathcal{L}_a$, $\mathcal{R}_a$, i.e., respectively the basic right- and left-invariant vector fields, are analytically given by

$$\mathcal{L}_a = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} + \frac{1}{2}\varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c} \tag{12}$$

$$\mathcal{R}_a = \frac{k}{2}\operatorname{ctg}\frac{k}{2}\frac{\partial}{\partial k^a} + \left(1 - \frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^b}{k}\frac{\partial}{\partial k^b} - \frac{1}{2}\varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c} \tag{13}$$

and therefore

$$\mathcal{A}_a = \mathcal{L}_a - \mathcal{R}_a = \varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}.$$

In terms of explicitly written components

$$\begin{aligned}
\mathcal{L}^i{}_a &= \frac{k}{2}\operatorname{ctg}\frac{k}{2}\delta^i{}_a + \left(1 - \frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^i}{k} + \frac{1}{2}\varepsilon_{ab}{}^i k^b\\
\mathcal{R}^i{}_a &= \frac{k}{2}\operatorname{ctg}\frac{k}{2}\delta^i{}_a + \left(1 - \frac{k}{2}\operatorname{ctg}\frac{k}{2}\right)\frac{k_a}{k}\frac{k^i}{k} - \frac{1}{2}\varepsilon_{ab}{}^i k^b\\
\mathcal{A}^i{}_a &= \varepsilon_{ab}{}^i k^b.
\end{aligned}$$

The shift of indices is meant here in the Kronecker-delta sense.

The corresponding Cartan one-forms are given by

$$\begin{aligned}
\mathcal{L}^a &= \frac{\sin k}{k}\mathrm{d}k^a + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_b}{k}\mathrm{d}k^b + \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bc}k^b\mathrm{d}k^c\\
\mathcal{R}^a &= \frac{\sin k}{k}\mathrm{d}k^a + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_b}{k}\mathrm{d}k^b - \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bc}k^b\mathrm{d}k^c
\end{aligned}$$

i.e., in terms of the components

$$\begin{aligned}
\mathcal{L}^a{}_i &= \frac{\sin k}{k}\delta^a{}_i + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_i}{k} + \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bi}k^b\\
\mathcal{R}^a{}_i &= \frac{\sin k}{k}\delta^a{}_i + \left(1 - \frac{\sin k}{k}\right)\frac{k^a}{k}\frac{k_i}{k} - \frac{2}{k^2}\sin^2\frac{k}{2}\varepsilon^a{}_{bi}k^b.
\end{aligned} \tag{14}$$

The central functions on $\mathrm{SU}(2)$ and on $\mathrm{SO}(3,\mathbb{R})$, in particular the idempotents $\varepsilon(j)$/characters $\chi(j)$ satisfy the obvious differential equations

$$\mathcal{A}_a f = 0, \qquad \text{i.e.,} \qquad \mathcal{L}_a f = \mathcal{R}_a f, \qquad a = 1, 2, 3.$$

The analytical formulas (12)-(14) are formally valid both on $\mathrm{SU}(2)$ and $\mathrm{SO}(3,\mathbb{R})$, and in general the calculus on $\mathrm{SU}(2)$ is simpler than that on $\mathrm{SO}(3,\mathbb{R})$. It is convenient to rewrite the formulas (12)-(14) so as to express them explicitly in terms of the angular and radial differential operations in the space of rotation vectors $\overline{k}$. After simple calculations one obtains

$$\begin{aligned}
\mathcal{L}_a &= n_a\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{abc}n^b\mathcal{A}^c + \frac{1}{2}\mathcal{A}_a\\
\mathcal{R}_a &= n_a\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\varepsilon_{abc}n^b\mathcal{A}^c - \frac{1}{2}\mathcal{A}_a\\
\mathcal{L}_a &= n^a\mathrm{d}k + 2\sin^2\frac{k}{2}\varepsilon^a{}_{bc}n^b\mathrm{d}n^c + \sin k\mathrm{d}n^a\\
\mathcal{R}_a &= n^a\mathrm{d}k - 2\sin^2\frac{k}{2}\varepsilon^a{}_{bc}n^b\mathrm{d}n^c + \sin k\mathrm{d}n^a.
\end{aligned}$$

Using the $\mathbb{R}^3$-vector notation, including also the vectors with operator components, we can denote briefly, without using indices and labels

$$\overline{\mathcal{L}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\overline{n}\times\overline{\mathcal{A}} + \frac{1}{2}\overline{\mathcal{A}} \tag{15}$$

$$\overline{\mathcal{R}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{2}\mathrm{ctg}\frac{k}{2}\overline{n}\times\overline{\mathcal{A}} - \frac{1}{2}\overline{\mathcal{A}} \tag{16}$$

$$\underline{\mathcal{L}} = \overline{n}\mathrm{d}k + 2\sin^2\frac{k}{2}\overline{n}\times\mathrm{d}\overline{n} + \sin k\mathrm{d}\overline{n} \tag{17}$$

$$\underline{\mathcal{R}} = \overline{n}\mathrm{d}k - 2\sin^2\frac{k}{2}\overline{n}\times\mathrm{d}\overline{n} + \sin k\mathrm{d}\overline{n} \tag{18}$$

$$\overline{\mathcal{A}} = \overline{k}\times\overline{\nabla} \tag{19}$$

where $\overline{\nabla}$ denotes the Euclidean gradient operator.

Let us note the following interesting and suggestive duality relations

$$\begin{aligned}
\langle \mathrm{d}k, \mathcal{A}_a\rangle &= \mathcal{A}_a k = 0, &\qquad \left\langle \mathrm{d}k, \frac{\partial}{\partial k}\right\rangle &= 1\\
\langle \mathrm{d}n_a, \mathcal{A}_b\rangle &= \mathcal{A}_b n_a = \varepsilon_{abc}n^c, &\qquad \left\langle \mathrm{d}n_a, \frac{\partial}{\partial k}\right\rangle &= \frac{\partial n_a}{\partial k} = 0.
\end{aligned}$$

In $\mathbb{R}^3$, considered as an Abelian group under addition of vectors, the right-invariant fields coincide with the left-invariant ones, and when using spherical variables we have then

$$\overline{\mathcal{L}} = \overline{\mathcal{R}} = \overline{\nabla} = \overline{n}\frac{\partial}{\partial r} - \frac{1}{r}\overline{n}\times\overline{\mathcal{A}}(\overline{r}) \tag{20}$$

$$\underline{\mathcal{L}} = \underline{\mathcal{R}} = \mathrm{d}\overline{r} = \overline{n}\mathrm{d}r + r\mathrm{d}\overline{n}. \tag{21}$$

This is in agreement with the formulas (15)–(19), namely, in a small neighbourhood of the group identity $I_2 \in \mathrm{SU}(2)$, i.e., for $\overline{k} \approx \overline{0}$, expressions (15)–(19) up to higher-order terms in $\overline{k}$, one obtains

$$\overline{\mathcal{L}} \approx \overline{\mathcal{R}} \approx \overline{\nabla}_{\overline{k}} = \overline{n}\frac{\partial}{\partial k} - \frac{1}{k}\overline{n} \times \overline{\mathcal{A}}$$
$$\underline{\mathcal{L}} \approx \underline{\mathcal{R}} \approx \mathrm{d}\overline{k} = \overline{n}\mathrm{d}k + k\mathrm{d}\overline{n}.$$

The quantities $\overline{n}$, $\mathrm{d}\overline{n}$, $\overline{\mathcal{A}}$ are non-sensitive to the asymptotics $k \to 0$, because they are purely angular (θ, φ) variables, independent of k.

$\mathrm{SU}(2)$ is the sphere $S^3(0,1)$ in $\mathbb{R}^4$. Taking the sphere of radius R, $S^3(0,R) \subset \mathbb{R}^4$, and performing the limit transition $R \to \infty$, one obtains also the relationships (20), (21) as an asymptotic limit.

The Killing metric tensor with the modified normalization (3) is given by

$$g_{ij} = \frac{4}{k^2}\sin^2\frac{k}{2}\delta_{ij} + \left(1 - \frac{4}{k^2}\sin^2\frac{k}{2}\right)\frac{k^i}{k}\frac{k^j}{k} \tag{22}$$

and its contravariant inverse by

$$g^{ij} = \frac{k^2}{4\sin^2 k/2}\delta^{ij} + \left(1 - \frac{k^2}{4\sin^2 k/2}\right)\frac{k^i}{k}\frac{k^j}{k}.$$

The corresponding metric element may be concisely written as

$$\mathrm{d}s^2 = \mathrm{d}k^2 + 4\sin^2\frac{k}{2}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) = \mathrm{d}k^2 + 4\sin^2\frac{k}{2}\mathrm{d}\overline{n}\cdot\mathrm{d}\overline{n} \tag{23}$$

or, in a more sophisticated way

$$g = \mathrm{d}k \otimes \mathrm{d}k + 4\sin^2\frac{k}{2}\delta_{AB}\mathrm{d}n^A \otimes \mathrm{d}n^A \tag{24}$$

and, similarly, for the inverse tensor

$$g^{-1} = \frac{\partial}{\partial k} \otimes \frac{\partial}{\partial k} + \frac{1}{4\sin^2 k/2}\delta^{AB}\mathcal{A}_A \otimes \mathcal{A}_B.$$

According to the standard procedure, the volume element on the Riemannian manifold is given by

$$\mathrm{d}\mu\left(\overline{k}\right) = \sqrt{|g|}\mathrm{d}^3\overline{k} = \sqrt{\det\left[g_{ij}\left(\overline{k}\right)\right]}\mathrm{d}^3\overline{k}.$$

It is easy to see that for our normalization of the metric tensor

$$\mathrm{d}\mu\left(\overline{k}\right) = 4\sin^2\frac{k}{2}\sin\theta\mathrm{d}k\mathrm{d}\theta\mathrm{d}\varphi = \frac{4\sin^2 k/2}{k^2}\,\mathrm{d}^3\overline{k} \tag{25}$$

where $\mathrm{d}^3\overline{k}$ is the usual volume element in $\mathbb{R}^3$ as the space of rotation vectors $\overline{k}$. This volume element is identical with that given by (25), (26), (27). The reason is that all these expressions are translationally-invariant and the Haar measure is unique. We assume here that G is unimodular. In fact, we mean only the compact semisimple groups and their products with Abelian groups (clearly, in the latter case it is not the Killing tensor that is meant in the Abelian factor). Nevertheless, any metric meant there is also assumed translationally-invariant, and so the total Riemann measure also coincides with (25), (26), (27).

Let us mention that when the Euler angles $(\varphi, \vartheta, \psi)$ are used as a parametrization, then the Riemann metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \mathrm{d}\varphi^2 + 2\cos\vartheta \mathrm{d}\varphi \mathrm{d}\psi + \mathrm{d}\psi^2. \tag{26}$$

The measure element is then expressed

$$\mathrm{d}\mu\left(\varphi, \vartheta, \psi\right) = \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi \mathrm{d}\psi. \tag{27}$$

The metric element expression (26) may be diagonalized by introducing the new "angles"

$$\alpha = \varphi + \psi, \qquad \beta = \varphi - \psi$$

however, this representation rather is not used practically.

Let us remind that on $\mathrm{SU}(2)$ the range of Euler angles is $[0, 4\pi]$ for φ, ψ, and $[0, 2\pi]$ for ϑ, on $\mathrm{SO}(3, \mathbb{R})$, it is respectively $[0, 2\pi]$ and $[0, \pi]$.

In some of earlier formulas we used the convention of the Haar measure on compact groups normalized to unity, $\mu(G) = 1$. When normalized in this way, it will be denoted as μ_1. The label "(1)" will be omitted when the normalization is clear from the context or when there is no danger of confusion.

After elementary integrations we find that on $\mathrm{SU}(2)$ the element of normalized measure is given by

$$\mathrm{d}\mu_{(1)} = \frac{1}{4\pi^2}\sin^2\frac{k}{2}\sin\theta \mathrm{d}k \mathrm{d}\theta \mathrm{d}\varphi = \frac{\sin^2 k/2}{4\pi^2 k^2}\,\mathrm{d}^3\overline{k}.$$

If we used the normalization (25), the "volume" of $\mathrm{SU}(2)$ would be $16\pi^2$. With the same normalization, the volume of $\mathrm{SO}(3, \mathbb{R})$ would be $8\pi^2$. It is intuitively clear: $\mathrm{SU}(2)$ is "twice larger" than $\mathrm{SO}(3, \mathbb{R})$. So, we would have

$$\mathrm{d}\mu_{(1)\mathrm{SO}(3,\mathbb{R})} = \frac{1}{2\pi^2}\sin^2\frac{k}{2}\sin\theta \mathrm{d}k \mathrm{d}\theta \mathrm{d}\varphi = \frac{\sin^2 k/2}{2\pi^2 k^2}\mathrm{d}^3\overline{k}.$$

However, as mentioned, all formulas will be meant in the covering group sense $\mathrm{SU}(2)$.

The metric tensor (22), (23) is conformally flat. It is seen when we introduce some new variables $\overline{\varrho}$ instead of $\overline{k}$, namely

$$\varrho = |\overline{\varrho}| = a\mathrm{tg}\frac{k}{4}, \qquad \frac{\overline{\varrho}}{\varrho} = \frac{\overline{k}}{k} = \overline{n} \tag{28}$$

where a denotes some positive constant. Then (23) becomes

$$\begin{aligned}\mathrm{d}s^2 &= \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho^2 + \varrho^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right)\\ &= \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\overline{n}\cdot\mathrm{d}\overline{n}\right)\end{aligned}$$

or, using again the "sophisticated" form (24)

$$g = \frac{16a^2}{(a^2+\varrho^2)^2}\left(\mathrm{d}\varrho\otimes\mathrm{d}\varrho + \varrho^2\delta_{ab}\mathrm{d}n^a\otimes\mathrm{d}n^b\right).$$

Apparently, (28) is a conformal mapping of $\mathrm{SU}(2)$ onto $\mathbb{R}^3$ with its usual Euclidean metric. The ball $K^2(0,2\pi)$ "blows up" to the total $\mathbb{R}^3$ and the sphere $S^2(0,2\pi)$ "blows up" to infinity. In other words, $\mathrm{SU}(2)$ is identified with the one-point compactification of $\mathbb{R}^3$ and the element $-I_2\in\mathrm{SU}(2)$ becomes just the compactifying point. The ball $K^2(0,\pi)$ corresponding to the manifold of $\mathrm{SO}(3,\mathbb{R})$ and its boundary sphere $S^2(0,\pi)$ (non-trivial square-root of identity) become respectively $K^2(0,a)$ and $S^2(0,a)$. If we put $a=\pi$, they are mapped onto themselves. From the conformal point of view the particular choice of the constant a does not matter.

The projective mapping (8) of $\mathrm{SO}(3,\mathbb{R})$ onto $\mathbb{RP}^3$ maps geodetics of (23) onto straight lines in $\mathbb{R}^3$. However, it is neither isometry nor the conformal transformation, instead we have that

$$\mathrm{d}s^2 = \frac{4}{4+\varkappa^2}\left(\frac{4}{4+\varkappa^2}\mathrm{d}\varkappa^2 + \varkappa^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right)\right).$$

On $\mathrm{SU}(2)$ the formulas (15), (17) in [28] take on the following form

$$\mathcal{L}^a(u) = R(u)^a{}_b\mathcal{R}^b(u), \qquad \mathcal{L}_a(u) = \mathcal{R}_b(u)R(u)^{-1b}{}_a$$

where the dependence

$$\mathrm{SU}(2)\ni u\mapsto R(u)\in\mathrm{SO}(3,\mathbb{R})$$

is given by (2), (9). In orthonormal coordinates this is the same formula, because inverses of orthogonal matrices coincide with their transposes. The corresponding symmetric operators $\mathbf{\Sigma}_a$, $\widehat{\mathbf{\Sigma}}_a$, denoted respectively by

$$\mathbf{S}_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a, \qquad \widehat{\mathbf{S}}_a = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a \tag{29}$$

are interrelated by the same formula

$$\mathbf{S}_a(u) = \widehat{\mathbf{S}}_b(u) R(u)^{-1b}{}_a.$$

When interpreted in terms of action on the wave functions on $\mathrm{SU}(2)$, they are operators of rotational angular momentum (spin) respectively in the spatial and co-moving representations. The corresponding operators of hyperspin

$$\mathbf{\Delta}_a = \frac{\hbar}{\mathrm{i}} \mathcal{A}_a = \frac{\hbar}{\mathrm{i}} \varepsilon_{ab}{}^c k^b \frac{\partial}{\partial k^c}$$

are given by

$$\mathbf{\Delta}_a = \mathbf{S}_a - \widehat{\mathbf{S}}_a, \qquad \mathcal{A}_a = \mathcal{L}_a - \mathcal{R}_a.$$

The term "hyper" is used because this quantity tells us "how much" the spatial components of spin exceed the corresponding laboratory ones. The operators $\mathcal{A}_a$ generate rotations of the rotation vector and this is just the meaning of "hyper".

According to (14) in [28], the corresponding classical quantities are given by

$$S_a = p_j \mathcal{L}^j{}_a, \qquad \widehat{S}_a = p_j \mathcal{R}^j{}_a = S_b R^b{}_a$$
$$\Delta_a = p_j \Delta^j{}_a = S_a - \widehat{S}_a = \varepsilon_{ab}{}^c k^b p_c$$

where p_j denote canonical momenta conjugate to k^j or rather to the corresponding generalized velocities $\mathrm{d}k^j/\mathrm{d}t$.

Evaluating differential forms on vector tangent to trajectories in the configuration spaces $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$, we obtain the following quantities

$$\omega^a = \mathcal{L}^a{}_i \left(\overline{k}\right) \frac{\mathrm{d}k^i}{\mathrm{d}t}, \qquad \widehat{\omega}^a = \mathcal{R}^a{}_i \left(\overline{k}\right) \frac{\mathrm{d}k^i}{\mathrm{d}t}, \qquad \omega^a\left(u, \dot{u}\right) = R(u)^a{}_b \widehat{\omega}^a\left(u, \dot{u}\right).$$

They are respectively spatial (ω^a) and co-moving ($\widehat{\omega}^a$) components of angular velocity. They are non-holonomic, i.e., fail to be time derivatives of any generalized coordinates. The following duality relations are satisfied

$$s_a \omega^a = \widehat{s}_a \widehat{\omega}^a = p_i \frac{\mathrm{d}k^i}{\mathrm{d}t}.$$

Let us quote the obvious commutators and Poisson brackets

$$[\mathcal{L}_a, \mathcal{L}_b] = -\,\varepsilon_{ab}{}^c \mathcal{L}_c, \qquad [\mathcal{R}_a, \mathcal{R}_b] = \varepsilon_{ab}{}^c \mathcal{R}_c, \qquad [\mathcal{L}_a, \mathcal{R}_b] = 0$$
$$[\mathcal{A}_a, \mathcal{L}_b] = -\,\varepsilon_{ab}{}^c \mathcal{L}_c, \qquad [\mathcal{A}_a, \mathcal{R}_b] = -\,\varepsilon_{ab}{}^c \mathcal{R}_c, \qquad [\mathcal{A}_a, \mathcal{A}_b] = -\,\varepsilon_{ab}{}^c \mathcal{A}_c$$
$$\frac{1}{\mathrm{i}\hbar}[\mathbf{S}_a, \mathbf{S}_b] = \varepsilon_{ab}{}^c \mathbf{S}_c, \qquad \frac{1}{\mathrm{i}\hbar}[\widehat{\mathbf{S}}_a, \widehat{\mathbf{S}}_b] = -\,\varepsilon_{ab}{}^c \widehat{\mathbf{S}}_c, \qquad \frac{1}{\mathrm{i}\hbar}[\mathbf{S}_a, \widehat{\mathbf{S}}_b] = 0$$
$$\frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a, \mathbf{S}_b] = \varepsilon_{ab}{}^c \mathbf{S}_c, \qquad \frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a, \widehat{\mathbf{S}}_b] = \varepsilon_{ab}{}^c \widehat{\mathbf{S}}_c, \qquad \frac{1}{\mathrm{i}\hbar}[\mathbf{\Delta}_a, \mathbf{\Delta}_b] = -\,\varepsilon_{ab}{}^c \mathbf{\Delta}_c$$
$$\{S_a, S_b\} = \varepsilon_{ab}{}^c S_c, \qquad \{\widehat{S}_a, \widehat{S}_b\} = -\,\varepsilon_{ab}{}^c \widehat{S}_c, \qquad \{S_a, \widehat{S}_b\} = 0$$
$$\{\Delta_a, S_b\} = \varepsilon_{ab}{}^c S_c, \qquad \{\Delta_a, \widehat{S}_b\} = \varepsilon_{ab}{}^c \widehat{S}_c, \qquad \{\Delta_a, \Delta_b\} = -\,\varepsilon_{ab}{}^c \Delta_c.$$

In the enveloping algebras built over Lie algebras of $\mathcal{L}$- and $\mathcal{R}$-operators there exists only one Casimir invariant, namely, the second-order one

$$C(\mathcal{L},2) = C(\mathcal{R},2) = \Delta = \delta^{ab}\mathcal{L}_a\mathcal{L}_b = \delta^{ab}\mathcal{R}_a\mathcal{R}_b. \tag{30}$$

In physical expressions like various kinetic energies and so on, one uses their $(-\hbar^2)$-multiplies

$$\mathbf{S}^2 := C(S,2) = C\left(\widehat{S},2\right) = -\hbar^2\Delta.$$

There is also only one Casimir in the associative algebra generated by the Lie algebra of $\mathcal{A}$-operators

$$\mathcal{A}^2 := C(\mathcal{A},2) = \delta^{ab}\mathcal{A}_a\mathcal{A}_b, \qquad \Delta^2 := -\hbar^2\mathcal{A}^2.$$

After some easy calculations one obtains for (30)

$$\Delta = \frac{\partial^2}{\partial k^2} + \mathrm{ctg}\frac{k}{2}\frac{\partial}{\partial k} + \frac{1}{4\sin^2 k/2}A^2.$$

For obvious reasons, when expressed by the spherical angular variables (θ,φ) in the space of rotation vector $\overline{k}$, Δ^2 has identical form with the operator of the squared magnitude of orbital angular momentum. Its spectrum consists of non-negative numbers $\hbar^2 l(l+1)$, where l denotes non-negative integers, $l = 0,1,2,\ldots$. As we saw in (45) in [28], $\Delta^2 = -\hbar^2\mathcal{A}^2$ does commute with the Laplace-Beltrami Casimir

$$\mathbf{S}^2 = -\hbar^2\delta^{ab}\mathcal{L}_a\mathcal{L}_b = -\hbar^2\delta^{ab}\mathcal{R}_a\mathcal{R}_b$$

so they have common wave functions. Spectrum of the Laplace-Beltrami operator consists of non-negative numbers $\hbar^2 j(j+1)$, where j runs over non-negative half-integers and integers

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \qquad \text{i.e.,} \qquad j \in \{0\} \bigcup \frac{\mathbb{N}}{2}$$

where $\mathbb{N}$ denotes the set of naturals and j is just the label of irreducible unitary representations of $\mathrm{SU}(2)$. When j is fixed, then l runs over the range $l = 0,1,\ldots,2j$ for the possible common eigenfunctions of $\mathbf{S}^2$ and Δ^2. According to (39)–(44), (46), (47) in [28], we have that

$$\mathbf{S}^2 D(j) = \hbar^2 j(j+1)D(j), \qquad \mathbf{S}^2\varepsilon(j) = \hbar^2 j(j+1)\varepsilon(j)$$

i.e., all matrix elements of the j-th irreducible unitary representation, or, equivalently, all elements of the minimal two-sided ideal $M(j)$, are eigenfunctions of

$$\mathbf{S}^2 = \delta^{ab}\mathbf{S}_a\mathbf{S}_b = \delta^{ab}\widehat{\mathbf{S}}_a\widehat{\mathbf{S}}_b = -\hbar^2\Delta[g]$$

with eigenvalues $\hbar^2 j(j+1)$.

Further, we have the following algebraization of operators

$$\mathbf{S}_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a, \qquad \widehat{\mathbf{S}}_a = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a$$

in this representation

$$S_a D(j) = S(j)_a D(j), \qquad \widehat{S}_a D(j) = D(j)S(j)_a \tag{31}$$

$$\Delta_a D(j) = [S(j)_a, D(j)] \tag{32}$$

and similarly for elements of the canonical basis, because, as we saw, there is a proportionality

$$\varepsilon(j)_{km} = (2j+1)D(j)_{km}.$$

Here $S(j)_a$ are standard $(2j+1)\times(2j+1)$ Hermitian matrices of the j-labelled angular momentum. According to (37), (38) in [28] we have that

$$D(j)\left(u\left(\overline{k}\right)\right) = \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}k^a S(j)_a\right).$$

This algebraization of differential operators is very convenient because the matrices of angular momentum are standard. Therefore, (50)–(53) in [28], or, alternatively, (54)–(57) in [28], may be used, where the label α to be replaced by j and the symbols $\Sigma(\alpha)_a$ by $S(j)_a$.

Representations $D(j)$ are irreducible, so, by definition

$$\delta^{ab}S(j)_a S(j)_b = \sum_a S(j)^2{}_a = \hbar^2 j(j+1)\mathrm{Id}_{\,(2j+1)}.$$

The only Abelian Lie subgroups of $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$ are one-dimensional, just the one-parameter subgroups. Therefore, one can choose only one $\mathcal{L}$-type operator and only one $\mathcal{R}$-type operator to form, together with $-\hbar^2\Delta[g]$, the complete system of eigenequations for the functions $\varepsilon(j)_{kl}/D(j)_{kl}$. Traditionally one chooses for $S(j)_a$ such a representation that $S(j)_3$ are diagonal. Then, of course, one should choose the operators $\mathcal{L}_3$, $\mathcal{R}_3$, or in terms of observables $\mathbf{S}_3$, $\widehat{\mathbf{S}}_3$. This is certainly the matter of convention. One could as well take any versor $\overline{n}\in\mathbb{R}^3$ and operators $n^a\mathcal{L}_a$, $n^a\mathcal{R}_a$ (or $n^a\mathbf{S}_a$, $n^a\widehat{\mathbf{S}}_a$), assuming only that $n^a S(j)_a$ is diagonal for any j. When we fix the quantum number j, then the eigenvalues of $\mathbf{S}_3$, $\widehat{\mathbf{S}}_3$ have the form $\hbar m$, where $m = -j, -j+1, \ldots, j-1, j$, jumping by one. Therefore, the matrix labels of $D(j)_{mk}$, $\varepsilon(j)_{mk}$ are not taken as $1,\ldots,2j+1$, but rather as $-j, -j+1, \ldots, j-1, j$. The matrices $S(j)_3$ are then chosen as

$$\begin{aligned} S(j)_3 &= \mathrm{diag}(-\hbar j, -\hbar(j-1), \ldots, \hbar(j-1), \hbar j) \\ &= \hbar\,\mathrm{diag}(-j, -(j-1), \ldots, (j-1), j). \end{aligned}$$

Therefore, the basic functions

$$D(j)_{mk}, \qquad \varepsilon(j)_{mk} = (2j+1)D(j)_{mk}$$

are defined by the following maximal system of compatible eigenequations

$$\mathbf{S}^2 D(j)_{mk} = \hbar^2 j(j+1)D(j)_{mk} \tag{33}$$

$$\mathbf{S}_3 D(j)_{mk} = m\hbar D(j)_{mk} \tag{34}$$

$$\widehat{\mathbf{S}}_3 D(j)_{mk} = k\hbar D(j)_{mk}. \tag{35}$$

The solution is unique up to normalization and this one is fixed by the first and third equations in (31), (32) with

$$n(\alpha) = n(j) = 2j+1.$$

Quite independently on the representation theory, the functions $D(j)_{mk}$ as solutions of (33)–(35) were found as basic stationary states of the free symmetric top, i.e., one with the following Hamiltonian (kinetic energy)

$$\mathbf{H} = \frac{1}{2I}\left(\widehat{\mathbf{S}}_1\right)^2 + \frac{1}{2I}\left(\widehat{\mathbf{S}}_2\right)^2 + \frac{1}{2K}\left(\widehat{\mathbf{S}}_3\right)^2.$$

The corresponding energy levels (eigenvalues of energy) are given by

$$E_{j,k} = \frac{1}{2I}\hbar^2 j(j+1) + \left(\frac{1}{2K} - \frac{1}{2I}\right)\hbar^2 k^2.$$

Certainly, they are $2(2j+1)$-fold degenerate, i,e, they do not depend on m at all and they do not distinguish the sign of k. If the top is spherical, $K = I$, they are $(2j+1)^2$-fold degenerate. When the top is completely asymmetric, the energy levels are $(2j+1)$-fold degenerate (independence on the spatial quantum number m).

Matrix elements $D(j)_{mk}$ of irreducible unitary representations, i.e., equivalently, elements of the canonical basis

$$\varepsilon(j)_{mk} = (2j+1)D(j)_{mk}$$

are common solutions of the system of eigenequations (33)–(35).

There is also another complete system of commuting operators, namely, $\mathbf{S}^2$, $\boldsymbol{\Delta}^2$, $\boldsymbol{\Delta}_3$. Of course, taking the third component is but just a custom, we could take as well $n^a \boldsymbol{\Delta}_a$, where $\overline{n}$ is an arbitrary unit vector in $\mathbb{R}^3$. Any common eigenfunction of Δ^2, Δ_3 has the following form

$$\psi\left(\overline{k}\right) = \psi\left(k, \theta, \varphi\right) = f(k) Y_{lm}\left(\overline{n}\left(\theta, \varphi\right)\right)$$

where f is an arbitrary function of the "rotation angle" $k = |\overline{k}|$, Y_{lm} is the standard symbol of spherical functions, and $\overline{n}$ is the unit vector of the oriented rotation axis. The eigenvalues are respectively given by $\hbar^2 l(l+1)$, where $l \in \{0\} \bigcup \mathbb{N}$ is an arbitrary non-negative integer, and $m\hbar$, where m runs over the range $m = -l, -l+1, \ldots, l-1, l$, jumping by one. The well-known system of eigenequations is satisfied

$$\Delta^2 \psi = \hbar^2 l(l+1)\psi, \qquad \Delta_3 \psi = \hbar m \psi. \tag{36}$$

The function f is arbitrary, because it is transparent for the action of Δ^2, Δ_3. The space of solutions of (36) is infinite-dimensional and this infinity is due to the arbitrariness of f. Roughly speaking, for any fixed value of l, such a system of functions represents an irreducible tensor of the group of automorphisms (9). The value $l = 0$ corresponds to scalars, i.e., functions constant on classes of adjoint elements. They are linear combinations or rather series of idempotents/characters $\varepsilon(j)/\chi(j)$. Similarly, all higher-order tensors may be combined from their orthogonal projections onto ideals $M(j)$. Those projections are common eigenfunctions

$$Q\{j\}_{lm} = f_{jl}(k) Y^l{}_m(\overline{n})$$

of $\mathbf{S}^2$, $\boldsymbol{\Delta}^2$, $\boldsymbol{\Delta}_3$, therefore, the "radial" functions f_{jl} satisfy the following reduced eigenequation

$$\frac{\mathrm{d}^2 f_{jl}}{\mathrm{d}k^2} + \operatorname{ctg}\frac{k}{2}\frac{\mathrm{d}f_{jl}}{\mathrm{d}k} + \left(j(j+1) - \frac{l(l+1)}{4\sin^2 k/2} \right) f_{jl} = 0. \tag{37}$$

When j is fixed, then l runs over the range

$$l = 0, 1, \ldots, 2j-1, 2j$$

i.e., integers from 0 to $2j$. It turn, any l-level is $(2j+1)$-fold degenerate, thus, for any fixed j, the number of independent functions $Q\{j\}_{lm}$ equals

$$\sum_{l=0}^{2j} (2l+1) = (2j+1)^2$$

just as expected, because $\dim M(j) = (2j+1)^2$.

This is an alternative choice of basis, or rather of orthonormal complete system in $L^2(\mathrm{SU}(2))$, tensorially ruled by irreducible representations of $\mathrm{SO}(3,\mathbb{R})$ as the automorphism group of $\mathrm{SU}(2)$.

The corresponding finite transformation rule reads

$$\begin{aligned} Q\{j\}_{lm}\left(g u\left(\overline{k}\right) g^{-1}\right) &= Q\{j\}_{lm}\left(u\left(R(g)\right)\overline{k}\right) = Q\{j\}_{lm}\left(k, R(g)\overline{n}\right) \\ &= \sum_n Q\{j\}_{ln}\left(k, \overline{n}\right) D(l)_{nm}\left(R(g)\right). \end{aligned}$$

Infinitesimally this is expressed as

$$\Delta_a Q\{j\}_{lm} = \sum_n Q\{j\}_{ln} S(l)_{anm}.$$

In terms of the convolution commutator

$$\left[\frac{\hbar}{\mathrm{i}}\mathcal{L}_a\delta, Q\{j\}_{lm}\right] = \left[\frac{\hbar}{\mathrm{i}}\mathcal{L}_a\varepsilon(j), Q\{j\}_{lm}\right] = \sum_n Q\{j\}_{ln} S(l)_{anm}.$$

Of course, the convolution commutator is meant in the sense

$$[f, g] = f * g - g * f. \tag{38}$$

The use of spherical functions ${Y^l}_m(\overline{n})$ in (36) expresses explicitly the fact that for a fixed l we are dealing with an irreducible object of the group of inner automorphisms. This is so-to-speak a non-redundant description of such objects, with all its advantages and disadvantages. The obvious disadvantage is that the tensorial structure is hidden. The point is that ${Y^l}_m(\overline{n})$ are independent quantities extracted from the l-th tensorial power of the unit versor $\overline{n}$, $\underset{l}{\otimes}\overline{n}$. Analytically such a symmetric tensor is given by the system of components

$$n^{a_1} \ldots n^{a_l}. \tag{39}$$

The transformation rule under $R \in \mathrm{SO}(3, \mathbb{R})$

$$(R\overline{n})^{a_1} \ldots (R\overline{n})^{a_l} = {R^{a_1}}_{b_1} \ldots {R^{a_l}}_{b_l} n^{b_1} \ldots n^{b_l}$$

is evidently tensorial and preserves the symmetry, however, it is reducible, because orthogonal transformations preserve all trace operations. Irreducible objects are obtained from (39) by eliminating all traces, e.g.,

$$\mathcal{Y}(1)^a = n^a \tag{40}$$

$$\mathcal{Y}(2)^{ab} = n^a n^b - \frac{1}{3}\delta^{ab} \tag{41}$$

$$\mathcal{Y}(3)^{abc} = n^a n^b n^c - \frac{1}{5}\left(n^a\delta^{bc} + n^b\delta^{ca} + n^c\delta^{ab}\right) \tag{42}$$

$$\begin{aligned}\mathcal{Y}(4)^{abcd} = {} & n^a n^b n^c n^d - \frac{1}{7}\Big(n^a n^b \delta^{cd} + n^a n^c \delta^{bd} + n^a n^d \delta^{bc} + n^b n^c \delta^{ad} \\ & + n^b n^d \delta^{ac} + n^c n^d \delta^{ab}\Big) + \frac{1}{35}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)\end{aligned} \tag{43}$$

and so on. The logic of those tensors is that they are algebraically built of n^a, δ^{bc}, are completely symmetric and traceless in any pair of indices (trace meant as a contraction with an appropriate δ_{ab}).

Any $\mathcal{Y}(l)$ has only $(2l+1)$ independent components, which are linear combinations of ${Y^l}_m$, $m = -l, \dots, l$. Therefore, the representation is very redundant, however, the tensorial structure is explicitly visible. Instead of functions $Q\{j\}_{lm}$ one can use tensorial objects

$$Q\{j,l\}^{a_1 \dots a_l} = f_{jl}(k)\mathcal{Y}(l)\,(\overline{n})^{a_1 \dots a_l}\,. \tag{44}$$

Infinitesimally, the tensorial character of quantities $Q\{j,l\}$ is represented by the following relationship

$$\mathcal{A}_b Q\{j,l\}^{a_1 \dots a_l} = [\mathcal{L}_b\delta, Q\{j,l\}^{a_1 \dots a_l}] = -\sum_c {\varepsilon_b}^{a_c}{}_d Q\{j,l\}^{a_1 \dots a_{c-1} d a_{c+1} \dots a_l}$$

for example

$$\mathcal{A}_b Q\{j,2\}^{km} = \left[\mathcal{L}_b\delta, Q\{j,2\}^{km}\right] = -{\varepsilon_b}^k{}_d Q\{j,2\}^{dm} - {\varepsilon_b}^m{}_d Q\{j,2\}^{kd}$$

and so on. Surely, $\mathcal{L}_b\delta$ in these equations may be replaced by $\mathcal{R}_b\delta$ and both may be replaced by $\mathcal{L}_a\varepsilon(j) = \mathcal{R}_a\varepsilon(j)$. Irreducibility implies that

$$\begin{aligned} \delta^{ab}\mathcal{A}_a\mathcal{A}_b Q\{j,l\}^{a_1 \dots a_l} &= \delta^{ab}\left[\mathcal{L}_a\delta, [\mathcal{L}_b\delta, Q\{j,l\}^{a_1 \dots a_l}]\right] \\ &= \delta^{ab}\left[\mathcal{L}_a\varepsilon(j), [\mathcal{L}_b\varepsilon(j), Q\{j,l\}^{a_1 \dots a_l}]\right] \\ &= -l(l+a)Q\{j,l\}^{a_1 \dots a_l} \end{aligned}$$

with the (38)-meaning of the convolution commutator.

4. Some Problems Concerning Irreducible Tensors of Automorphism Group

There are some subtle points concerning irreducible tensors of the automorphism group, which were partially mentioned earlier in the paper devoted to general Lie groups [28]. Namely, the tensorial quantities (64) in [28] were introduced there. They were obtained as convolution monomials of

$$Q_a = \mathcal{L}_a\delta = \mathcal{R}_a\delta \qquad \text{or} \qquad \Sigma_a = \frac{\hbar}{\mathrm{i}}Q_a$$

or rather as symmetrizations of these monomials. The symmetrizations of monomials built of Σ_a are Hermitian in the sense of group algebra, just as Σ_a themselves. However, in general they are not irreducible tensors, just the traces (in the sense of Killing metric tensor) must be subtracted. The symmetrized monomials are represented in the Peter-Weyl sense by matrices (63) in [28] alternatively, depending on whether the convention (50) or (54) in [28] is used. And an important point is of course that the monomials (58) in [28] are different from the pointwise products $Q_a Q_b \dots Q_k$. In particular, the pointwise products $Q(\alpha)_a Q(\alpha)_b \dots Q(\alpha)_k$ of $M(\alpha)$-projections do not belong to $M(\alpha)$, whereas the convolutions $Q(\alpha)_a * Q(\alpha)_b * \dots * Q(\alpha)_k$ certainly do.

Let us specialize the problem to $\mathrm{SU}(2)$. The distribution $\Sigma_a = (\hbar/\mathrm{i})Q_a$, physically corresponding to the angular momentum, is suggestively expressed by the operators (29)

$$\Sigma_a = \mathbf{S}_a\delta = \widehat{\mathbf{S}}_a\delta = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a\delta = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a\delta \tag{45}$$

and its projections onto ideals $M(j)$ are given by

$$\Sigma(j)_a = \frac{\hbar}{\mathrm{i}}\mathcal{L}_a\varepsilon(j) = \frac{\hbar}{\mathrm{i}}\mathcal{R}_a\varepsilon(j). \tag{46}$$

The above expression (45) is a series built of (46) with all possible values of $j = 0, 1/2, 1, 3/2, \dots$ and the limit is meant in the distribution sense. But of course $\Sigma(j)_a$ themselves are well-defined smooth functions and

$$\Sigma(j)_a = \frac{\mathrm{d}\varepsilon(j)}{\mathrm{d}k}\frac{k_a}{k} = (2j+1)\frac{\mathrm{d}\chi(j)}{\mathrm{d}k}n_a$$

because the idempotents $\varepsilon(j)$/characters $\chi(j)$ depend only on k. The Peter-Weyl coefficients of Σ_a are given by the usual $(2j+1)\times(2j+1)$ matrices $S(j)_a$ of angular momentum or by their transposes $S(j)^T_a$, depending on which one of conventions (54) or (50) in [28] is used.

The higher-order Hermitian $\mathrm{SO}(3,\mathbb{R})$-tensors are again given by (64) in [28] and the corresponding Peter-Weyl matrices (63) in [28] will be denoted by

$$\begin{aligned} S(j,l)_{a_1\dots a_l} &= S(j)_{(a_1}\dots S(j)_{a_l)} \\ S(j,l)^T_{a_1\dots a_l} &= S(j)^T_{(a_1}\dots S(j)^T_{a_l)}. \end{aligned}$$

They are tensorial and symmetric, nevertheless, just like (39), they are still reducible. To obtain irreducible objects, one must eliminate traces, in analogy to (40)–(43). The corresponding traceless parts of (matrix-valued) tensors $S(j,l)$, $S(j,l)^T$ will be denoted by

$$S^\circ(j,l) = \mathrm{Traceless}\left(S(j,l)\right), \qquad S^\circ(j,l)^T = \mathrm{Traceless}\left(S(j,l)^T\right). \tag{47}$$

Let us observe that the very literal analogy with (40)–(43) is, nevertheless, misleading, because in (36), (39), (40)–(43) we are dealing with the pointwise products

$$n^a n^b \dots n^r \qquad \text{or} \qquad k^a k^b \dots k^r.$$

Because of this the shape factor $f_{jl}(k)$ in (36), (44) must be introduced and subject to the "radial" Schrödinger-type equation. Unlike this, there is no problem of "radial" equation when one deals with functions $\Sigma(j,l)$ on $\mathrm{SU}(2)$ with the Peter-Weyl coefficients (47). Namely, for any fixed half/integer j and any $l \leq 2j$, the following functions on $\mathrm{SU}(2)$

$$T(j,l)_{a_1\ldots a_l} = \mathrm{Tr}\,(S^\circ(j,l)_{a_1\ldots a_l}\widehat{\varepsilon}(j)) = \mathrm{Tr}\,(S^\circ(j,l)_{a_1\ldots a_l}D(j)(2j+1)) \quad (48)$$

are eigenfunctions of $\mathbf{S}^2 = -\hbar^2\mathbf{\Delta}$ with the eigenvalue $\hbar^2 j(j+1)$, thus, they are elements of $M(j)$ and simultaneously are the eigenfunctions of $\mathbf{\Delta}^2 = \delta^{ab}\mathbf{\Delta}_a\mathbf{\Delta}_b$ with the eigenvalue $\hbar^2 l(l+1)$. Any element of $M(j)$ may be uniquely expanded as follows

$$F = \sum_{l=0}^{2j} P(l)^{a_1\ldots a_l} T(j,l)_{a_1\ldots a_l} \quad (49)$$

where the tensor $P(l)$ is totally symmetric and traceless.

Its Peter-Weyl matrix of coefficients $\widehat{F}$ in the convention (54) in [28] has the following form

$$\widehat{F} = \sum_{l=0}^{2j} P(l)^{a_1\ldots a_l}\; S^\circ(j,l)_{a_1\ldots a_l}. \quad (50)$$

Evidently, the function (49) is Hermitian in the sense of group algebra if and only if the coefficients $P(l)^{a_1\ldots a_l}$ are real, because all the matrices $S^\circ(j,l)_{a_1\ldots a_l}$ combined in (50) are Hermitian.

The above representation is tensorially symmetric, however, informationally redundant. In non-redundant description, based on spherical functions Y_{lm}, we have instead of (49) the representation

$$F = \sum_{l=0}^{2j}\sum_{m=-l}^{l} P_{lm} Q\{j\}_{lm} \quad (51)$$

where the functions $Q\{j\}_{lm}$ are given by (36).

The obvious properties of spherical functions, i.e.,

$$Y^l{}_m(-\overline{n}) = (-1)^l Y_{lm}(\overline{n}), \qquad \overline{Y}^l{}_m = Y^l{}_{-m}$$

imply that F is Hermitian in the sense of group algebra over $\mathrm{SU}(2)$ if and only if

$$\overline{P}_{lm} = (-1)^l P_{l(-m)}. \quad (52)$$

The Hermitian elements of the group algebra of $\mathrm{SU}(2)$ given by (48) are assumed to represent some important physical quantities. They have very suggestive tensorial structure and for $l = 1$ they represent the angular momentum. Because of this there is a natural temptation to interpret them physically in terms of magnetic multipole momenta [29]. Although in tensorial representation their system is redundant, it is convenient to expand with respect to them the density operators. The corresponding coefficients $P(l)^{a_1 \cdots a_l}$ are directly related to the expectation values of multipoles, and it is reasonable to interpret them physically as magnetic polarizations of the corresponding order [29]. It is clear that the physical situations, characterized by the fixed label j of the Casimir invariant, possess multipole momenta and polarizations of the orders $l = 0, 1, \ldots, 2j$. The algebraically non-redundant description of these objects is based on (51)–(52).

5. Quasiclassical Asymptotic of "Large Quantum Numbers"

Let us now discuss the quasiclassical limit. By this we mean the limit of "large quantum numbers" in equations like (36), (37) and others. An important aspect of this asymptotics is that the corresponding basic solutions are superposed with coefficients which are "slowly varying" functions of their arguments in some "wide" range of their values and practically vanishing outside this range. It is important that the range is simultaneously "wide" in the sense "much wider than one", but at the same time concentrated about some "large" mean value. This enables one to perform approximate "continuization" of discrete labels/(quantum numbers) and to replace their summation by integration.

For $l = 0$ the substitution of

$$f_{j0} = A \frac{\chi_{j0}}{\sin k/2}, \qquad A = \mathrm{const}$$

to (37) leads immediately to the following result

$$f_{j0} = A \frac{\sin(2j+1)k/2}{\sin k/2}.$$

But

$$\varepsilon(j)(0) = (2j+1)^2$$

thus, $A = 2j + 1$, and finally

$$\varepsilon(j)(k) = (2j+1) \frac{\sin(2j+1)k/2}{\sin k/2}, \quad \delta(k) = \sum_{j=0}^{\infty} (2j+1) \frac{\sin(2j+1)k/2}{\sin k/2}. \tag{53}$$

One can easily show that

$$f_{j,l+1} = \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{l}{2} \mathrm{ctg} \frac{k}{2} \right) f_{jl} \tag{54}$$

therefore, iterating this recurrence formula one obtains the explicit formula for the multipole basis (36)

$$f_{jl} = \prod_{n=l-1}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{n}{2} \mathrm{ctg} \frac{k}{2} \right) \varepsilon(j). \tag{55}$$

Let us discuss the asymptotic expansions of such expressions in a domain $[0, a]$ where $a < 2\pi$. One can show that for continuous functions f on $[0, a]$ the following holds

$$\lim_{j\to\infty} \int_0^a f(k) \left(\frac{\sin(2j+1)k/2}{\sin k/2} - \frac{\sin(2j+1)k/2}{k/2} \right) \mathrm{d}k$$
$$= \lim_{j\to\infty} \int_0^a f(k) \frac{k/2 - \sin k/2}{(k/2)\sin k/2} \sin(2j+1)\frac{k}{2} \mathrm{d}k = 0. \tag{56}$$

Incidentally, this statement is true for more general "sufficiently regular" functions f, i.e., they need not be continuous. The equation (56) means that in the integral mean-value sense in $[0, a]$ the functions $\varepsilon(j)$ with "sufficiently large" j-s may be asymptotically replaced by

$$(2j+1)\frac{\sin(2j+1)k/2}{k/2}. \tag{57}$$

And for "sufficiently large" values of j the functions (57) are essentially concentrated about $k = 0$.

Therefore, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$ the following holds

$$\left| \int_0^a f(k) \frac{\sin nk/2}{k/2} \mathrm{d}k - \int_0^\infty f(k) \frac{\sin nk/2}{k/2} \mathrm{d}k \right| < \varepsilon$$

and

$$\lim_{n\to\infty} \int_0^a f(k) \frac{\sin nk/2}{k/2} \mathrm{d}k = \pi f(0)$$

i.e.,

$$\mathbb{N} \ni n \mapsto \frac{1}{\pi} \frac{\sin nk/2}{k/2}$$

is a "Dirac-delta sequence".

The functions $\varepsilon(j) = (2j+1)\chi(j)$ are concentrated about $k = 0$ and have there the maxima $(2j+1)^2$. At $k = 2\pi$ they have the extrema $\pm(2j+1)^2$ depending on whether j is respectively integer $(+)$ or half-integer $(-)$. For $j \to \infty$, $\varepsilon(j)$ may be replaced by

$$\varepsilon^\circ(j) = (2j+1)\frac{2}{k} \sin \frac{(2j+1)k}{2} \tag{58}$$

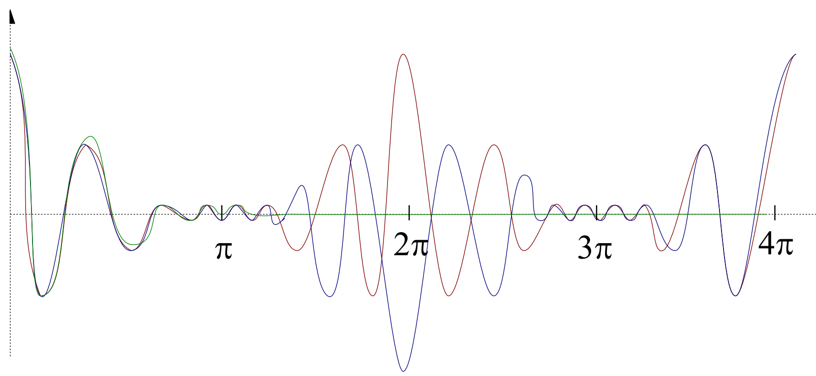

Figure 1. Asymptotic behaviour of functions $\varepsilon(j)$.

in any interval $[0, a]$, $a < 2\pi$. But it is also seen that $\varepsilon(j)$ may be replaced by

$$\varepsilon^{2\pi}(j) = \pm(2j+1)\frac{2}{2\pi - k}\sin\frac{(2j+1)k}{2} \tag{59}$$

in any interval $[a, 2\pi]$, $a > 0$. The signs $+/-$ appear respectively for integer/half-integer values of j. Therefore, globally, in the total $\mathrm{SU}(2)$-range $k \in [0, 2\pi]$, we have the following asymptotics for $j \to \infty$

$$\varepsilon(j) \approx (2j+1)\sin\frac{(2j+1)k}{2}\left(\frac{2}{k} + (-1)^{2j}\frac{2}{2\pi - k}\right). \tag{60}$$

The oscillating, sign-changing extremum at $k = 2\pi$ is a purely quantum, spinorial effect. Such an effect does not appear on $\mathrm{SO}(3, \mathbb{R})$, when the range of k is given by $[0, \pi] \subset \mathbb{R}$. However, when the functions $\varepsilon(j)$ are superposed with slowly-varying coefficients concentrated at large values of j, then the subsequent peaks approximately cancel each other. Nevertheless, for any fixed j, it does not matter how large one, we have the asymptotic formula (60) with both peaks. We shall write it symbolically

$$\varepsilon(j) \approx \varepsilon^{\circ}(j) + \varepsilon^{2\pi}(j) \tag{61}$$

where $\varepsilon_0(j)$, $\varepsilon_{2\pi}(j)$ are concentrated respectively about $k = 0$ and $k = 2\pi$. The same is true for all other "radial" functions appearing in the multipole expansion (36).

Approximate equation for f_{jl} about $k = 0$ and for large values of j has the following form

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2} f_{jl}^\circ + \frac{2}{k}\frac{\mathrm{d}}{\mathrm{d}k} f_{jl}^\circ + \left(j(j+1) - \frac{l(l+1)}{k^2} \right) f_{jl}^\circ = 0. \tag{62}$$

For $\varepsilon^\circ(j) = f_{j0}^\circ$ one re-obtains the known expression

$$\varepsilon^\circ(j) = (2j+1)\frac{\sin(2j+1)k/2}{k/2}.$$

One can easily show that

$$f_{j,l+1}^\circ = \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{l}{k} \right) f_{jl}^\circ$$
$$f_{jl}^\circ = \left(\prod_{n=l-1}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{n}{k} \right) \right) \varepsilon^\circ(j)$$

in a complete analogy to (54), (55).

Another often used approximation for large j is

$$j(j+1) \mapsto \left(j + \frac{1}{2} \right)^2.$$

Then the differential equation (62) becomes approximately

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2} f_{jl}^\circ + \frac{2}{k}\frac{\mathrm{d}}{\mathrm{d}k} f_{jl}^\circ + \left(\left(j + \frac{1}{2} \right)^2 - \frac{l\,(l+1)}{k^2} \right) f_{jl}^\circ = 0.$$

Again one can show that the approximate solutions of rigorous equations for the large values of j have the following form

$$f_{jl} = f_{jl}^\circ + f_{jl}^{2\pi}$$

where

$$f_{jl}^\circ = \left(\prod_{n=l-1}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{n}{k} \right) \right) \varepsilon^\circ\,(j)$$
$$f_{jl}^{2\pi} = \left(\prod_{n=l-1}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}k} - \frac{n}{2\pi - k} \right) \right) \varepsilon^{2\pi}\,(j) \approx -\, f_{j+\frac{1}{2},l}^{2\pi}.$$

One can note that $\varepsilon(j)$ in (53) has the profound geometric interpretation of the generated unit of $M(j)$ and $\chi(j) = (1/(2j+1))\varepsilon(j)$ is the character of the j-th irreducible unitary representation of $\mathrm{SU}(2)$. And seemingly, one might have an impression that the asymptotic counterpart (58) is something "accidental", non-interpretable in geometric terms. However, as a matter of fact, it is an important object of the Fourier analysis on $\mathbb{R}^3 \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3,\mathbb{R})$.

Indeed, it may be easily shown that the Fourier representation of the Dirac delta distribution on $\mathbb{R}^3$ (as the Fourier transform of unity)

$$\delta\left(\overline{\omega}\right) = \frac{1}{\left(2\pi\right)^3} \int \mathrm{e}\left(\mathrm{i}\underline{\varkappa}\overline{\omega}\right) \mathrm{d}^3\underline{\varkappa}$$

after performing the integration over angels becomes

$$\delta\left(\overline{\omega}\right) = \frac{1}{\left(2\pi\right)^3} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} \mathrm{d}\varkappa\varkappa^2 \int_0^{\pi} \mathrm{d}\vartheta \cos\vartheta \mathrm{e}\left(\mathrm{i}\varkappa\omega\cos\vartheta\right)$$

where $(\varkappa, \vartheta, \varphi)$ are spherical variables in the space $\mathbb{R}^3$ of vectors $\underline{\varkappa}$, adapted to the direction of $\overline{\omega}$ as the "z-axis direction". After the substitution $x = \cos\vartheta \in [1, -1]$ and $\varkappa = \zeta/2$ and elementary integrations, one obtains the following formula

$$\delta\left(\overline{\omega}\right) = \frac{1}{16\pi^2} \int_0^{\infty} \mathrm{d}\zeta \frac{\zeta \sin\zeta\omega/2}{\omega/2}$$

Under substituting $\zeta = (2j+1)$ it is turned into

$$\delta\left(\overline{\omega}\right) = \frac{1}{8\pi^2} \int_{-1/2}^{\infty} \mathrm{d}j\,(2j+1)\,\frac{\sin\left(2j+1\right)\omega/2}{\omega/2} = \int \mathrm{d}j\ \varepsilon_{\mathrm{class}}(j)(\omega). \tag{63}$$

Expression

$$\varepsilon_{\mathrm{class}}(j)(\omega) = (2j+1)\,\frac{\sin\left(2j+1\right)\omega/2}{\omega/2} \tag{64}$$

is an obvious counterpart of (53) and of its expression for the Dirac distribution on $\mathrm{SU}(2)$, and the all other analogies are easily readable. They are not merely formal analogies, the point is that they are really true asymptotic approximations and geometric counterparts. Discrete summation over the "quantum number" j is now replaced by the integration over the continuous label j corresponding to the non-compactness of $\mathbb{R}^3 \simeq \mathfrak{su}(2)$ and well suited to the "classical" nature of expressions.

The superposed functions (64) play in the commutative group algebra $\mathfrak{su}(2) \approx \mathbb{R}^3$ the role of generating units of ideals $M(j)$ composed of functions with the fixed "square of linear momentum"

$$\left(j+1/2\right)^2 \hbar^2 \approx j\left(j+1\right)\hbar^2.$$

The last approximate "equality" corresponding to the "large" values of j. This ideal is not minimal. The minimal ones just correspond to the single exponents with the wave vectors $\underline{\varkappa}$, i.e., "linear momenta" $\hbar\underline{\varkappa}$. In $\varepsilon_{\text{class}}(j)$ superposed are (with equal "coefficients") all exponents $\mathrm{e}\left(\mathrm{i}\,(j+1/2)\,\overline{n}\cdot\overline{\omega}\right)$, where $\overline{n}$ runs over the manifold $S^2(0,1) \subset \mathbb{R}^3$ of all unit vectors. The ideals $M(j)$ are minimal ones invariant under the group $\mathrm{SO}(3,\mathbb{R})$ of outer automorphisms of $\mathrm{SU}(2)$. Those outer automorphisms are not only algebraic automorphisms of $\mathbb{R}^3$ as an Abelian additive group. In addition they preserve the standard Euclidean metric in $\mathbb{R}^3$. This metric just corresponds up to multiplicative constant factor to the Killing metric of $\mathfrak{su}(2) \approx \mathbb{R}^3$. It worth to note that the above terms like "group algebra" are now used rather in a metaphoric sense, because we are dealing with continuous spectrum and are outside of $L^2\left(\mathbb{R}^3\right)$ and $L^1\left(\mathbb{R}^3\right)$. Everything may be rigorously formulated in terms of rigged Hilbert spaces and direct integrals of Hilbert spaces, however, there is no place for that here.

The above limit transition and asymptotics are meant in the sense of truncation procedure in the rigorous group algebra of $\mathrm{SU}(2)$.

Quasiclassical limit is based on the truncation procedure of the group algebra of $\mathrm{SU}(2)$. Namely, we take the subalgebra consisting of all ideals $M(j)$ with $j \geq j_0$ for some fixed j_0

$$M\left(j \geq j_0\right) := \bigoplus_{j \geq j_0} M(j).$$

As mentioned, for large values of j, the generated units $\varepsilon(j) \in M(j)$ are essentially concentrated about $k=0$, $k=2\pi$

$$\varepsilon(j) \approx \varepsilon^{\circ}(j) + \varepsilon^{2\pi}(j)$$

cf. (58), (59), (60), (61), and the following holds

$$\varepsilon^{\circ}(j)(0) = (2j+1)^2, \qquad \varepsilon^{2\pi}(j)(2\pi) = (-1)^{2j}\,(2j+1)^2.$$

The larger truncation threshold j_0, the better the generated unit of $M\left(j \geq j_0\right)$

$$\varepsilon\left(j \geq j_0\right) := \sum_{j=j_0}^{\infty} \varepsilon(j) \tag{65}$$

is approximated by

$$\varepsilon_{\text{class}}\left(j \geq j_0\right) := \int_{j_0}^{\infty} \mathrm{d}j\; \varepsilon_{\text{class}}(j) \tag{66}$$

where $\varepsilon_{\text{class}}(j)$ is given by (64). Of course, the convergence of series (65) and integral (66) is meant in the distribution sense.

Projections of functions A, B on $\mathrm{SU}(2)$ onto the truncated ideal $M\,(j \geq j_0)$ will be denoted by

$$\widetilde{A} = A\,(j \geq j_0)\,, \qquad \widetilde{B} = B\,(j \geq j_0)\,.$$

The abbreviations $\widetilde{A}$, $\widetilde{B}$ are used when there is no danger of confusion.

For physically relevant functions A, B, the Peter-Weyl series expansions of $\widetilde{A}$, $\widetilde{B}$ may be reasonably approximated by continuous integral representations like (63), (66).

Let us go back to quantum states-densities represented in terms of non-redundant multipole expansions as follows

$$\varrho = \sum_{j=j_0}^{\infty} \sum_{l=0}^{2j} \sum_{m=-l}^{l} P(j)_{lm} Q\,\{j\}_{lm} \tag{67}$$

$$Q\{j\}_{lm}\left(\overline{k}\right) = f_{jk}(k) Y_{lm}\left(\frac{\overline{k}}{k}\right) \tag{68}$$

where the expansion coefficients $P(j)_{lm}$ may be roughly interpreted as magnetic multipole moments.

Quasiclassical states are represented by expressions (67), (68), where

- j_0 is "large"
- $P(j)_{lm}$ as functions of j are concentrated in some ranges

$$\left[\overline{j} - \Delta j/2, \overline{j} + \Delta j/2\right], \quad \overline{j} \gg \Delta j \gg 1$$

- within this range, $P(j)_{lm}$ are slowly varying functions of j

$$|P(j+1/2)_{lm} - P(j)_{lm}| \ll |P(j)_{lm}|\,.$$

Algebraic operations of group algebra on $\mathrm{SU}(2)$ attain some very peculiar representation in quasiclassical limit in the above sense. So, let us write down the convolution formula for "truncated" functions

$$\begin{aligned}&(A(j \geq j_0) * B(j \geq j_0))\left(u\left(\overline{k}\right)\right)\\ &\qquad = \int A(j \geq j_0)\left(u\left(\overline{l}\right)\right) B(j \geq j_0)\left(u\left(-\overline{l}\right) u\left(\overline{k}\right)\right) \frac{4\sin^2 l/2}{l^2}\frac{\mathrm{d}^3\overline{l}}{16\pi^2}.\end{aligned}$$

The terms concentrated about $k = 2\pi$, as it was seen, approximately cancel each other. One can assume that the integrated functions are essentially concentrated in a close neighbourhood of the unity in $\mathrm{SU}(2)$, i.e., the null of $\mathfrak{su}(2)$. There, in the lowest order of approximation, we have

$$u\left(\bar{l}\right)u\left(\bar{k}\right) \approx u\left(\bar{l}+\bar{k}+\frac{1}{2}\,\bar{l}\times\bar{k}\right).$$

Performing the corresponding Taylor expansions in our integral formulas and making use of the earlier mentioned relationship between the variables $\overline{k}$ and $\overline{\omega}$, we finally obtain

$$A(j \geq j_0) *_{\mathrm{SU}(2)} B(j \geq j_0) \approx A(j \geq j_0) *_{\mathbb{R}^3} B(j \geq j_0) \tag{69}$$

where the convolution symbols on the left- and right-hand sides are meant in the non-commutative $\mathrm{SU}(2)$- and commutative $\mathbb{R}^3 \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3,\mathbb{R})$-senses, respectively. Surely, (69) is meant in the sense of lowest-order approximation, the terms with higher-order derivatives are neglected.

Similarly, for the quantum Poisson bracket we obtain the familiar expression

$$\left[\widetilde{A},\widetilde{B}\right] = \frac{1}{\mathrm{i}\hbar}\left(\widetilde{A} *_{\mathrm{SU}(2)} \widetilde{B} - \widetilde{B} *_{\mathrm{SU}(2)} \widetilde{A}\right) \approx \frac{1}{\mathrm{i}\hbar}\left(\left(\mathcal{A}_a\widetilde{A}\right) *_{\mathbb{R}^3} \left(\omega^a\widetilde{B}\right)\right). \tag{70}$$

Here again we mean the lowest-order approximation, when the higher-derivatives terms following from the Taylor expansions are neglected. As usual, $\mathcal{A}_a$ is the generator of inner automorphisms in $\mathrm{SU}(2)$, i.e., equivalently, of Killing rotations in $\mathfrak{su}(2) \approx \mathbb{R}^3$

$$\mathcal{A}_a = \varepsilon_{ab}{}^{c}\omega^b\frac{\partial}{\partial\omega^c}.$$

So, in terms of Fourier representants $\widehat{\widetilde{A}}(\underline{\sigma})$, we have

$$\{\sigma_i,\sigma_j\} = \sigma_k\varepsilon^k{}_{ij}, \qquad \left\{\widehat{\widetilde{A}},\widehat{\widetilde{B}}\right\} = \sigma_k\varepsilon^k{}_{ij}\frac{\partial\widehat{\widetilde{A}}}{\partial\sigma_i}\frac{\partial\widehat{\widetilde{B}}}{\partial\sigma_j}.$$

In particular, for the evolution of density $\widetilde{\varrho}$ we obtain

$$\frac{\partial\widetilde{\varrho}}{\partial t} = \left[\widetilde{H},\widetilde{\varrho}\right]_{\mathbb{R}^3}, \qquad \frac{\partial\widehat{\widetilde{\varrho}}}{\partial t} = \left[\widehat{\widetilde{H}},\widehat{\widetilde{\varrho}}\right]$$

where H denotes the Hamiltonian. Taking appropriate Hamiltonians one obtains classical asymptotics of various dynamical models of the evolution of quantum angular momentum, or rather systems of quantum angular momenta. This includes complicated interactions between magnetic multipoles as described above.

6. Final Comments Concerning Quasiclassical Limit

Let finish with some comments concerning quasiclassical formulas which may be helpful when operating with some geometrically and physically important quantities.

First of all, let us observe that (69) is a merely zeroth-order approximation. The first-order approximation is given by

$$\widetilde{A} *_{\mathrm{SU}(2)} \widetilde{B} \approx \widetilde{A} *_{\mathbb{R}^3} \widetilde{B} + \frac{1}{2}\left[\widetilde{A}, \widetilde{B}\right]_{\mathbb{R}^3} \tag{71}$$

where, let us remind $\left[\widetilde{A}, \widetilde{B}\right]_{\mathbb{R}^3}$ is the extreme right-hand side of (70). The second term is the lowest-order approximation to the $\mathrm{SU}(2)$-convolution commutator. It is well known that the commutator, or more precisely quantum Poisson bracket, describes infinitesimal transformations, in particular symmetries of quantum states (as described by density operators). It is well known that the operator eigenequation for density operators

$$\mathbf{A}\varrho = a\varrho$$

implies that the operators $\mathbf{A}$, ϱ do commute, thus, their quantum Poisson bracket vanishes

$$\{\mathbf{A}, \varrho\}_Q = \frac{1}{\mathrm{i}\hbar}(\mathbf{A}\varrho - \varrho\mathbf{A}) = 0.$$

It is assumed here that $\mathbf{A}$ represents a physical quantity, thus, it is self-adjoint, $\mathbf{A}^+ = \mathbf{A}$. Therefore, the concept of eigenstate, in particular that of pure state (one satisfying a maximal possible system of compatible eigenconditions), unifies in a very peculiar way two logically distinct concepts: information and symmetry. Information aspect is that the physical quantity $\mathbf{A}$ has a sharply defined value on the state ϱ, there is no statistical spread of measurement results. Symmetry aspect is that ϱ is invariant under the one-parameter group of unitaries, i.e., quantum automorphisms, generated by $\mathbf{A}$. On the quantum level, symmetry properties are implied by information properties, because the quantum Poisson bracket is algebraically built of the associative product. This is no longer the case in quasiclassical limit and on the classical level, where the Poisson bracket and (commutative) associative product are algebraically independent on each other. But information and symmetry are qualitatively different things, therefore, on the quasiclassical level, the two first terms of the expansion for the non-commutative associative product should be taken into account when discussing classical concepts corresponding to eigenequations. Otherwise the physical interpretation of eigenconditions would be damaged.

Let us remind, following (33)–(35), that differential equations satisfied by the functions $\varepsilon(j)_{mk}$ may be written in the following form

$$\begin{aligned}\mathbf{S}^2\,\varepsilon(j)_{mk} &= j(j+1)\hbar^2\,\varepsilon(j)_{mk}\\ \mathbf{S}_3\,\varepsilon(j)_{mk} &= m\hbar\,\varepsilon(j)_{mk}\\ \widehat{\mathbf{S}}_3\,\varepsilon(j)_{mk} &= k\hbar\,\varepsilon(j)_{mk}\end{aligned}$$

with the known spectra of quantum numbers j, m, k. Rewriting these equations in terms of $\mathrm{SU}(2)$-convolutions we obtain

$$\Sigma^2 * \varepsilon(j)_{mk} = \varepsilon(j)_{mk} * \Sigma^2 = j(j+1)\hbar^2\,\varepsilon(j)_{mk} \tag{72}$$

$$\Sigma_3 * \varepsilon(j)_{mk} = m\hbar\,\varepsilon(j)_{mk} \tag{73}$$

$$\varepsilon(j)_{mk} * \Sigma_3 = k\hbar\,\varepsilon(j)_{mk} \tag{74}$$

where Σ_a are given by (45) and Σ^2 denotes the convolution-squared vector Σ_a

$$\Sigma^2 = \Sigma_1 * \Sigma_1 + \Sigma_2 * \Sigma_2 + \Sigma_3 * \Sigma_3.$$

To obtain the quasiclassical counterparts of (72)–(74) we must use the asymptotic formulas (71). It is more convenient to express them in terms of Fourier transforms, which were defined as functions on the Lie co-algebra $(\mathfrak{su}(2))^* \simeq \mathbb{R}^3$. So, we shall use the coordinates σ_i introduced above and the functions $\widehat{\varepsilon}(j)(\underline{\sigma})$ such that

$$\varepsilon(j)_{mn}(\overline{\varkappa}) = \frac{1}{(2\pi\hbar)^3}\int \widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\underline{\sigma}\overline{\varkappa}\right)\mathrm{d}^3\underline{\sigma}. \tag{75}$$

The left-hand sides of (75) are functions on the Lie algebra $\mathfrak{su}(2) \simeq \mathbb{R}^3$ used to represent the approximate expressions for the elements of canonical basis (matrix elements of irreducible UNIREPS) as functions on $\mathrm{SU}(2)$. The system (72)–(74) is expressed in terms of these Fourier transforms as follows

$$\sigma^2\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) = j(j+1)\hbar^2\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) \tag{76}$$

$$\sigma_3\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) + \frac{1}{2}\left\{\sigma_3, \widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\right\} = m\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) \tag{77}$$

$$\sigma_3\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}) - \frac{1}{2}\left\{\sigma_3, \widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\right\} = n\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}). \tag{78}$$

The last two equations imply that

$$\left\{\sigma_3, \widehat{\varepsilon}(j)_{mn}(\underline{\sigma})\right\} = (m-n)\hbar\,\widehat{\varepsilon}(j)_{mn}(\underline{\sigma}).$$

It is convenient to use the polar angle φ in the plane $\sigma_3 = 0$ of variables σ_1, σ_2 in $\mathfrak{su}(2) \simeq \mathbb{R}^3$

$$\mathrm{tg}\,\varphi = \frac{\sigma_2}{\sigma_1}\cdot$$

Instead of cylindrical variables σ_3, $\varrho = \sqrt{{\sigma_1}^2 + {\sigma_2}^2}$, φ in the Lie co-algebra $(\mathfrak{su}(2))^* \simeq \mathbb{R}^3$, we shall use the modified, spherically-cylindrical coordinates

$$\sigma = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2}, \qquad \sigma_3, \qquad \varphi = \operatorname{arctg} \frac{\sigma_2}{\sigma_1}\cdot$$

They coincide with the canonical Darboux coordinates in $(\mathfrak{su}(2))^*$ as a Poisson manifold. Their Poisson brackets have the following form

$$\{\varphi, \sigma_3\} = 1, \qquad \{\sigma, \varphi\} = 0, \qquad \{\sigma, \sigma_3\} = 0.$$

In particular, $\sigma^2 = \underline{\sigma} \cdot \underline{\sigma}$ is the basic Casimir invariant. Its value surfaces $\sigma = \mathrm{const}$ are canonically two-dimensional symplectic manifolds. The exceptional "s-state" value $\sigma = 0$ is the singular co-adjoint orbit of dimension zero, just the origin of coordinates.

In these coordinates the following holds

$$\{\sigma_3, f(\underline{\sigma})\} = \frac{\partial}{\partial \varphi} f(\sigma, \sigma_3, \varphi)$$

therefore, the system (76)–(78) is solved as follows

$$\begin{aligned} \widehat{\varepsilon}(j)_{mn} &= N(j)\delta\left(\sigma^2 - \hbar^2 j(j+1)\right)\delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right) \\ &= \frac{N(j)}{2\hbar\sqrt{j(j+1)}}\delta\left(\sigma - \hbar\sqrt{j(j+1)}\right)\delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right)\mathrm{e}\left(\mathrm{i}(m-n)\varphi\right) \end{aligned} \tag{79}$$

where $N(j)$ is a j-dependent normalization factor. It is defined by the demand that

$$\varepsilon(j)_{mn}|_{\overline{\varkappa}=0} = (2j+1)\delta_{mn}. \tag{80}$$

As already mentioned above, in quasiclassical situations the quantum-mysterious $j(j+1)$ is not very essential and may be replaced by $(j+1/2)^2$ or just by j^2.

Let us observe an interesting analogy with some formulas from the Weyl-Wigner-Moyal formalism for quantum systems with classical analogy. The "basis" of the wave function space consisting of non-normalizable, or rather "Dirac-δ-normalized", states $|\underline{\pi}\rangle$ of fixed linear momentum $\underline{\pi}$ implies the following H^+-algebra "basis" in the space of phase-space functions (including the Moyal quasi-probability distributions)

$$\varrho_{\underline{\pi}_1,\underline{\pi}_2}\left(\overline{q}, \underline{p}\right) = \delta\left(p - \frac{1}{2}\left(\underline{\pi}_1 + \underline{\pi}_2\right)\right)\mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left(\underline{\pi}_1 - \underline{\pi}_2\right)\overline{q}\right).$$

There is an obvious analogy with the term

$$\delta\left(\sigma_3 - \frac{1}{2}\left(\mu_1 + \mu_2\right)\right) \mathrm{e}\left(\frac{\mathrm{i}}{\hbar}\left(\mu_1 - \mu_2\right)\varphi\right)$$

in (79), if we put $\mu_1 = \hbar m$, $\mu_2 = \hbar n$. This analogy is not accidental. However, there is no place here for more details.

Equation (79) and normalization condition (80) imply finally that

$$\widehat{\varepsilon}(j)_{mn} \approx 16\pi^2\hbar^4\left(j + \frac{1}{2}\right)^2 \delta\left(\sigma^2 - \hbar^2\left(j + \frac{1}{2}\right)^2\right) \delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right) \mathrm{e}\left(\mathrm{i}(m-n)\varphi\right)$$

therefore

$$\widehat{\mathcal{D}}(j)_{mn} \approx 8\pi^2\hbar^4\left(j + \frac{1}{2}\right) \delta\left(\sigma^2 - \hbar^2\left(j + \frac{1}{2}\right)^2\right) \delta\left(\sigma_3 - \hbar\frac{m+n}{2}\right) \mathrm{e}\left(\mathrm{i}(m-n)\varphi\right).$$

In the above formulas we mean the same as previously asymptotic "indifference" concerning $j(j+1)$, $(j+1/2)^2$, j^2 for large values of j.

Warning: It must be stressed that the above functions are not literally meant asymptotic expressions for $\varepsilon(j)_{mn}$, $\mathcal{D}(j)_{mn}$ for "large" values of j. They may be used instead of rigorous $\varepsilon(j)_{mn}$, $\mathcal{D}(j)_{mn}$ when superposing them with coefficients "slowly varying" as functions of j. And the very important point: The discrete quantum number j may be then formally admitted to be a continuous variable and the summation with "slowly-varying" coefficients may be approximated by integration. In this way the compactness of $\mathrm{SU}(2)$ is seemingly "lost". This procedure is well known in practical applications of Fourier analysis, where often Fourier series may be approximated by Fourier transforms.

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