



NAMBU DYNAMICS, N -LIE ALGEBRAS AND INTEGRABILITY

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Abstract. We present a generalized formulation of Poisson dynamics suitable to describe *the n -bodies interactions*. Examples are given of physical systems endowed with such a general structure.

1. Introduction

The Nambu dynamics is an example of n -Poisson structure which is a special n -Lie algebra. The latter was introduced for the first time by Filippov [4] in 1985 who gave first examples, developed first structural concepts, like simplicity, in this context and classified n -Lie algebras of dimensions $n + 1$ which is parallel to the Bianchi classification of three-dimensional Lie algebras.

Filippov defines an n -Lie algebra structure to be an n -ary multi-linear and anti-symmetric operation

$$[v_1, \dots, v_{n-1}]$$

which satisfies the n -ary Jacobi identity:

$$\begin{aligned}
 & [v_1, \dots, v_{n-1}, [u_1, \dots, u_n]] \\
 &= \sum_{i=1}^n [u_1, \dots, u_{i-1}, [v_1, \dots, v_{n-1}, u_i], u_{i+1}, \dots, u_n]. \quad (1)
 \end{aligned}$$

Such an operation, realized on the smooth function algebra of a manifold and additionally assumed to be an n -derivation, is an n -Poisson structure.

This general concept, however, was not introduced by Filippov. A first proposal goes back to Albeggiani [13] who introduced, in a different context, the *Poisson bracket of the n -th order* by the formula

$$\{H_1, H_2, \dots, H_n\} = \sum_{i=1}^{\nu} \frac{\partial (H_1, H_2, \dots, H_n)}{\partial (x_{1i}, x_{2i}, \dots, x_{ni})}$$

where H_1, H_2, \dots, H_n (the ‘‘Hamiltonians’’) are functions of $n\nu$ variables x_{ji} ($j = 1, 2, \dots, n; i = 1, 2, \dots, \nu$).

However, the above bracket does not satisfy generally Jacobi identity.

The case $\nu = 1$ was considered much later (in 1994) by L. Takhtajan [10] in order to formalize mathematically the n -ary generalization of Hamiltonian mechanics proposed by Nambu [9] in 1973. The n -bracket operation considered by Nambu was

$$\{f_1, \dots, f_n\} = \det \left\| \frac{\partial f_i}{\partial x_j} \right\|. \quad (2)$$

But Nambu himself as well as his followers do not mention that n -bracket (2) satisfies the n -Jacobi identity (1). On the other hand, Filippov reports (2) in his paper among other examples of n -Lie algebras.

In what follows, n -Lie algebra structures on smooth function algebras of smooth manifolds which are given by means of multi-differential operators will be called *local n -Lie algebras*. A theorem by Kirillov shows that these multi-differential operators are of first order.

A local n -Lie algebra structure on a manifold is said to be an *n -Jacobi structure*. In the case when the structure multi-differential operator is a multi-derivation one gets an *n -Poisson structure*. Thus, n -Poisson manifolds form a subclass of n -Jacobi ones.

A full local description of n -Jacobi and, in particular, of n -Poisson manifolds can be found in [8]. This is an n -ary analogue of the Darboux lemma. In what concerns n -Poisson manifolds the same result was obtained by Alexeevsky and Guha [1] (see also [5]).

An important consequence of the n -Darboux lemma is that the Cartesian product of two n -Jacobi, or two n -Poisson manifolds does not produce manifold of the same type if $n > 2$.

It is not our goal to fully describe the local structure of local n -Lie algebras, but we will try to be systematic in what concerns the relevant basic formulae and constructions. Moreover, possible applications to integrable systems and related problems of dynamics will be illustrated on some examples of current interest.

More precisely, the content of the talk is as follows.

Section 2 is devoted to a short review of Hamiltonian, Symplectic and Poisson dynamics as a natural introduction to Nambu dynamics. Section 3 is devoted to general Jacobi-Poisson dynamics and their properties. There using concrete examples some simple applications of n -ary structures to dynamics will be given.

First, we use the hyper-integrable type dynamics to show how the constants of motion can be put in relation with multi-Poisson structures. Second, alternative Poisson realizations of a spinning particle dynamics Γ are given by using ternary structures preserved by Γ .

2. Poisson Dynamics and Nambu Dynamics

Let us now consider a generic dynamics described by the equation

$$\frac{du^h}{dt} = \Lambda^{hk}(u) \frac{\partial \mathcal{H}}{\partial u^k} \quad (3)$$

where $\Lambda^{hk}(u)$ and \mathcal{H} are given function of coordinates u .

The evolution of a generic functions f , defined on the phase space Φ , will be given by

$$\frac{df}{dt} = \frac{\partial f}{\partial u^h} \frac{du^h}{dt} = \frac{\partial f}{\partial u^h} \Lambda^{hk} \frac{\partial \mathcal{H}}{\partial u^k} = (\nabla f, \Lambda \nabla \mathcal{H}).$$

In order to have a Jacobi-Poisson theorem for this type of dynamics, we must require:

- skew-symmetry

$$(\nabla f, \Lambda \nabla g) = -(\nabla g, \Lambda \nabla f)$$

- Jacobi identity

$$(\nabla(\nabla f, \Lambda \nabla g), \Lambda \nabla h) + (\nabla(\nabla g, \Lambda \nabla h), \Lambda \nabla f) + (\nabla(\nabla h, \Lambda \nabla f), \Lambda \nabla g) = 0.$$

In this case the bracket $(\nabla f, \Lambda \nabla g)$ will be called the Poisson bracket of f and g and will be denoted with $\{f, g\}_\Lambda$, or simply, if no ambiguity arises, with $\{f, g\}$. In terms of the matrix Λ , the previous requirements are expressed by the following:

- skew-symmetry: $\Lambda = -\Lambda^T$
- Jacobi identity: $\Lambda^{ij} \frac{\partial \Lambda^{hk}}{\partial u^j} + \Lambda^{hj} \frac{\partial \Lambda^{ki}}{\partial u^j} + \Lambda^{kj} \frac{\partial \Lambda^{ih}}{\partial u^j} = 0.$

We shall call Hamiltonian a dynamical system with n degrees of freedom, and then with a $2n$ dimensional phase space, if it is described by the equation

$$\frac{df}{dt} = \{f, \mathcal{H}\}_\Lambda$$

where the bracket satisfies the properties

$$\begin{aligned} \{f, g\}_\Lambda &= -\{g, f\}_\Lambda \\ \{\{f, g\}_\Lambda, h\}_\Lambda + \{\{g, h\}_\Lambda, f\}_\Lambda + \{\{h, f\}_\Lambda, g\}_\Lambda &= 0 \\ \{f, c\}_\Lambda &= 0, \quad \text{for all } c \in \mathbb{R} \\ \{f, gh\}_\Lambda &= \{f, g\}_\Lambda h + g\{f, h\}_\Lambda \end{aligned}$$

and, moreover

$$\det \Lambda \neq 0.$$

Because of the above condition, coordinates exist (Darboux theorem) such that

$$\Lambda = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \quad (4)$$

where 0 and I denote the $n \times n$ nul and identity matrices.

In this case one speaks of *symplectic dynamics*, since the *symplectic structure*

$$\omega = \Lambda^{-1}$$

is invariant under the flow of canonical vector field X_f defined by

$$X_f g \equiv \{f, g\}, \quad \text{for all } g \in \mathcal{F}(\mathcal{M})$$

and, in particular under the Hamiltonian flow generated by the Hamiltonian vector field

$$X_H = \frac{\partial \mathcal{H}}{\partial u^k} \Lambda^{kh} \frac{\partial}{\partial u^h}.$$

2.1. Poisson Dynamics

Let us finally observe that in the previous definition no role is played by the even dimensionality of the phase space. Thus, it is natural to define more general dynamics according to the following definition.

A dynamics, described by the equations

$$\frac{df}{dt} = \{f, \mathcal{H}\}_P$$

with the bracket satisfying the properties

$$\begin{aligned} \{f, g\}_P &= -\{g, f\}_P \\ \{\{f, g\}_P, h\}_P + \{\{g, h\}_P, f\}_P + \{\{h, f\}_P, g\}_P &= 0 \\ \{f, gh\}_P &= \{f, g\}_P h + g\{f, h\}_P \\ \{f, c\}_P &= 0, \quad \text{for all } c \in \mathbb{R} \end{aligned}$$

is called a *Poisson dynamics*.

Of course, a Hamiltonian dynamics is a symplectic dynamics and then also a Poisson dynamics.

We notice that

- properties expressed by first two relations, namely

$$\{f, g\} = -\{g, f\} \quad (5)$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad (6)$$

endow the set \mathcal{F} of differentiable functions defined on Φ with a Lie algebra structure.

- properties expressed by last two relations, namely

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad (7)$$

$$\{f, c\} = 0, \quad \text{for all } c \in \mathbb{R} \quad (8)$$

gives to the bracket a natural compatibility with the usual associative product of functions.

Since it can be written in the following equivalent alternative forms:

$$X_{\{f, g\}}h = [X_f, X_g]h \quad (9)$$

$$X_f\{g, h\} = \{X_f g, h\} + \{g, X_f h\} \quad (10)$$

the Jacobi identity is equivalent to the following alternative statements suggested by (9) and (10) respectively:

- The map

$$\begin{aligned} f &\mapsto X_f = \{f, \cdot\} \\ \{f, g\} &\mapsto X_{\{f, g\}} \end{aligned}$$

is a Lie algebra morphism

$$(\mathcal{F}, \{\cdot, \cdot\}) \mapsto (\mathcal{X}_{\mathcal{F}}, [\cdot, \cdot])$$

between $(\mathcal{F}, \{\cdot, \cdot\})$ and the set of Hamiltonian vector fields $\mathcal{X}_{\mathcal{F}}$ endowed with the Lie Bracket given by the commutator $[\cdot, \cdot]$.

- The operator $X_f = \{f, \cdot\}$ is a derivation of the Poisson bracket.

We observe that properties expressed by equations (5), (6) and (8) are purely algebraic in nature, so that the following abstract algebraic formulation can be introduced.

Let M be a Poisson manifold and \mathcal{F} the ring of functions defined on it. This means that on M a bracket $\{\cdot, \cdot\}$ is defined such that:

1. it yields the structure of a Lie algebra on \mathcal{F} , i.e.,

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0 \end{aligned}$$

2. it has a natural compatibility with the usual associative product of functions, which is

$$\begin{aligned} \{h, c\} &= 0, \quad \text{for all } c \in \mathbb{R} \\ \{h, fg\} &= \{h, f\}g + f\{h, g\}. \end{aligned}$$

Therefore, we can define an abstract Poisson algebra as an associative commutative algebra endowed with a Lie bracket satisfying (5), (6) and (8).

It is natural to generalize the notion of a Poisson manifold by relaxing condition (2) and requiring only that $\{f, g\}$ be just a local type operation

$$\text{support } \{f, g\} \subseteq (\text{support } f) \cap (\text{support } g).$$

The bracket $\{f, g\}$ is then called a *Jacobi bracket* and the corresponding manifold a *Jacobi manifold*.

2.2. Nambu Dynamics

The possibility of further generalizations of Poisson dynamics rely on the possibility to generalize the Poisson bracket.

Let us consider a dynamical system described by the equations

$$\frac{df}{dt} = \{f, \mathcal{H}_1, \mathcal{H}_2\}$$

where the ternary bracket in the right hand side is supposed to be skew-symmetric. This dynamics will be called a ternary Poisson dynamics if the ternary bracket allows for a Poisson theorem on first integrals. In such a case the ternary bracket

will be called ternary Poisson bracket. We are thus looking for a property of the ternary bracket such that

$$\{f_h, H_1, H_2\} = 0, \quad h = 1, 2, 3 \quad \Rightarrow \quad \{\{f_1, f_2, f_3\}, H_1, H_2\} = 0. \quad (11)$$

For this purpose it is useful to recall the form of Jacobi identity, for binary bracket, given in equation (10)

$$X_f\{g, h\} = \{X_f g, h\} + \{g, X_f h\}.$$

This form can be immediately generalized to skew-symmetric brackets with an arbitrary number of entries.

Indeed, given the ternary bracket $\{f, g, h\}$, we require that the operator (vector field) X_{fg} , defined by

$$X_{fg}h := \{f, g, h\}$$

is a derivation of the bracket, that is

$$X_{fg}\{h_1, h_2, h_3\} = \{X_{fg}h_1, h_2, h_3\} + \{h_1, X_{fg}h_2, h_3\} + \{h_1, h_2, X_{fg}h_3\}. \quad (12)$$

The above formula can be explicitly written as follows:

$$\begin{aligned} \{f, g, \{h_1, h_2, h_3\}\} &= \{\{f, g, h_1\}, h_2, h_3\} + \{h_1, \{f, g, h_2\}, h_3\} \\ &\quad + \{h_1, h_2, \{f, g, h_3\}\} \end{aligned}$$

which should be difficult to invent without a deep understanding of the significance of the usual Jacobi identity.

An interesting example given by Nambu is the following:

$$\frac{d\vec{r}}{dt} = \nabla H \wedge \nabla G$$

or

$$\frac{dx}{dt} = \frac{\partial(H, G)}{\partial(y, z)}, \quad \frac{dy}{dt} = \frac{\partial(H, G)}{\partial(z, x)}, \quad \frac{dz}{dt} = \frac{\partial(H, G)}{\partial(x, y)}.$$

Since

$$\text{Div } \vec{X} = \nabla \cdot (\nabla H \wedge \nabla G) = 0$$

we obtain from

$$\frac{d}{dt} \int_V \rho d\mu = \int_V \left(\frac{d\rho}{dt} + \rho \text{Div } \vec{X} \right)$$

that

$$V = \int_V d\mu = \text{const.}$$

A specific particular case is given by the Euler rotator for which

$$H = L^2 = \frac{1}{2} (L_x^2 + L_y^2 + L_z^2), \quad G = \mathcal{H} = \frac{1}{2} \left(\frac{L_x^2}{I_x^2} + \frac{L_y^2}{I_y^2} + \frac{L_z^2}{I_z^2} \right).$$

It is not difficult to prove that equation (11) is equivalent to equation (12). We will not go on further on this subject. Much more details can be found in [8] and references therein, where examples of n -ary Poisson dynamics are explicitly given and the following important property, here reported just for the case $n = 3$, is proven:

If $\{f, g, h\}$ is ternary Poisson bracket, the binary bracket $\{f, g\}_h = \{f, g, h\}$, obtained by fixing one of the functions, is a binary Jacobi-Poisson bracket. Furthermore, a linear combination of two of them $c_1\{f, g\}_{h_1} + c_2\{f, g\}_{h_2}$ is again a binary Jacobi-Poisson bracket.

3. n -Jacobi-Poisson Dynamics

An n -ary Jacobi-Poisson dynamics is a dynamics for which the evolution of any observable is described by the differential equation

$$\frac{df}{dt} = \{f, H_1, H_2, \dots, H_{n-1}\}$$

where the bracket $\{\cdot, \dots, \cdot\}$ satisfies the properties:

$$\begin{aligned} \{f, g, h\} &= -\{f, h, g\} = -\{g, f, h\} \\ \{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} &= \sum_{i=1}^n \{g_1, \dots, g_{i-1}, \{f_1, \dots, f_{n-1}, g_i\}, \\ &\quad g_{i+1}, \dots, g_n\} \\ \{f_1, \dots, f_{n-1}, h_1 h_2\} &= \{f_1, \dots, f_{n-1}, h_1\} h_2 + h_1 \{f_1, \dots, f_{n-1}, h_2\} \\ \{f_1, \dots, f_{n-1}, c\} &= 0, \quad \text{for all } c \in \mathbb{R}. \end{aligned}$$

In other words the Hamiltonian vector field $X_{f_1, f_2, \dots, f_{n-1}}$, associated with the $n - 1$ functions f_1, f_2, \dots, f_{n-1} , via

$$X_{f_1, f_2, \dots, f_{n-1}} g = \{f_1, \dots, f_{n-1}, g\}$$

is a derivation both of the bracket and the product of functions.

3.1. Hereditary Structure of the n -Lie Bracket

The Hamiltonian vector field $X_{f_1, f_2, \dots, f_{n-1}}$ will also be called sometimes a *Nambu vector field* or a *Nambu dynamics*.

Few physically relevant properties are:

- The flow of a Nambu dynamics preserves the Nambu bracket.
- *Hamiltonian* functions H_1, H_2, \dots, H_{n-1} are first integrals and the Nambu bracket of n first integrals is again a first integral (*Jacobi-Poisson Theorem*).

We find the important consequence that a dynamical vector field which is Nambu for a k -ary bracket must possess at least $k - 1$ first integrals.

This observation and the hereditary structure of the n -Lie bracket

$$\begin{aligned} \{f_1, \dots, f_n\} \text{ } n\text{-Poisson} & \Rightarrow \{f_1, \dots, f_{n-1}\}|_F = \{f_1, \dots, f_{n-1}, F\} \text{ } (n-1)\text{-Poisson} \\ \{f_1, \dots, f_n\} \text{ } n\text{-Poisson} & \Rightarrow \{f_1, \dots, f_{n-1}\}|_F + \{f_1, \dots, f_{n-1}\}|_G \text{ } (n-1)\text{-Poisson} \end{aligned}$$

explain why completely integrable systems are likely to be found among Nambu dynamics.

3.2. Hyperintegrable Dynamics

Occasionally, a dynamical vector field Γ admitting $2n - 1$ constants of the motion on a $2n$ dimensional manifold M , is called *hyper-integrable* or *degenerate*.

In these cases

$$i_\Gamma \Omega = df_1 \wedge df_2 \wedge \dots \wedge df_{2n-1}$$

with $f_1, f_2, \dots, f_{2n-1}$ first integrals for Γ .

Here it is possible to define a Nambu bracket by setting:

$$\{h_1, h_2, \dots, h_{2n}\}|_F \Omega = F(f_1, f_2, \dots, f_{2n-1}) dh_1 \wedge dh_2 \wedge \dots \wedge dh_{2n}. \quad (13)$$

In other words, denoting with L_Γ the Lie derivative with respect to vector field Γ , with $f_1, f_2, \dots, f_{2n-1}$ first integrals for Γ and $f_{2n} \in C^\infty(M)$ is such that $\Gamma(f_{2n}) = 1$, then the $2n$ -Poisson bracket

$$\{h_1, h_2, \dots, h_{2n}\} = \det \left\| \frac{\partial h_i}{\partial f_j} \right\|, \quad i, j = 1, \dots, 2n \quad (14)$$

is preserved by Γ which becomes Hamiltonian with respect to (14) with the Hamiltonian function $(f_1, f_2, \dots, f_{2n-1})$.

Of course the corresponding $2n$ -Poisson vector is

$$\Lambda = \frac{\partial}{\partial f_1} \wedge \frac{\partial}{\partial f_2} \wedge \dots \wedge \frac{\partial}{\partial f_{2n}}.$$

More generally $2n$ -Poisson bracket

$$\{h_1, h_2, \dots, h_{2n}\}_F = F \det \left\| \frac{\partial h_i}{\partial f_j} \right\|, \quad i, j = 1, \dots, 2n$$

is preserved by Γ iff F is a first integral, i.e., $F = F(f_1, f_2, \dots, f_{2n-1})$.

By changing F we find all possible $2n$ -ary bracket for the given vector field Γ .

3.3. Additional Examples

3.3.1. The Spinning Particle

Given a dynamics, i.e., a vector field Γ on a manifold M , it could be interesting to realize it as a Hamiltonian field with respect to a Poisson structure [3]. Below it will be shown how multi-Poisson structures can be used in this connection.

We shall ignore the spatial degrees of freedom of the particle and study only the spin variables. Let us consider the spin variables $\mathbf{S} = (S_1, S_2, S_3)$ as elements in \mathbb{R}^3 . The equations for these variables when the particle interacts with an external magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ are given by

$$\frac{dS_i}{dt} = \mu \epsilon_{ijk} S_j B_k \quad (15)$$

where μ denotes the magnetic moment.

This dynamics has two first integrals, namely, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2$ and $\mathbf{S} \cdot \mathbf{B} = S_1 B_1 + S_2 B_2 + S_3 B_3$ and, in addition, is canonical for the ternary bracket associated with the three-vector field

$$\frac{\partial}{\partial S_1} \wedge \frac{\partial}{\partial S_2} \wedge \frac{\partial}{\partial S_3}.$$

The most general ternary bracket preserved by dynamics (15), is associated with the three-vector field

$$f \frac{\partial}{\partial S_1} \wedge \frac{\partial}{\partial S_2} \wedge \frac{\partial}{\partial S_3} \quad (16)$$

where f is a first integral of it.

All Poisson structures obtained by fixing a function $F = F(\mathbf{S}^2, \mathbf{S} \cdot \mathbf{B})$, are preserved by the dynamics and are mutually compatible. The corresponding Poisson bracket is

$$\{S_j, S_k\}_F^f = f \varepsilon_{jkl} \frac{\partial F}{\partial S_l}.$$

Now we show how the ternary Poisson structure (16) allows for the alternative ordinary Poisson brackets described in [3]

Standard description

$$f = \frac{1}{2}, \quad F = S^2.$$

For this choice the algebra generated by the Poisson brackets on linear functions is the $\mathfrak{su}(2)$ Lie algebra. The Hamiltonian function for the dynamics is the standard one $H = -\mu \mathbf{S} \cdot \mathbf{B}$.

Non-standard description

Now we take

$$f = \frac{1}{2}, \quad F = S_1^2 + S_2^2 + \frac{1}{2\lambda} \left[\frac{\cosh 2\lambda}{\sinh \lambda} S_3 - \frac{1}{\lambda} \right]$$

with Hamiltonian $H = -\mu\lambda S_3$. Here for simplicity we have taken the magnetic field along the third axis. The parameter λ is a deformation parameter and the standard description is recovered for $\lambda \mapsto 0$. The hereditary Poisson brackets are:

$$\{S_2, S_3\}_F^f = S_1, \quad \{S_1, S_3\}_F^f = S_2, \quad \{S_1, S_2\}_F^f = \frac{1}{2} \frac{\sinh 2\lambda}{\sinh \lambda} S_3.$$

These brackets are a classical realization of the quantum commutation relations for generators of the $U_q(\mathfrak{sl}(2))$ Hopf algebra.

We also notice that this Poisson Bracket is compatible with the previous one as they are hereditary from the same ternary structure (16).

Another non-standard description

There is another choice for f and F which is known to correspond to the classical limit of the $U_q(\mathfrak{sl}(2))$ Hopf algebra.

It is

$$f = \frac{\lambda}{4} S_3, \quad F = S_1^2 + S_2^2 + S_3^2 + S_3^{-2}.$$

It leads to the following brackets:

$$\{S_2, S_3\}_F^f = \frac{\lambda}{2} S_1 S_3, \quad \{S_1, S_3\}_F^f = \lambda S_2 S_3, \quad \{S_1, S_2\}_F^f = \frac{\lambda}{2} [S_3^2 - S_3^{-2}].$$

3.4. The n -Darboux Theorem

For the construction of Nambu descriptions for a given dynamical field it turns out to be very useful the already cited result by Alekseevsky and Guha [1].

It provides, similarly to the role of Darboux theorem for the local normal form of symplectic structures, a local normal form, namely:

For any Nambu n -bracket, with $n \geq 2$, it is possible to find, in a neighborhood of a point x where $\Lambda(x) \neq 0$, a local coordinate system such that

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}. \quad (17)$$

This result turns out to be very useful for the construction of Nambu descriptions for a given dynamical field.

Further interesting results on n -Lie algebra structures and their applications can be found in [2].

Conclusions

The main structure regarding n -Poisson structures has been there reported. It tells that the structure n -vector of an n -Poisson structure is of rank n (decomposable) if $n > 2$. This leads directly to the n -Darboux lemma: given an n -Poisson structure, $n > 2$, on a manifold M there exists a local chart x_1, \dots, x_m , $m = \dim M \geq n$, on M such that the corresponding n -Poisson bracket is given by (2). Two consequences of this result are worth mentioning. First, the n -bracket defined naturally on the dual of an n -Lie algebra \mathcal{V} is not generally an n -Poisson structure if $n > 2$. This is in sharp contrast with usual (i.e., $n = 2$) Lie algebras. However, it can be proven that it is still so for n -dimensional and $(n + 1)$ -dimensional n -Lie algebras.

The n -Darboux lemma for general n -Jacobi manifolds with $n > 2$ is also reported. The proof can be found in [8]. The key idea in doing that is to split a first order multi-differential operator into two parts similarly to the canonical representation of a scalar first order differential operator as the sum of a derivation and a function.

The multi-generalization of the concept of local Lie algebra studied in this paper is not, in fact, unique and there are other natural alternatives (see [6,7,12]). All these generalizations are mutually interrelated and open very promising perspectives for particle and field dynamics.

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