



GEOMETRICAL FRAMEWORK OF QUANTIZATION PROBLEM

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Abstract. The basic elements of the geometric approach to a consistent quantization formalism are summarized, with reference to the methods of the old quantum mechanics and the induced representations theory of Lie groups. A possible relationship between quantization and phase-space discretization is briefly discussed.

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1. Introduction

The notion of quantization has appeared at the beginning of the last century in the theory of thermal radiation, when M. Planck has formulated the hypothesis of the energy quanta: $\epsilon = h\nu$, $h = 6.626 \times 10^{-34}$ J·s [21]. The existence of h was considered in statistical mechanics as evidence for a granular structure of the $2n$ -dimensional phase-space, composed of elementary cells (“quantum states”) of volume h^n . For the integrable systems with multiple periodic motions, such as the hydrogen atom, this structure was provided by the quantization rules of the old quantum mechanics. Relativistic effects have also been included, as a correction to Balmer’s formula, due to the variation of mass with velocity, was introduced by Bohr [3], and the relativistic Kepler problem was quantized by Sommerfeld, applying integrality constraints to the action invariants [25].

In the algebraic (Dirac) approach to quantum mechanics, the observables are represented by elements of the set $\mathcal{F}(M)$ of the smooth real functions over the classical (momentum) phase space (M, ω) , $M = T^*Q$, ($Q = \mathbb{R}^3$), with ω the globally defined symplectic form. Let X_f be the vector field determined by $i_{X_f}\omega = df$, and L_X the Lie derivative with respect to X . As $\mathcal{F}(M)$ becomes a Lie algebra with respect to the Poisson bracket $\{\cdot, \cdot\}$, $\{f, g\} = \langle df, X_g \rangle = \omega(X_f, X_g) = -L_{X_f}g$, $f, g \in \mathcal{F}(M)$, the full quantization of M was defined as a \mathbb{R} -linear map $f \mapsto \hat{f}$ from $\mathcal{F}(M)$ to a set $\mathcal{A}(\mathcal{H})$ of symmetric operators on the Hilbert space \mathcal{H} , having the following properties [1]

- 1) the map $\hat{\cdot} : \mathcal{F}(M) \mapsto \mathcal{A}(\mathcal{H})$ is injective
- 2) $[\hat{f}, \hat{g}] = i\hbar\{f, g\}^\wedge$, $f, g \in \mathcal{F}(M)$
- 3) $\hat{1} = I$, for $f = 1$, constant on M , and I the identity operator on \mathcal{H}
- 4) \hat{q}_k, \hat{p}_k , $k = 1, 2, 3$ act irreducibly on \mathcal{H} .

It is presumed that once $\hat{\cdot}$ and \mathcal{H} are found, the quantum dynamics with respect to the classical time [4,6] is given by the Schrödinger equation, and the scalar product in \mathcal{H} has the statistical interpretation of probability amplitude.

In classical nonrelativistic statistical mechanics, a Brownian particle can be described by a time-dependent distribution function $f \geq 0$ defined on the phase-space $M = T^*\mathbb{R}^3$, evolving according to the Fokker-Planck equation. Though, at zero temperature both classical and quantum distributions arise as two different types of “functional coherent states” f_0, f_ψ for the classical Liouville equation [8]. These are solutions associated with “action waves” $n^{[S]}$, respectively “quantum waves”

$\psi = \sqrt{n} \exp(iS/\hbar)$, expressed in terms of only two functions of coordinates and time: the localization probability density in the coordinate space $n(\mathbf{q}, t)$, and the local “momentum potential” $S(\mathbf{q}, t)$. Moreover, these classes are related, as the action distributions turn into quantum distributions (Wigner functions) when the configuration space $Q = \mathbb{R}^3$ is discretized. However, by contrast to the action distributions, the Wigner distributions f_ψ remain the same as functionals of ψ during time evolution only for polynomial potentials of degree at most two. This limitation also appears in the canonical quantization, as the van Hove theorem [1, 10] states the incompatibility between the conditions 1), 2), 3), 4). Thus, it is possible to fulfill the first three conditions, obtaining a “prequantization”, but then the Heisenberg algebra $\mathfrak{H} \equiv \{q, p, 1\}$ is represented with infinite multiplicity. Also, if only the last three conditions are retained, then the map $\hat{\cdot}$ should be restricted to some subalgebra $\mathcal{F}_P \subset \mathcal{F}(M)$, containing \mathfrak{H} .

This work presents, following [14] as main reference, the geometrical framework in which the “action” and the “quantum” phase-space distributions are defined. The next two sections are based on the notes from the seminar “Classical Limit and Quantization Methods” given in 1989 at the Institute of Atomic Physics in Bucharest. Section 2 recalls the main concepts applied to the prequantization of Hamiltonian dynamical systems. The reduction of the prequantum Hilbert space $\mathcal{H} \simeq L^2(M, \omega^n)$, to the quantum Hilbert space $\mathcal{H}_P \simeq L^2(Q)$, is considered in Section 3. The transition from classical to quantum distributions by phase-space discretization is outlined in Section 4.

2. The Prequantization

2.1. Equivalence Classes of Line Bundles

Let M be a C^∞ differentiable manifold, separable and connected. A line bundle on M is a vector bundle

$$\begin{array}{c} \mathbb{C} \mapsto L \\ \downarrow \pi . \\ M \end{array}$$

The projection map π is smooth, and for any $p \in M$, $L_p = \pi^{-1}(p)$ (the fiber over p) is a one-dimensional vector space over \mathbb{C} .

On L , as manifold, we can introduce local coordinates. Let $\mathcal{U} = \{U_i, i \in I\}$, be an open covering of M , and $s_i : U_i \mapsto L$ smooth non-vanishing sections, such that the map $\sigma_i : \mathbb{C} \times U_i \mapsto \pi^{-1}(U_i)$, $\sigma_i(z, p) = z s_i(p)$, is a diffeomorphism. The set of pairs $\{(U_i, s_i), i \in I\}$ defines a local system for the line bundle L .

Let Γ_L be the space of smooth sections $s : M \mapsto L$. For the local system (U_i, s_i) any $s \in \Gamma_L$ can be written on U_i as $s = \psi_i s_i$, where $\psi_i \in \mathcal{F}_c(U_i)$ is a complex function on U_i . The collection $\{\psi_i\}_{i \in I}$ represents the local coordinates of s .

On $U_i \cap U_j$ the local system defines by the relation $s_i = c_{ij} s_j$ the transition functions $c_{ij} \in \mathcal{F}_c^*(U_i \cap U_j)$. These functions should satisfy the relationships

$$c_{ij} = c_{ji}^{-1}, \quad c_{ij} c_{jk} = c_{ik} \quad (1)$$

on $U_i \cap U_j$, respectively on $U_i \cap U_j \cap U_k$. If expressed in the form $c_{ij} = \exp(iq_{ij}/\hbar)$, ($\hbar = h/2\pi = 1/2\pi$, as we take $h = 1$), we can see that the new functions q_{ij} provide a constant with integer values, denoted by a_{ijk}

$$a_{ijk} = q_{ij} + q_{jk} - q_{ik} \in \mathbb{Z} \quad (2)$$

on any intersection $U_i \cap U_j \cap U_k \neq \emptyset$.

Two line bundles L^1 and L^2 on M are equivalent if there exists a diffeomorphism $\tau : L^1 \mapsto L^2$ such that for any $p \in M$, the map τ induces a linear isomorphism $L_p^1 \mapsto L_p^2$. The set of equivalence classes of line bundles on M is denoted $\mathcal{L}(M)$.

If c_{ij}^1, c_{ij}^2 are the transition functions for L^1 , respectively L^2 , then the two are equivalent iff there exists $\lambda_i = s_i^2/s_i^1, \lambda_i \in \mathcal{F}_c^*(U_i)$ (the set of nonvanishing complex functions on U_i), such that $c_{ij}^2 = \lambda_i c_{ij}^1 \lambda_j^{-1}$. Using this result it can be proved [14] that there exists a one-to-one mapping $\kappa : \mathcal{L}(M) \mapsto H^2(M, \mathbb{Z})$, which associates to any element $\ell = [L] \in \mathcal{L}(M)$ the Čech cohomology class $[a] \in H^2(M, \mathbb{Z})$ [12] of the function a_{ijk} associated with L . In particular, L is called trivial if equivalent to $\mathbb{C} \times M$ (Γ_L contains a nonvanishing global section).

2.2. Line Bundles with Connection

Let $\chi_c(M)$ be the Lie algebra of complex fields on M , and L a line bundle on M . A connection in the line bundle $\pi : L \mapsto M$ is a linear map $\nabla : \chi_c(M) \mapsto \text{End}(\Gamma_L)$ such that

$$\nabla_{\Phi\xi} = \Phi\nabla_{\xi} \quad (3)$$

$$\nabla_{\xi}(\Phi s) = (L_{\xi}\Phi)s + \Phi\nabla_{\xi}s, \quad \Phi \in \mathcal{F}_c(M), \quad s \in \Gamma_L. \quad (4)$$

If $\{(U_i, s_i), i \in I\}$ is a local system for L , then ∇ is completely specified by its action on the sections $\{s_i\}_{i \in I}$

$$\nabla_{\xi}s_i = 2\pi i \alpha_i(\xi)s_i, \quad \xi \in \chi_c(M), \quad i \in I. \quad (5)$$

The condition (3) implies $\alpha_i(\Phi\xi) = \Phi\alpha_i(\xi)$, such that the collection of functions $\{\alpha_i(\xi) ; i \in I, \xi \in \chi_c(M)\}$ defines a family of complex one-forms $\{\alpha_i\}_{i \in I}$,

$i_\xi \alpha_i \equiv \langle \alpha_i, \xi \rangle = \alpha_i(\xi)$, associated with the connection ∇ . On $U_i \cap U_j$ we get

$$\alpha_i = \alpha_j + dq_{ij}$$

and conversely, any family of one-forms having this property specifies uniquely a connection ∇ . Such a family arises by the pull-back of an unique \mathbb{C}^* - invariant one-form $\alpha \in \Omega^1(L^*)$, called connection form. Here $\Omega^k(L^*)$ denotes the set of k -forms on the manifold $L^* = \{\cup_{p \in M} L_p^*; L_p^* = L_p - \{0\}\}$. The form α is globally defined on L^* , and $s_i^* \alpha = \alpha_i, i \in I$.

If $(L^1, \alpha^1), (L^2, \alpha^2)$, are line bundles on M with connection forms α^1, α^2 , then there exists a diffeomorphism $\tau : L^1 \mapsto L^2$ such that τ induces a linear isomorphism

$$L_p^1 \mapsto L_{\pi(\tau(L_p^1))}^2$$

and $\tau^* \alpha^2 = \alpha^1$.

One should note that any equivalence $(L, \alpha) \mapsto (L, \alpha^1)$ is specified by a function $\Phi \in \mathcal{F}_c^*(M)$, such that

$$\tau_\Phi^* \alpha^1 = \alpha^1 - \frac{1}{2\pi i} \frac{d\tilde{\Phi}}{\tilde{\Phi}} = \alpha, \quad \tilde{\Phi} = \pi^*(\Phi)$$

and $\tau_\Phi(x) = \Phi(\pi x)x, x \in L$. In particular, $\tau_\Phi : (L, \alpha) \mapsto (L, \alpha)$ is an equivalence iff Φ is a complex constant on M .

The family of one-forms $\{\alpha_i\}_{i \in I}$, associated to the connection ∇ determines an unique complex two-form ω^∇ on M , such that

$$d\omega^\nabla = 0, \quad \omega|_{U_i} = d\alpha_i, \quad \pi^* \omega = d\alpha. \quad (6)$$

Because

$$[\nabla_\xi, \nabla_\eta] - \nabla_{[\xi, \eta]} = 2\pi i \omega^\nabla(\xi, \eta), \quad \xi, \eta \in \chi_c(M)$$

and ω^∇ is called the curvature form of the connection ∇ . If it is real and nondegenerate, ω^∇ provides a symplectic structure on M .

2.3. Line Bundles with Connection and Hermitian Structure

A Hermitian structure on L is a function $(\cdot, \cdot) : L \times L \mapsto \mathbb{C}$ having the properties:

- i) (\cdot, \cdot) induces a structure of 1-dimensional Hilbert space on L_p , for all $p \in M$
- ii) $|\cdot|^2$ is a positive function on L^* , $|x|^2 \equiv (x, x), x \in L^*$.

Let γ be a smooth curve on M . The covariant derivative of the section $s \in \Gamma_L$ along γ is defined by

$$\frac{Ds}{Dt} = \nabla_{\dot{\gamma}(t)} s. \quad (7)$$

For any smooth curve γ on M , $\{\gamma_t; t \in (a, b)\}$, the covariant derivative defines a linear isomorphism $\tau_{t',t} : L_{\gamma_t} \mapsto L_{\gamma_{t'}}$, called a parallel transport by

$$\frac{Ds}{Dt} \Big|_{\gamma_t} = \frac{d}{dt'} \tau_{t,t'} s(\gamma_{t'}) \Big|_{t'=t}. \quad (8)$$

A section $r = \psi s_i$ is autoparallel along γ if $\nabla_{\dot{\gamma}} r = 0$, or

$$\psi(\gamma_t) = e^{-2\pi i \int_0^t \langle \alpha_i, \dot{\gamma} \rangle} \psi(\gamma_0).$$

If $\gamma = \partial\Sigma \subset M$ is closed, contractible on Σ to a point, then

$$\mathcal{Q}_\gamma = e^{-2\pi i \int_\Sigma \omega}$$

is the scalar function of the parallel transport. Applications to autoparallel sections for the constrained quantum dynamics are presented in [7].

The Hermitian form (\cdot, \cdot) is called ∇ -invariant if the parallel transport leaves invariant the inner product on the fiber

$$\frac{d}{dt} (\tau_{t,t'} s^1_{(\gamma_{t'})}, \tau_{t,t'} s^2_{(\gamma_{t'})}) \Big|_{t'=t} = 0 \quad (9)$$

or

$$\mathbb{L}_\xi(s^1, s^2) = (\nabla_\xi s^1, s^2) + (s^1, \nabla_\xi s^2). \quad (10)$$

When $s^1 = s^2 = s_i$ this reduces to

$$d \ln |s_i|^2 = 2\pi i (\alpha_i - \bar{\alpha}_i) \quad (11)$$

where $\bar{\alpha}_i$ is the complex conjugate form of α_i . Thus, $\alpha_i - \bar{\alpha}_i$ is a real one-form, exact on U_i , so that the curvature form

$$\omega|_{U_i} = d\alpha_i = d\bar{\alpha}_i$$

is real, and $\mathcal{Q}_\gamma \in U(1)$. Let $[\omega]_{\text{dR}} \in H^2_{\text{dR}}(M, \mathbb{R})$ be the de Rahm cohomology class of ω . In general, the isomorphism between $H^2_{\text{dR}}(M, \mathbb{R})$ and $H^2(M, \mathbb{R})$ associates to a real, closed two-form ω on M , expressed locally as

$$\omega|_{U_i} = d\alpha_i, \quad \alpha_i = \alpha_j + df_{ij}, \quad f_{ij} : M \mapsto \mathbb{R}$$

the class $[\omega] \equiv [a^\omega] \in H^2(M, \mathbb{R})$, where $a^\omega_{ijk} = f_{ij} + f_{jk} - f_{ik}$ is a real constant on $U_i \cap U_j \cap U_k$. However, if ω is the curvature of the connection ∇ on the line bundle

L with ∇ -invariant Hermitian structure, then a_{ijk}^ω is an integer, and ω specifies an integral cohomology class in $H^2(M, \mathbb{R})$.

Conversely, the problem is to what extent a closed, real two-form ω , satisfying the integrality condition, determines a Hermitian line bundle with a connection on M . If ω is integral, then in general a_{ijk}^ω are not integers, but we can find real constants $x_{ij} = -x_{ji}$ on $U_i \cap U_j \neq \emptyset$, such that

$$z_{ijk} = a_{ijk} + x_{ij} + x_{jk} - x_{ik}$$

are integers on $U_i \cap U_j \cap U_k \neq \emptyset$. This result allows to define a line bundle L on M with the transition functions

$$c_{ij} = \exp(2\pi i q_{ij}), \quad q_{ij} = f_{ij} + x_{ij}$$

on $U_i \cap U_j \neq \emptyset$. Because

$$\alpha_i = \alpha_j + df_{ij} = \alpha_j + dq_{ij} = \alpha_j + \frac{1}{2\pi i} \frac{dc_{ij}}{c_{ij}}$$

with α_i, α_j real, on L there exists a connection ∇ defined by the family of one-forms $\{\alpha_i\}_{i \in I}$, and a ∇ -invariant Hermitian structure.

In this formulation, the one-forms α_i are defined by ω up to a total differential $d\Phi_i$. If $\alpha'_i = \alpha_i + d\Phi_i$, then $f'_{ij} = f_{ij} + \Phi_i - \Phi_j$, and

$$c'_{ij} = \lambda_i c_{ij} \lambda_j^{-1}, \quad \lambda_i = e^{2\pi i \Phi_i}$$

define a Hermitian line bundle with connection (L', ∇') , equivalent to (L, ∇) . In specifying this equivalence class there is still an arbitrariness due to the way of choosing the constants x_{ij} . Thus, the integrality condition allows one to replace x_{ij} by new real constants $x'_{ij} = x_{ij} + y_{ij}$, where $y_{ij} + y_{jk} + y_{ki} \in \mathbb{Z}$, and $y_{ij} = -y_{ji}$. The line bundle L' , specified by the transition functions

$$c'_{ij} = \exp 2\pi i (f_{ij} + x'_{ij}) = e^{2\pi i y_{ij}} c_{ij}$$

is equivalent to L if only the functions y_{ij} have the form $y_{ij} = c_i - c_j$. Because $y_{ij} + y_{jk} + y_{ki} \neq 0$, y_{ij} does not specify a cocycle in $C^1(M, \mathbb{R})$, but its exponential determines a cocycle in $C^1(M, U(1))$. The bundles L' and L are equivalent only if this cocycle is coboundary, such that the set of equivalence classes of the Hermitian line bundles whose connection has the same curvature form ω parameterized by $H^1(M, U(1))$. This set of equivalence classes is denoted by $\mathcal{L}_c(M, \omega)$, and the result presented above states the isomorphism $\mathcal{L}_c(M, \omega) \simeq H^1(M, U(1)) \simeq H^2(M, \mathbb{Z})$.

Let $\epsilon : H^2(M, \mathbb{Z}) \mapsto H^2(M, \mathbb{R})$ be the homomorphism induced by the injection $\epsilon : \mathbb{Z} \mapsto \mathbb{R}$, $\kappa : \mathcal{L}(M) \mapsto H^2(M, \mathbb{Z})$ the bijection introduced in Subsection 2.1, and $\sigma : \mathcal{L}_c \mapsto \mathcal{L}$ the mapping given by $\sigma[(L, \alpha)] = [L]$. In this case, the Weil integrality condition states that if ω is any real, closed two-form on M , then

- i) $\mathcal{L}_c(M, \omega) \neq \{\emptyset\}$ iff $[\omega] \in H^2(M, \mathbb{R})$ is integral
- ii) $\sigma\mathcal{L}_c(M, \omega) = \{[L] \in \mathcal{L}; \epsilon\kappa[L] = [\omega]\}$.

Applications to the calculus of the energy levels of the multidimensional Kepler problem are presented in [16–18].

2.4. The BWS Condition

Let (N, ω) be a reducible presymplectic manifold, and (M', ω') , with $M' = N/K$, the reduced space. Here K is a smooth distribution on N , with the tangent space

$$T_m K = \{x \in T_m N; i_x \omega_m = 0\}.$$

Proposition 1. *A sufficient condition to obtain a quantizable reduction (M', ω') of (N, ω) is*

$$\oint_{\gamma} \theta \in \mathbb{Z} \tag{12}$$

where θ is a global one-form such that $\omega = d\theta$, and γ is any closed curve contained in a leaf of K . If N is simply connected, then (12) is also necessary [30].

For the proof we take a contractible covering $\mathcal{U} = \{U_i, i \in I\}$ of M' , such that for any $i \in I$ there exists a section Σ_i in K over U_i and a diffeomorphism $\rho_i : U_i \mapsto \Sigma_i$. If $m_1, m_2 \in U_i \cap U_j$ are two points joined by the curve c , then $\rho_i(c)$ is a curve in Σ_i , and $\rho_j(c)$ is a curve in Σ_j . Moreover, $\rho_i(m_1)$ and $\rho_j(m_1)$ can be joined by a curve γ_1 in the leaf of K through m_1 , respectively $\rho_i(m_2)$ and $\rho_j(m_2)$ can be joined by a curve γ_2 in the leaf of K through m_2 . Let S be the surface bounded by $\rho_i(c), \rho_j(c), \gamma_1, \gamma_2$, so that $\pi(S) = c$. Because $S \in \ker(\omega)$

$$\begin{aligned} \int_S \omega &= \oint_{\partial S} \theta = \int_{\rho_i(m_1)}^{\rho_i(m_2)} \theta - \int_{\rho_j(m_1)}^{\rho_j(m_2)} \theta + \int_{\rho_i(m_2)}^{\rho_j(m_2)} \theta - \int_{\rho_i(m_1)}^{\rho_j(m_1)} \theta \\ &= f_{ji}(m_2) - f_{ji}(m_1) + \int_{m_1}^{m_2} (\rho_i^* \theta - \rho_j^* \theta) = 0 \end{aligned}$$

which yields

$$\rho_i^* \theta - \rho_j^* \theta \equiv \theta_i - \theta_j = df_{ij}.$$

The one-forms $\theta_i = \rho_i^* \theta$, $\theta_j = \rho_j^* \theta$ on $U_i \cap U_j \neq \{\emptyset\}$ are related to the symplectic form $\omega' = d\theta_i = d\theta_j$. The functions $f_{ij} = -f_{ji}$ on $U_i \cap U_j$ can be defined by integration along an arbitrary curve contained in the leaf of K over m ,

$$f_{ij}(m) = \int_{\rho_i(m)}^{\rho_j(m)} \theta.$$

Thus, $f_{ij} + f_{jk} + f_{ik} \in \mathbb{Z}$ as an integral (12) of the one-form θ along a closed curve in the leaf of K through m , proving that the class $[\omega'] \in H^2(M', \mathbb{R})$ is integral. When $N = h^{-1}(E) \subset M$ is the constant energy surface of a classical system on (M, ω) with Hamiltonian h , then (12) is similar to the Bohr-Wilson-Sommerfeld (BWS) condition from the old quantum mechanics.

2.5. The Prequantum Hilbert Space and Operators Related to Observables

Let (M, ω) be a quantizable classical phase-space, in the sense that $[\omega] \in H^2(M, \mathbb{R})$ is integral. In this case, on M we can define a Hermitian line bundle with connection (L, α) . The natural volume element on M is $\epsilon_\omega = \omega^n$, and for $\omega = \sum_{k=1}^n dq_k \wedge dp_k$

$$\epsilon_\omega = dq_1 \wedge \dots \wedge dq_n \wedge dp_1 \wedge \dots \wedge dp_n.$$

If (\cdot, \cdot) denotes the ∇ -invariant Hermitian form on L , the prequantum Hilbert space \mathcal{H} is defined as the space of all sections $s \in \Gamma_L(M)$ for which

$$\int_M \epsilon_\omega (s, s)$$

exists and is finite. The inner product in \mathcal{H} is

$$\langle s_1, s_2 \rangle \equiv \int_M \epsilon_\omega (s_1, s_2), \quad s_1, s_2 \in \mathcal{H}. \quad (13)$$

Let $\mathfrak{e}(L)$ be the Lie algebra of the \mathbb{C}^* -invariant, real fields on L^* . By the existence of the connection form α and the projection $\pi_* : TL \mapsto TM$, there exists also a linear isomorphism

$$\mathfrak{e}(L) \mapsto \mathcal{F}_c(M) \times \chi(M) \quad (14)$$

which associates to $\eta \in \mathfrak{e}(L)$ a function $\Phi \in \mathcal{F}_c(M)$ and a vector $\xi \in \chi(M)$ such that

$$\pi^* \Phi = -\langle \alpha, \eta \rangle, \quad \xi = \pi_* \eta. \quad (15)$$

Conversely, any function $\Phi \in \mathcal{F}_c(M)$ specifies a unique field $\eta_\Phi \in \mathfrak{e}(L)$, $\eta_\Phi \in \ker(\pi_*)$ by the relation

$$\langle \alpha, \eta_\Phi \rangle = -\pi^* \Phi$$

and for any $\xi \in \chi(M)$ there exists a unique field $\hat{\xi} \in \mathfrak{e}(L)$, $\hat{\xi}_x \in \ker(\alpha_x)$, $x \in L^*$ [14]

$$\pi_* \hat{\xi} = \xi.$$

Proposition 2. *Therefore $\mathfrak{e}(L)$ is parameterized by $\mathcal{F}_c(M) \times \chi(M)$, such that for any $\Phi \in \mathcal{F}_c(M)$ and $\xi \in \chi(M)$*

$$\eta_{(\Phi, \xi)} = \eta_\Phi + \hat{\xi} \in \mathfrak{e}(L) \quad (16)$$

$$[\eta_{(\Phi_1, \xi_1)}, \eta_{(\Phi_2, \xi_2)}] = \eta_{(\xi_1 \Phi_2 - \xi_2 \Phi_1 + \omega(\xi_1, \xi_2), [\xi_1, \xi_2])} \quad (17)$$

To prove this let us denote by $\eta_{(\Phi, \xi)}$ the commutator $[\eta_{(\Phi_1, \xi_1)}, \eta_{(\Phi_2, \xi_2)}]$. Then

$$\pi_* [\eta_{(\Phi_1, \xi_1)}, \eta_{(\Phi_2, \xi_2)}] = [\xi_1, \xi_2] \equiv \xi \quad (18)$$

and

$$\langle \alpha, [\eta_{(\Phi_1, \xi_1)}, \eta_{(\Phi_2, \xi_2)}] \rangle \equiv -\pi^* \Phi. \quad (19)$$

With the identity

$$\langle \alpha, [\eta_1, \eta_2] \rangle = L_{\eta_1} \langle \alpha, \eta_2 \rangle - L_{\eta_2} \langle \alpha, \eta_1 \rangle - d\alpha(\eta_1, \eta_2) \quad (20)$$

the equality (19) becomes

$$\Phi = L_{\xi_1} \Phi_2 - L_{\xi_2} \Phi_1 + \omega(\xi_1, \xi_2). \quad (21)$$

The elements of the algebra $\mathfrak{e}(L)$ act on functions on L^* , but we can also find a representation of $\mathfrak{e}(L)$ in the space of the sections Γ_L . Thus, we can define an $\mathfrak{e}(L)$ -isomorphism $\tilde{\cdot} : \Gamma_L(M) \mapsto \mathcal{F}(L^*)$ associating to any section $s \in \Gamma_L(M)$ a function $\tilde{s} \in \mathcal{F}(L^*)$,

$$\tilde{s}(x) = \frac{s(\pi x)}{x}, \quad x \in L^*.$$

Proposition 3. *If $(\Phi, \xi) \in \mathcal{F}_c(M) \times \chi(M)$ then $\eta_{(\Phi, \xi)} \tilde{s} = \tilde{t}$, where*

$$t = (\nabla_\xi + 2\pi i \Phi) s \equiv \hat{\eta} s.$$

With respect to a local system $\{(U_i, s_i); i \in I\}$ on L , the elements of Γ_L are represented by functions $\psi_i : U_i \mapsto \mathbb{C}$, provided by

$$s|_{U_i} = \psi_i s_i, \quad s \in \Gamma_L.$$

For this local trivialization, the operator $\hat{\eta}_{(\Phi, \xi)} = \nabla_\xi + 2\pi i \Phi$ determines an operator $\hat{\eta}_i(\Phi, \xi)$ on $\mathcal{F}_c(U_i)$

$$\hat{\eta}_i(\Phi, \xi) = L_\xi + 2\pi i(\langle \alpha_i, \xi \rangle + \Phi).$$

The tangent fields to L which preserve the connection and the Hermitian structure form a subalgebra in $\mathfrak{e}(L)$, denoted $\mathfrak{e}(L, \alpha)$. It can be proved [14] that if $\eta_{(\Phi, \xi)} \in \mathfrak{e}(L)$, then

$$L_{\eta_{(\Phi, \xi)}} | \cdot |^2 = 0 \quad \text{iff} \quad \Phi \in \mathcal{F}(M) \quad (22)$$

and

$$L_{\eta_{(\Phi, \xi)}} \alpha = \pi^*(i_\xi \omega - d\Phi). \quad (23)$$

This shows that

$$\eta_{(\Phi, \xi)} \in \mathfrak{e}(L, \alpha) \quad \text{iff} \quad \Phi \in \mathcal{F}(M) \quad \text{and} \quad i_\xi \omega = d\Phi.$$

Therefore, the mapping

$$\delta : \mathcal{F}(M) \mapsto \mathfrak{e}(L, \alpha), \quad \delta(\Phi) = \eta_{(\Phi, \xi_\Phi)}$$

(or $\mathcal{F}(M) \mapsto \mathfrak{en}\mathfrak{d}(\Gamma_L)$), called map of pneumatization, is an isomorphism of Lie algebras.

These results indicate that we can obtain a representation of the Lie algebra of the observables, $\mathcal{F}(M)$, in the prequantum Hilbert space \mathcal{H} . In this representation each function Φ has an associated operator

$$\hat{\eta}_{(\Phi, \xi_\Phi)} = \nabla_{\xi_\Phi} + 2\pi i \Phi$$

on Γ_L , or on the space of the local representatives ψ_i of the sections

$$\hat{\eta}_{i(\Phi, \xi_\Phi)} = L_{\xi_\Phi} + 2\pi i(\langle \alpha_i, \xi_\Phi \rangle + \Phi).$$

Thus, defining the local operator associated with the observable f as

$$\hat{f} = \frac{1}{2\pi i} \hat{\eta}_{(f, \xi_f)} \equiv \frac{1}{2\pi i} L_{\xi_f} + \langle \alpha_i, \xi_f \rangle + f \quad (24)$$

one obtains a map which satisfies the conditions 1,2,3 stated in the Section 1, discussed in detail in [6]. In particular, if $M = \mathbb{R}^2$, $\alpha_i = -pdq$, then $\xi_p = \partial_q$, $\xi_q = -\partial_p$, and $\hat{p} = -i\hbar\partial_q$, $\hat{q} = q + i\hbar\partial_p$.

2.6. The Prequantization of Classical Dynamical Systems

The classical dynamical systems defined on the phase-space (M, ω) are subgroups of $\mathcal{D}(M)$, the group of diffeomorphisms on M . The symplectic diffeomorphisms form a subgroup denoted $\mathcal{D}(M, \omega)$, of diffeomorphisms which act by canonical transformations

$$\mathcal{D}(M, \omega) = \{\rho \in \mathcal{D}(M); \rho^* \omega = \omega\}.$$

This subgroup contains $\text{Ham}(M, \omega)$, the subgroup of all Hamiltonian diffeomorphisms, and if M is simply connected, then $\mathcal{D}(M, \omega) = \text{Ham}(M, \omega)$.

The phase-space (M, ω) has also an associated set of equivalence classes of Hermitian line bundles with connection, $\mathcal{L}_c(M, \omega)$. The group symplectic diffeomorphisms $\mathcal{D}(M, \omega)$ acts on $\mathcal{L}_c(M, \omega)$, but prequantum representations in a class $\ell \in \mathcal{L}_c(M, \omega)$, can be obtained only for the elements of the stability group $\mathcal{D}_\ell(M, \omega)$ of ℓ with respect to the action of $\mathcal{D}(M, \omega)$

$$\mathcal{D}_\ell(M, \omega) = \{\rho \in \mathcal{D}(M, \omega); \rho_{\mathcal{L}}^* \ell = \ell, \ell \in \mathcal{L}_c(M, \omega)\}.$$

Thus, if $[L] = \ell$ and $\rho \in \mathcal{D}_\ell(M, \omega)$, then $\rho_{\mathcal{L}}^* L$ and L are equivalent, and there exists an equivalence of line bundles with connection $\epsilon : \rho_{\mathcal{L}}^* L \mapsto L$, uniquely specified up to a phase factor.

In general, if G is a group acting on (M, ω) by canonical transformations, there are operators $\hat{g} : \mathcal{H} \mapsto \mathcal{H}$ which define a projective representation of G in \mathcal{H} , such that for $g_1, g_2 \in G$

$$g_1 \hat{g}_2 = \tau_{12} \hat{g}_1 \hat{g}_2, \quad |\tau_{12}| = 1. \quad (25)$$

In the prequantum Hilbert space $\mathcal{H} \subset \Gamma_L$, $[L] = \ell$, the operator associated to $\rho \in \mathcal{D}_\ell(M, \omega)$ is defined up to a phase-factor by the equality

$$(\hat{\rho}^{-1} s)_{(p)} = \epsilon(\rho^* s_{(p)}). \quad (26)$$

If $\rho_h(t) \in \text{Ham}(M, \omega)$ is generated by the Hamiltonian h , then $\rho^* L = L$ and the operator $\hat{\rho}_t : \Gamma_L \mapsto \Gamma_L$ will be defined by

$$\tilde{\rho}_\eta(t) \circ \hat{\rho}_t^{-1} s = s \circ \rho_h(t) \quad (27)$$

where $\tilde{\rho}_\eta(t)$ is the group of one-parameter diffeomorphisms on L^* generated by $\eta_{(h, X_h)} \in \mathfrak{e}(L, \alpha)$.

Theorem 4. *The field $\eta_{(h, X_h)}$ is globally integrable on L^* iff X_h is globally integrable on M , and the diagram*

$$\begin{array}{ccc} \tilde{\rho}_\eta(t) : L^* & \mapsto & L^* \\ \pi \downarrow & & \pi \downarrow \\ \rho_t : M & \mapsto & M \end{array}$$

commutes.

To obtain explicitly the operator $\hat{\rho}_t$, we can write (27) in local coordinates. Let $\{(U_i, s_i); i \in I\}$ be a local system, and σ the diffeomorphism

$$\sigma : \mathbb{C} \times U_i \mapsto \pi^{-1}(U_i), \quad \sigma(z, p) = z s_i(p).$$

The functions $\tilde{s}(x) \equiv s(\pi x)/x$ on L^* associated to the sections $s \in \Gamma_L(U_i)$ are represented locally by the functions \tilde{s}^b on $\mathbb{C} \times U_i$

$$\tilde{s}^b(z, p) = \tilde{s}(zs_i(p)) = \frac{1}{z}\tilde{s}(s_i(p)) = \frac{1}{z}\psi(p).$$

Also, the connection form α and the field $\eta_{(\Phi, \xi)}$ have the local expressions

$$\alpha^b = \alpha_i + \frac{1}{2\pi i} \frac{dz}{z}, \quad \alpha_i = s_i^* \alpha \quad (28)$$

$$\eta^b = \xi^b - 2\pi i \Phi z \partial_z \quad (29)$$

where $\xi^b = \xi - 2\pi i (\langle \alpha_i, \xi \rangle z \partial_z - \langle \bar{\alpha}_i, \xi \rangle \bar{z} \partial_{\bar{z}}) \in \ker(\alpha^b)$. The flow of η^b determines the time-evolution of the functions \tilde{s}_i^b by the equation

$$\frac{d\tilde{s}_i^b}{dt} = \eta^b \tilde{s}_i^b. \quad (30)$$

This provides the dependence on time of the coordinate z and of the point p in terms of the local expression, denoted $\tilde{\rho}_{\eta^b}^b(t)$

$$\tilde{\rho}_{\eta^b}^b(t)(z_0, p_0) = (z_0 e^{-2\pi i \int_0^t dt' (\langle \alpha_i, \xi \rangle + \Phi)}, \rho_t(p_0)) \quad (31)$$

of the flow $\tilde{\rho}_\eta(t)$.

The operator $\hat{\rho}(t)$ defines an operator \hat{U}_t acting on the complex functions $\psi(p) = s(p)/s_i(p)$, $p \in U_i$ representing the sections $s \in \Gamma_L(U_i)$, by

$$\hat{U}_t \psi = \frac{\hat{\rho}_t s}{s_i}. \quad (32)$$

Explicitly, this is obtained from (27) in local form

$$\tilde{\rho}_\eta^b(t)(\hat{U}_t^{-1} \psi, p) = (\psi(\rho_t p), \rho_t(p)) \quad (33)$$

where the action of $\tilde{\rho}_\eta^b$ is given by (31)

$$\tilde{\rho}_\eta^b(t)(\hat{U}_t^{-1} \psi, p) = (e^{-2\pi i \int_0^t dt' (\langle \alpha_i, \xi \rangle + \Phi)} (\hat{U}_t^{-1} \psi)_p, \rho_t(p)). \quad (34)$$

The result

$$(\hat{U}_t^{-1} \psi)_p = e^{2\pi i \int_0^t dt' (\langle \alpha_i, \xi \rangle + \Phi)} \psi(\rho_t(p)) \quad (35)$$

agrees with the expression derived in the previous subsection for the local operator

$$\hat{\Phi} = \frac{1}{2\pi i} L_{\xi_\Phi} + \langle \alpha_i, \xi_\Phi \rangle + \Phi$$

because

$$i\hbar \frac{d}{dt} (\hat{U}_t \psi) = \hat{\Phi} \hat{U}_t \psi. \quad (36)$$

2.7. Applications to Elementary Systems

Let G be a simply connected Lie group with the Lie algebra \mathfrak{g} , and \mathfrak{g}^* is the dual space of \mathfrak{g} . For $f \in \mathfrak{g}^*$ one can define on G a right (R_g) -invariant one-form θ_f , and a closed two-form $\omega_f = d\theta_f$ defined by

$$\omega_f(x, y)|_e = \langle f, [x, y] \rangle, \quad x, y \in \mathfrak{g}.$$

The leaf K_f determined by the kernel of ω_f on G has as tangent space at the identity the space

$$T_e K_f = \{x \in \mathfrak{g}; \langle f, [x, y] \rangle = 0 \text{ for any } y \in \mathfrak{g}\}$$

namely the Lie algebra \mathfrak{g}_f of the stability group G_f of f with respect to the coadjoint action of G . Thus, the leaf through e is the connected component $(G_f)_0$ of G_f , which contains e , and if closed $M' \cong G/K_f$ is covering space for the orbit $M_f = G/G_f$ of f in \mathfrak{g}^* .

Theorem 5. *Let (M', ω') , $M' = G/K_f$, be the reduced phase-space associated with the reducible presymplectic manifold (G, ω_f) . Then (M', ω') is quantizable iff f can be integrated to a character of $(G_f)_0$.*

Proof: Let us assume first that

$$\oint_{\gamma} \theta_f \in \mathbb{Z}$$

(the BWS condition) with $\gamma \subset (G_f)_0$. Thus, one can define

$$\chi_f(h) = e^{2\pi i \int_e^h \theta_f} \quad (37)$$

where the integral can be taken along any curve in $(G_f)_0$, which joins e to h . Because

$$\chi_f(h_1 h_2) = e^{2\pi i \int_e^{h_1 h_2} \theta_f} = e^{2\pi i \int_e^{h_2} \theta_f + 2\pi i \int_{h_2}^{h_1 h_2} \theta_f} \quad (38)$$

independently of the integration path, from the BWS condition, while

$$\int_{h_2}^{h_1 h_2} \theta_f = \int_e^{h_1} \theta_f$$

from the R_g -invariance of θ_f , one obtains

$$\chi_f(h_1 h_2) = \chi_f(h_1) \chi_f(h_2) \quad (39)$$

so that χ_f is a character for $(G_f)_0$. If $h = e^{tx}$, with $x \in \mathfrak{g}_f$, then

$$\chi_f(e^{tx}) = e^{2\pi i \langle f, x \rangle t}$$

from the R_g -invariance, an f appears as an infinitesimal character in the sense that

$$\frac{d}{dt}\chi_f(e^{tx})|_{t=0} = 2\pi i\langle f, x \rangle, \quad x \in \mathfrak{g}_f \quad (40)$$

$$\langle f, [x, y] \rangle = 0, \quad x, y \in \mathfrak{g}_f. \quad (41)$$

Conversely, applied to (37) the condition for integrability of the infinitesimal character f to a character of $(G_f)_0$ independently of the path (40), leads to the BWS condition. \blacksquare

Let us consider

$$G = \text{SU}(2) = \left\{ \begin{bmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{bmatrix}; |z_0|^2 + |z_1|^2 = 1 \right\} \simeq \mathbb{S}^3 \subset \mathbb{C}^2. \quad (42)$$

The algebra $\mathfrak{g} \equiv T_e G$ of G consists of the matrices

$$x_a = -\frac{i}{2} \begin{bmatrix} a_1 & a_2 - ia_3 \\ a_2 + ia_3 & -a_1 \end{bmatrix} = \sum_{i=1}^3 a_i E_i$$

with $(a_1, a_2, a_3) \equiv \mathbf{a} \in \mathbb{R}^3$ and $[E_i, E_j] = \epsilon_{ijk} E_k$.

The right (left) - invariant vector fields Y_a (Z_a) generated by $x_a \in \mathfrak{g}$ are

$$Y_a = -\frac{i}{2} [(a_1 z_0 - (a_2 - ia_3)\bar{z}_1)\partial_{z_0} + (a_1 z_1 + (a_2 - ia_3)\bar{z}_0)\partial_{z_1}] + c.c. \quad (43)$$

$$Z_a = -\frac{i}{2} [(a_1 z_0 + (a_2 + ia_3)z_1)\partial_{z_0} + (-a_1 z_1 + (a_2 - ia_3)z_0)\partial_{z_1}] + c.c. \quad (44)$$

and the right-invariant one-form θ_f associated to $f \in \mathfrak{g}^* \simeq \mathfrak{g}$ is

$$\theta_f = i[(f_1 \bar{z}_0 - (f_2 + if_3)z_1)dz_0 + (f_1 \bar{z}_1 + (f_2 + if_3)z_0)dz_1] + c.c. \quad (45)$$

where $c.c.$ means the complex conjugate of the previous term. In particular, for $\mathbf{f} \equiv (f_1, f_2, f_3) = (-l, 0, 0)$ we get

$$(G_f)_0 = G_f = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}; t \in \mathbb{R} \right\} \subset G$$

and

$$\theta_f = il \sum_{k=0}^1 (z_k d\bar{z}_k - \bar{z}_k dz_k), \quad \omega_f = 2il \sum_{k=0}^1 dz_k \wedge d\bar{z}_k. \quad (46)$$

Each $h_t \in G_f$ is generated by $x_a \in \mathfrak{g}$ with $\mathbf{a} = (-2, 0, 0)$

$$x_a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad h_t = e^{tx_a} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

and therefore

$$\langle \theta_f, Y_a \rangle = \langle f, x_a \rangle = \mathbf{f} \cdot \mathbf{a} = -a_1 l = 2l$$

determines the character χ_f

$$\chi_f(h_t) = e^{2\pi i \langle \theta_f, Y_a \rangle t} = e^{4\pi i l t}.$$

This character allows one to define a line bundle L' on $M' = G/G_f$ by factorizing the trivial bundle $G \times \mathbb{C}$ with respect to the equivalence relation “ \sim ”

$$(g, z) \sim (hg, \chi_f(h)z)$$

where $g \in G, h \in G_f$ and $z \in \mathbb{C}$. The sections in $\Gamma_{L'}$ are represented by functions $\psi : G \mapsto \mathbb{C}$ (sections in $G \times \mathbb{C}$) which satisfy the global relation

$$\psi(hg) = \chi_f(h)\psi(g) \quad (47)$$

or locally

$$Y_a \psi(g) = 2\pi i \mathbf{a} \cdot \mathbf{f} \psi(g). \quad (48)$$

Thus, the sections of $\Gamma_{L'}$ are represented in the coordinates (z_i, \bar{z}_i) by the functions $\psi : \mathbb{S}^3 \mapsto \mathbb{C}$ which satisfy

$$\sum_{k=0}^1 (z_k \partial_{z_k} - \bar{z}_k \partial_{\bar{z}_k}) \psi(z, \bar{z}) = 4\pi l \psi(z, \bar{z}). \quad (49)$$

The equivalence relation “ \sim ” is well defined, and M' is quantizable if

$$\chi_f(h_{2\pi}) = e^{8\pi^2 i l} = 1, \quad 4\pi l \in \mathbb{Z}$$

namely $l = n\hbar/2$ (here $\hbar = 1/2\pi$), with $n \in \mathbb{Z}$. A physical application to the intrinsic angular momentum (spin) is presented in [24].

The points of the phase-space M' correspond to equivalence classes in G defined by

$$[g] = \{hg; h \in G_f, g \in G\}.$$

Let $\text{pr} : G \mapsto M'$ be the projection $\text{pr}(g) = [g], g \in G$. A canonical action

$$g_1 [g] = [gg_1^{-1}]$$

of G on M' can be defined by the projection on M' of the action to the right of G on G (because the equivalence necessary for projection is obtained by the action to the left), and M' becomes a homogeneous phase-space for G . Locally, the action of G on M' arises by the projection of the left-invariant fields, $-Z_a$, on TM' , $\text{pr}_*(-Z_a) = X_a$, and because the algebra \mathfrak{g} is semisimple, there exists a lift λ of this action such that the diagram

$$\begin{array}{ccccccc}
 0 & \mapsto & \mathcal{F}(M') & \mapsto & \mathfrak{ham}(M') & \mapsto & 0 \\
 & & & & \lambda \swarrow \uparrow & & \\
 & & & & \mathfrak{g} & &
 \end{array}$$

commutes. Explicitly, for any $x_a \in \mathfrak{g}$ one can find $h_a : M' \mapsto \mathbb{C}$, $\text{pr}^* h_a = \langle \theta_f, Z_a \rangle$, representing the Hamiltonian of the vector field X_a

$$i_{X_a} \omega' = dh_a.$$

To get the time-evolution of the sections from the line bundle L' , associated with the dynamical system generated on M' by the Hamiltonian h_a , we project on L' the trajectory in $G \times \mathbb{C}$ of the dynamical system generated by the Hamiltonian $\langle \theta_f, Z_a \rangle$. This trajectory is given by (35) in which

$$\begin{aligned}
 \alpha_i &= \theta_f, & \xi &= -Z_a, & \Phi &= \langle \theta_f, Z_a \rangle \\
 (\hat{U}_{g_t} \psi)(g) &= \psi(gg_t), & g &\in G
 \end{aligned} \tag{50}$$

while the projection on $\Gamma_{L'}$ requires ψ constrained by (47)

$$\psi(hg) = \chi_f(h) \psi(g), \quad g \in G, \quad h \in G_f. \tag{51}$$

The result indicates that the prequantization of the phase space (M', ω') is equivalent to the derivation of the representations of the group G induced by the character χ_f of the subgroup G_f [15]. In general these representations are not irreducible (do not provide a quantization for (M', ω')), but imposing the condition on ψ to be holomorphic, we obtain irreducible representations. Thus, the holomorphy condition, by introducing a complex polarization, represents a way of restricting the prequantum Hilbert space.

The technique of the induced representations was successfully applied to quantize the relativistic free particle or the liquid drop. In both cases the classical configuration space is the orbit of a group H in a linear space V , and the quantization consists in finding induced representations for the semidirect product $G = H \times V$. In the first case $H = O(3, 1)$ is the Lorentz group, $V = \mathbb{R}^{3,1}$ is the Minkowski space, and G is the Poincaré group, while in the second case $H = SL(3, \mathbb{R})$, $V = \mathbb{R}^4$, and $G = CM(3)$ [20, 22, 23]. The case of a massive free particle in the anti-de Sitter spacetime is considered in [2].

3. Elements of Quantization

3.1. Complex Polarizations

A complex polarization of the $2n$ -dimensional manifold (M, ω) is a complex distribution P having the following properties

- i) for any $m \in M$, $P_m \subset T_m^c M$ is a complex Lagrangian subspace
- ii) $D_m = P_m \cap \bar{P}_m \cap T_m M$ has a constant dimension
- iii) P is integrable, in the sense that for any $m \in M$ there exists a collection of functions $\{z_k \in \mathcal{F}_c(M), k = 1, \dots, n\}$, such that $\{\bar{X}_{z_k}, k = 1, \dots, n\}$ generate P_m .

Let us introduce the notation

$$\begin{aligned}\chi_c(U, P) &= \{X \in \chi_c(U); X_m \in P_m, m \in U \subset M\} \\ \mathcal{F}_c(U, P) &= \{f \in \mathcal{F}_c(U); \bar{X}f = 0, X \in \chi_c(U, P)\} \\ &= \{f \in \mathcal{F}_c(U); \bar{X}_f \in \chi_c(U, P)\}\end{aligned}$$

$$\mathcal{F}_c(U, P, 1) = \{f \in \mathcal{F}_c(U); \{f, g\} \in \mathcal{F}_c(U \cap V, P) \forall V \subset M, g \in \mathcal{F}_c(V, P)\}.$$

The set $\mathcal{F}_c(U, P, 1)$ consists of functions having the property that generate flows which preserve the polarization

$$\mathbb{L}_{X_f} \bar{P} \subset \bar{P} \Leftrightarrow f \in \mathcal{F}_c(U, P, 1). \quad (52)$$

When $f \in \mathcal{F}_c(U, P, 1)$ is real, the flow generated by X_f preserves both \bar{P} and ω . The polarization P is called admissible if on a neighborhood of any $m \in M$, there exists a symplectic potential β , which is adapted to P in the sense that

$$i_{\bar{X}} \beta = 0, \quad X \in \chi_c(M, P).$$

The polarization P is of Kähler type if $P_m \cap \bar{P}_m = \{0\}$ and $T_m M = P_m + \bar{P}_m$. In the Kähler case any $X \in T_m M$ can be written as $X = Z + \bar{Z}$, with $Z \in P_m$, and $T_m M$ carries a complex structure

$$J_m : T_m M \mapsto T_m M, \quad J_m X = iZ - i\bar{Z}$$

compatible with the symplectic form ω in the sense that $\omega(JX, JY) = \omega(X, Y)$.

Let (M, ω, J) be a Kähler manifold (Appendix [10]), and $\{z_k, k = 1, \dots, n\}$ are local complex coordinates such that

$$J\partial_{z_k} = i\partial_{z_k}, \quad J\partial_{\bar{z}_k} = -i\partial_{\bar{z}_k}. \quad (53)$$

On M can be introduced two polarizations: the holomorphic polarization P generated at any point by the vectors $\{\partial_{z_k}, k = 1, \dots, n\}$, and the antiholomorphic polarization \bar{P} generated by $\{\partial_{\bar{z}_k}, k = 1, \dots, n\}$. Thus, for any $U \subset M$

$$\mathcal{F}_c(U, P) = \{f : U \mapsto \mathbb{C}; f = \text{holomorphic}\}.$$

3.2. Phase-Space Quantization for Kähler Polarizations

Let (M, ω) be a symplectic manifold, $\{q_k, p_k, k = 1, \dots, n\}$ the local canonical coordinates for ω ($\omega = \sum_{k=1}^n dq_k \wedge dp_k$), and P a Kähler polarization on M locally generated by the vectors

$$\{\partial_{z_k}; z_k = (q_k - ip_k)/\sqrt{2}, k = 1, \dots, n\}.$$

The complex potential adapted to P is

$$\beta = i \sum_{k=1}^n \bar{z}_k dz_k \quad (54)$$

and $\omega = d\beta$. If (M, ω) is quantizable, then there exists a Hermitian line bundle with a connection (L, α) on M , and the associated prequantum Hilbert space \mathcal{H} . We can further define the space of the polarized sections

$$\Gamma_L(M, P) = \{s \in \Gamma_L(M); \nabla_{\bar{X}} s = 0, X \in \chi_c(M, P)\} \quad (55)$$

and the quantum Hilbert space $\mathcal{H}_P = \mathcal{H} \cap \Gamma_L(M, P)$. The space $\Gamma_L(M, P)$ is well defined because the local integrability condition for the sections $s \in \Gamma_L(M)$ is satisfied. Thus, if

$$\nabla_{\bar{X}} s = \nabla_{\bar{Y}} s = 0, \quad X, Y \in \chi_c(M, P)$$

then

$$\nabla_{[\bar{X}, \bar{Y}]} s = [\nabla_{\bar{X}}, \nabla_{\bar{Y}}] s - 2\pi i \omega(\bar{X}, \bar{Y}) s = 0 \quad (56)$$

because $\omega(\bar{X}, \bar{Y}) = 0$.

The Hilbert space \mathcal{H}_P is not invariant with respect to the action of the operator associated with any classical observable, and one should specify which classical observables provide operators on \mathcal{H}_P . If $f \in \mathcal{F}(M)$, then

$$\hat{f} = \frac{1}{2\pi i} \nabla_{X_f} + f$$

and the condition $\hat{f}\mathcal{H}_P \subset \mathcal{H}_P$ yields

$$\nabla_{\bar{X}} \hat{f} s = 0, \quad X \in \chi_c(M, P), \quad s \in \mathcal{H}_P. \quad (57)$$

However, because

$$\begin{aligned} 2\pi i \nabla_{\bar{X}} \hat{f} s &= \nabla_{\bar{X}} (\nabla_{X_f} + 2\pi i f) s = ([\nabla_{\bar{X}}, \nabla_{X_f}] + 2\pi i \mathbf{L}_{\bar{X}} f) s \\ &= (\nabla_{[\bar{X}, X_f]} + 2\pi i \omega(\bar{X}, X_f) + 2\pi i \mathbf{L}_{\bar{X}} f) s = \nabla_{[\bar{X}, X_f]} s \end{aligned}$$

the condition (57) is equivalent to $L_{X_f} \bar{P} \subset \bar{P}$. Thus, the observable f determines an operator on \mathcal{H}_P if only if $f \in \mathcal{F}_c(M, P, 1)$.

Let s be a section of $\Gamma_L(M, P)$ for which $s^* \alpha = \beta$, and r the unit section in $\Gamma_L(U)$ such that $r^* \alpha = \sum_{k=1}^n q_k dp_k$. Then, $s = e^\varphi r$, with φ specified (up to an additive constant φ_0) by

$$\nabla_X s = 2\pi i \langle \beta, X \rangle s = \nabla_X (e^\varphi r) = (L_X \varphi) s + 2\pi i \langle r^* \alpha, X \rangle s$$

with

$$\langle \beta, X \rangle = \langle r^* \alpha, X \rangle + \frac{1}{2\pi i} \langle d\varphi, X \rangle, \quad X \in \chi(M)$$

such that

$$\begin{aligned} d\varphi &= 2\pi i (\beta - r^* \alpha) = 2\pi i \sum_{k=1}^n \left[\frac{i}{2} (q_k + ip_k) d(q_k - ip_k) - q_k dp_k \right] \\ &= -\frac{\pi}{2} \sum_{k=1}^n d(q_k^2 + p_k^2 + 2iq_k p_k) = -\pi \sum_{k=1}^n d(|z_k|^2 + iq_k p_k). \end{aligned}$$

Considering $\varphi_0 = 0$, we get $\varphi = -\pi \sum_{k=1}^n (|z_k|^2 + ip_k q_k)$ and

$$(s, s) = e^{-2\pi \sum_{k=1}^n |z_k|^2}.$$

Thus, with respect to the local system specified by s , the elements of the space \mathcal{H}_P are sections of the form $\{\psi_p s_p, p \in U \subset M\}$, where ψ_p are holomorphic functions of $\{z_k, k = 1, \dots, n\}$, and the inner product (13) is given by the formula

$$\langle \psi_1, \psi_2 \rangle \sim \int \epsilon_\omega \bar{\psi}_1(z) \psi_2(z) e^{-2\pi \sum_{k=1}^n |z_k|^2}.$$

This Hilbert space coincides with the representation introduced by Fock, in his study of the quantum states of the harmonic oscillator. Though, its domain of applicability remains limited because the only observables quantizable in \mathcal{H}_P are polynomials in coordinates and momenta of degree at most two.

For the harmonic oscillator the classical Hamiltonian is $h = 2\pi\nu\bar{z}z$, and

$$X_h = 2\pi i \nu (z \partial_z - \bar{z} \partial_{\bar{z}}).$$

The operator \hat{h} in \mathcal{H} associated with h

$$\hat{h} = \frac{1}{2\pi i} L_{X_h} + \langle \beta, X_h \rangle + h = \nu (z \partial_z - \bar{z} \partial_{\bar{z}})$$

becomes $\hat{h}_P = \nu z \partial_z$ when restricted to \mathcal{H}_P . Its eigenvalues are $n\nu$, $n \in \mathbb{Z}$, showing that this approach yields the same result, physically incomplete, as the old quantum mechanics. To obtain the missing term $\nu/2$ the bundle of polarized sections should be extended by the line bundle of half-forms providing the measure for the inner product in \mathcal{H}_P , as indicated by examples in [19].

3.3. Real Polarizations and Asymptotic Solutions

A real polarization of the symplectic manifold (M, ω) is a foliation of M by Lagrangian (maximal isotropic) submanifolds. If $M = T^*Q$, and $\omega = \sum_{k=1}^n dq_k \wedge dp_k$ is the canonical two-form, then the vertical foliation P is a real polarization, and the leaves of P are the surfaces $q_k = \text{constant}$, $k = 1, \dots, n$.

Let P be a real polarization of the symplectic manifold (M, ω) . Then, on a neighborhood of any point m of M one can find canonical coordinates $(x, y) \equiv (x_k, y_k)_{k=1, \dots, n}$ such that the leaves of P coincide locally with the surfaces $x = \text{constant}$ or $y = \text{constant}$. The canonical coordinates having this property are called “adapted to P ”.

Let $\Lambda \subset M$ be a Lagrangian submanifold and $U \subset M$ such that $\omega|_U = d\theta$. Because $\omega|_\Lambda = 0$ then also $d\theta|_\Lambda = 0$, and locally there exists a function \wp on Λ , called “local phase function”, $\wp : \Lambda \mapsto \mathbb{R}$, such that $\theta|_\Lambda = -d\wp$.

If $M = T^*Q$, then Λ is transversal to the vertical polarization P if the restriction to Λ of the projection

$$\begin{array}{c} \Lambda \subset M \\ \pi \downarrow \\ Q \end{array}$$

is a diffeomorphism. In this case $S \in \mathcal{F}(W)$, $\pi(\Lambda) = W \subset Q$, $\pi^*S = \wp$, is called “generating function of the first kind” of Λ . Moreover, $\Lambda \cap T^*Q$ determines a one-form on W with the local coordinates

$$(p, q) \equiv \left(\frac{\partial S}{\partial q}, q \right).$$

Thus, a foliation of the phase-space $M = T^*Q$ by Lagrangian submanifolds corresponds to a family of generating functions $S(q, y)$, $y \equiv \{y_k, k = 1, \dots, n\}$, parameterized by the variables y . This type of foliation appears naturally in classical mechanics by the Hamilton-Jacobi equation,

$$h(\partial_q S, q) = \text{constant}$$

which represents the condition $h|_{\Lambda_S} = \text{constant}$ for the Lagrangian submanifold Λ_S of T^*Q generated by S , such that X_h is tangent to Λ .

If we denote $x \equiv \{x_k = \partial S / \partial y_k, k = 1, \dots, n\}$, then (x, y) is a local coordinate system on T^*Q adapted to the polarization Λ_S determined by $S(q, y)$. In this system h is a function only of y , and the equations of motion are

$$\dot{y} = 0, \quad \dot{x} = \text{constant}.$$

In particular, when $Q \simeq \mathbb{R}^n$ and $\Lambda_S \subset \mathbb{T}^n \equiv \mathbb{R}^n / \mathbb{Z}^n$ is part of an invariant torus, then y - x are the ‘‘action-angle’’ coordinates used to express the BWS conditions.

To quantize a classical system described by the Hamiltonian h it is convenient to find a Hilbert space \mathcal{H}_{Λ_S} associated to the polarization determined by the solution S of the Hamilton-Jacobi equation. Because in the case $\Lambda_S \subset \mathbb{T}^n$ the BWS conditions provide constraints on the phase-space reduction (to a point) and the stationary states, it is natural to select the sections from $\Gamma_L(M, \Lambda_S)$ by $\nabla_X r = 0$, where X is tangent to Λ_S and $r \in \Gamma_L(M)$. Let s be a section in $\Gamma_L(M)$ such that

$$s^* \alpha|_{U_i} = -dS$$

and $r = \psi s$ an arbitrary element in $\Gamma_L(M, \Lambda_S)$. The equation

$$\nabla_X r = (\mathbf{L}_X \psi) s - 2\pi i \langle dS, X \rangle \psi s = (\mathbf{L}_X \psi) s - 2\pi i (\mathbf{L}_X S) \psi s = 0 \quad (58)$$

has the solution $\ln \psi - 2\pi i S = f(y)$, where f is an arbitrary function of y , or

$$\psi(q, y) = a(y) e^{2\pi i S(q, y)}. \quad (59)$$

The sections from $\Gamma_L(M, \Lambda_S)$ can be transferred to the space $\Gamma_L(M, P)$, where P is the vertical polarization associated to the Schrödinger representation. The function obtained in [30]

$$\Psi(q) = A(q) e^{2\pi i S(q)} \quad (60)$$

can be interpreted as asymptotic solution of the Schrödinger equation in the WKB [27] approximation.

As h may contain any potential, in general S is multiple-valued, and Ψ should be defined by a sum $\sum_{\mu} A_{\mu} \exp(2\pi i S_{\mu})$ over different branches. Presuming that A_{μ} can have the branches $\pm |A_{\mu}|$, but Ψ remains single-valued, one obtains corrected BWS quantum conditions [13], which yield for the harmonic oscillator the exact energy levels $\nu(n + 1/2)$, $n = 0, 1, 2, \dots$.

4. Quantization and Discretization

The geometric elements presented in the previous sections also appear in the formalism of statistical mechanics. Let $f \geq 0$ be the distribution function [26] of a classical system composed of N identical, non-interacting particles, defined on the one-particle phase-space (M, ω) , normalized by

$$\int_M \epsilon_{\omega} f(\mathbf{q}, \mathbf{p}, t) = N, \quad N \geq 1. \quad (61)$$

For a one-particle Hamilton function $h : M \mapsto \mathbb{R}$, at zero temperature and without friction, $f(\mathbf{q}, \mathbf{p}, t)$ evolves according to the transport equation

$$\partial_t f + L_{X_h} f = 0. \quad (62)$$

Let us consider $M = T^*\mathbb{R}^3$, $\omega = \sum_{i=1}^3 dq_i \wedge dp_i$, $h(\mathbf{q}, \mathbf{p}) = \mathbf{p}^2/2m + V(\mathbf{q})$, and

$$f(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi)^3} \int d^3k e^{-i\mathbf{k}\cdot\mathbf{p}} \tilde{f}(\mathbf{q}, \mathbf{k}, t) \quad (63)$$

where $\tilde{f}(\mathbf{q}, \mathbf{k}, t)$ is the Fourier transform of $f(\mathbf{q}, \mathbf{p}, t)$. In this case, a particular class of exact solutions for (62) are the ‘‘action distributions’’ $f_0(\mathbf{q}, \mathbf{p}, t)$, provided by

$$\tilde{f}_0(\mathbf{q}, \mathbf{k}, t) = n(\mathbf{q}, t) e^{i\mathbf{k}\cdot\partial_{\mathbf{q}}S(\mathbf{q}, t)} \quad (64)$$

where $n(\mathbf{q}, t) \geq 0$ (the particle density) and $S(\mathbf{q}, t)$ satisfy the continuity, respectively the Hamilton-Jacobi equations [8].

The partial derivative $\mathbf{k} \cdot \partial_{\mathbf{q}}S(\mathbf{q}, t)$ in (64) is the limit of

$$\frac{k}{\ell} [S(\mathbf{q} + \frac{\ell}{2k}\mathbf{k}, t) - S(\mathbf{q} - \frac{\ell}{2k}\mathbf{k}, t)] \quad (65)$$

$k = |\mathbf{k}|$, when $\ell \rightarrow 0$. If a new parameter $\sigma = \ell/k$ is introduced, then

$$\tilde{f}_0(\mathbf{q}, \mathbf{k}, t) = \lim_{\sigma \rightarrow 0} \tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t) \quad (66)$$

where

$$\tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t) \equiv \psi^*(\mathbf{q} - \frac{\sigma\mathbf{k}}{2}, t) \psi(\mathbf{q} + \frac{\sigma\mathbf{k}}{2}, t) \quad (67)$$

and $\psi = \sqrt{n} \exp(iS/\sigma)$. However, when $k \rightarrow 0$

$$S(\mathbf{q} \pm \frac{\sigma_0}{2}\mathbf{k}, t) = S(\mathbf{q}, t) \pm \frac{\sigma_0}{2}\mathbf{k} \cdot \partial_{\mathbf{q}}S(\mathbf{q}, t) + \frac{\sigma_0^2}{8}(\mathbf{k} \cdot \partial_{\mathbf{q}})^2 S(\mathbf{q}, t) \pm \dots$$

and if the terms containing $(\sigma_0 k)^m$, $m \geq 3$ are neglected, then

$$\mathbf{k} \cdot \partial_{\mathbf{q}}S(\mathbf{q}, t) = \frac{1}{\sigma_0} [S(\mathbf{q} + \frac{\sigma_0}{2}\mathbf{k}, t) - S(\mathbf{q} - \frac{\sigma_0}{2}\mathbf{k}, t)]$$

for any dimensional constant σ_0 . Thus, within a suitable domain of \mathbf{k} , we may also consider in (67) σ as a finite constant, (e.g. $\sigma = \hbar$), such that f_ψ defined by (63)

$$f_\psi(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi)^3} \int d^3k e^{-i\mathbf{k}\cdot\mathbf{p}} \tilde{f}_\psi(\mathbf{q}, \mathbf{k}, t) \quad (68)$$

is the Wigner transform [11] of $\psi(\mathbf{q}, t)$. In this case, the normalization condition (61) takes the form

$$\int d^3q d^3p f_\psi(\mathbf{q}, \mathbf{p}, t) = \int d^3q |\psi(\mathbf{q}, t)|^2 = \langle \psi | \psi \rangle = N \quad (69)$$

and the phase-space overlap between two distributions f_{ψ_1} , f_{ψ_2} , (resembling the inner product (13)), is [9]

$$\langle f_{\psi_1} f_{\psi_2} \rangle \equiv \int d^3q d^3p f_{\psi_1} f_{\psi_2} = \frac{|\langle \psi_1 | \psi_2 \rangle|^2}{(2\pi\sigma)^3} \quad (70)$$

where

$$\langle \psi_1 | \psi_2 \rangle \equiv \int d^3q \psi_1^*(\mathbf{q}, t) \psi_2(\mathbf{q}, t). \quad (71)$$

Worth noting, within this framework can be defined overlaps $\langle f_{\psi_1} f_{\psi_2} \rangle$ between “action distributions”, or mixed overlaps $\langle f_\psi f_0 \rangle$, while f can be a sum $f = f_0 + f_\psi$. Usually, derivatives are replaced by finite differences such as (65) in numerical or lattice [28] calculations, as a result of discretization. Variations in the length unit (measure), with respect to a connection form provided by the electromagnetic potentials, have also been introduced by H. Weyl [5, 29]. Presuming that canonical and mechanical momentum coincide, let $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be the (covariant) fundamental vectors of a Bravais lattice \mathcal{B}_P in the momentum space, $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$ the (contravariant) fundamental vectors of the reciprocal lattice \mathcal{B}_P^*

$$\mathbf{a}^l = 2\pi\epsilon_{lmn} \frac{\mathbf{a}_m \times \mathbf{a}_n}{\Omega_P}, \quad \Omega_P = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)$$

and $\{\mathbf{c}^i = \hbar\mathbf{a}^i, i = 1, 2, 3\}$ the fundamental vectors of a Bravais lattice \mathcal{B}_Q in the coordinate space. Thus, if $\tilde{f}(\mathbf{q}, \mathbf{k}, t)$ has significant values only when \mathbf{k} relates two nodes of \mathcal{B}_P^* and $\sigma\mathbf{k}$ in (67) relates two nodes of \mathcal{B}_Q , then $\sigma = \hbar$.

These considerations indicate that a natural relationship between the classical distribution function f_0 and the quantum WKB wave function arises by phase-space discretization, using for the canonically conjugate variables a direct lattice and its reciprocal. It can also be shown [8] that the Wigner function f_ψ is an exact solution of (62) with ψ an exact solution of the Schrödinger equation, only if the potential V is a polynomial of degree at most two.

5. Summary and Conclusions

The integrality conditions of the old quantum mechanics, as well as the correspondence between observables and operators, sought in algebraic quantization, may

receive a geometrical interpretation in the theory of complex line bundles with connection and Hermitian structure. The basic elements of this “prequantization” theory have been presented in Section 2. However, the meaning of the complex line over a (physical or reduced) phase-space (M, ω) is not obvious. The distribution function $f \geq 0$ used in classical statistical mechanics is integrable over M , but real, while the quantum wave functions ψ are complex, but $\bar{\psi}\psi$ is integrable only over the configuration space. To retrieve this property, the prequantum Hilbert space can be reduced, as indicated in Section 3, by selecting a polarization.

For the real polarization Λ_S generated by the solution S of the Hamilton-Jacobi equation, the polarized sections take the form of the quantum wave functions in the WKB approximation. However, similar functions appear in the structure of the exact solution f_0 for the classical one-particle Liouville equation, which becomes the Wigner transform f_ψ of ψ if the configuration space is discretized. Although f_ψ is not positive definite, it is integrable over M , and the phase-space overlap between two such functions f_{ψ_1}, f_{ψ_2} is proportional to $|\langle \psi_1 | \psi_2 \rangle|^2$.

By discretization, f acquires a nonlocal character, as $f_\psi(\mathbf{q}, \mathbf{p})$ depends not only on $|\psi(\mathbf{q})|^2$, but also on $\bar{\phi}_{\mathbf{p}}(\mathbf{q}_-)\phi_{\mathbf{p}}(\mathbf{q}_+)$ with $\phi_{\mathbf{p}}(\mathbf{q}) = \exp(-i\mathbf{q} \cdot \mathbf{p}/\hbar)\psi(\mathbf{q})$ and $\mathbf{q}_\pm = \mathbf{q} \pm \hbar\mathbf{k}/2$ covering a whole domain of “quantum coherence”, containing \mathbf{q} . The relationship between a possible lattice structure of the phase-space and the statistical interpretation of f_ψ remains a subject worth of further consideration.

6. Appendix

Definition 6. M is a complex manifold if it possesses an atlas $\{(U_i, \varphi_i), i \in I\}$ where U are open sets covering M , $\varphi_i : U_i \mapsto \mathcal{O}_i \subset \mathbb{C}^n$ is a diffeomorphism, and the transition functions $c_{ij} = \varphi_j \circ \varphi_i^{-1}$ are holomorphic. If $p \in U_i \cap U_j$ then $T_p\varphi_i : T_pM \mapsto \mathbb{C}^n$, $T_p\varphi_j : T_pM \mapsto \mathbb{C}^n$, and $T_p\varphi_i \circ (T_p\varphi_j)^{-1} \in \text{GL}(n, \mathbb{C})$.

Definition 7. Let (M, ω) be a complex symplectic manifold. Then M is called a Kähler manifold if for any $p \in M$ the complex structure $J_p \in \text{Sp}(T_pM)$ and ω_p define a Kähler structure on T_pM , $\omega_p(J_px, J_py) = \omega_p(x, y)$. M is a positive Kähler manifold if $(x, y)_p \equiv \omega_p(x, J_py)$ is positive definite.

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