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CHARACTERIZATION AND COMPUTATION OF CLOSED GEODESICS ON TOROÏDAL SURFACES

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Abstract. The aim of the present study is to characterize and compute closed geodesics on toroïdal surfaces. We show that a closed geodesic must make a number of rotations about the equatorial part (k rotations) and the axis of revolution (k' rotations) of the surface. We give the relation that exists between the numbers k and k', and the Clairaut's constant C corresponding to the geodesic. Moreover, we prove that the numbers k and k' are relatively prime. We validate our findings by constructing closed geodesics on some examples of toroïdal surfaces using MAPLE. Finally, using experimental data on cardiac fiber direction, we show that fibers run as geodesics in the left ventricle whose geometrical shape looks like a toroïdal surface.

1. Introduction

Geodesics are of high importance due to their wide applications in many fields such as topography, biology, etc. For instance, according to Streeter [14], in the equatorial part of the left ventricle free wall, fibers are organized into toroïdal surfaces on which they run as geodesics.

From a mathematical point of view, a geodesic on a parameterized surface is a solution to a nonlinear system of two second order ordinary differential equations. Consequently, this representation leads to the local existence of a geodesic starting from a given point and tangent to a given vector belonging to the tangent plane at the point, see Berger and Gostiaux [3]. On the other hand, global existence results are available in the case of closed surfaces. Indeed, there is a global result that guaranties that every local geodesic can be extended to a geodesic defined over $\mathbb{R}$, see Schwartz [13]. Moreover, another result states that every two points of a closed surface can always be connected by at least a geodesic with minimal length (particular case of the Hopf-Rinow theorem, [6]). Closed geodesics in particular, have been studied intensively, however, their existence is not straightforward. We mention the following result in a particular case of complete surfaces homeomorphic to a plane, a cylinder, or a Mobius band. If such a surface has a finite area, then there exist infinitely many closed geodesics lying on it, see Bangert, [2].

In the present research, we study the characteristics of closed geodesics lying on a particular case of surfaces: the toroïdal surfaces. The origin of our study was initiated in 2000 by looking at geometrical organization of cardiac fibers in order to check Streeter's conjecture about geodesics, for more details see Mourad *et al* [9] and Mourad [10]. Actually, the left ventricle has approximately the shape of a body of revolution [14] (about a half of an ellipsoid) with the possibility that a fiber running on the external surface can pass through the bottom of the left ventricle (the apex) in order to run on the internal surface (that of the cavity) and it ends by closing on itself. This led us to focus on toroïdal surfaces and closed geodesics.

This paper is organized as follows. In Section 2, we present the theoretical results about closed geodesics on toroïdal surfaces. We prove that such a geodesic makes a number of rotations k' about the axis of revolution and k about the equator of the surface that are related to the Clairaut's constant C of the geodesic. The numbers k and k' could be zeros but not both at the same time. The obtained relation between k, k' and C allows us to determine the Clairaut's constant from the numbers of rotations already mentioned, which in turn, for a given initial point, leads to the computation of the initial tangent to the geodesic. Consequently, we get all the ingredients in order to solve numerically the system of differential equations corresponding to the geodesic. Moreover, we show that these numbers of rotations, if they are both different from zero, are relatively prime. Recently, related studies have been published by Mladenov and Oprea [7], [8] and Alexander [1]. In [1], Alexander has studied closed geodesics on certain surfaces of revolution and has got the same relation between the Clairaut's constant and the numbers of rotation. The third section concerns itself with the validation of the results obtained in section 2. We compute closed geodesics on some examples of toroïdal surfaces using MAPLE. In the last section, we present some results obtained on experimental data about cardiac fiber directions and we show the validation of Streeter's conjecture in the case of the left ventricle.

2. Theoretical Results

Let Σ be a smooth toroïdal surface parameterized as follows:

$$\mathbf{S}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)) \tag{1}$$

where f and g are two smooth real functions. Geometrically, Σ can be obtained by rotating a planar merdian curve C_0 about the Z-axis, where C_0 is defined in the XOZ-plane by $v \mapsto (f(v), g(v))$ and it does not cross the z-axis. We suppose that the curve C_0 is regular, i.e., for all $v,\ f'^2(v) + g'^2(v) \neq 0$, periodic of period p, and that every pair (v_1, v_2) such that $(f(v_1), g(v_1)) = (f(v_2), g(v_2))$ satisfies $v_2 - v_1 \in p\mathbb{Z}$. This means that the curve C_0 is simple. Moreover, we assume that the function f is strictly positive.

Let Γ be a geodesic curve lying on Σ. The geodesic Γ can be parameterized by

$$s \mapsto \boldsymbol{\phi}(s) = \Big(f(v(s)) \cos u(s), f(v(s)) \sin u(s), g(v(s))\Big)$$

where s is the arclength. Consequently, we have

$$(f^{'2} + g^{'2})\dot{v}^2 + f^2\dot{u}^2 = 1. \tag{2}$$

It can be shown that any geodesic Γ satisfies the following system

$$(f^{'2} + g^{'2})\dot{v}^2 + f^2\dot{u}^2 = 1, \qquad f^2(v)\dot{u} = C \tag{3}$$

where C is a constant called the Clairaut's constant. For further information regarding geodesics lying on a surface of revolution, we refer to do Carmo [4].

Now with system (3) we can compute at any given point and for a given Clairaut's constant, the initial tangent to the geodesic.

It is obvious that the meridians defined by $\boldsymbol{\phi}(s) = \mathbf{S}(u_0, v(s))$ are the geodesics of Σ corresponding to $C = 0$. On the other hand, a parallel $\boldsymbol{\phi}(s) = \mathbf{S}(u(s), v_0)$ is a geodesic if and only if $f'(v_0) = 0$, see do Carmo [4].

In this paper, we are in particular interested in the closed geodesics lying on toroïdal surfaces.

Definition 1. *A geodesic is said to be closed* (*or periodic*) *if it is a periodic curve.*

Remark 2. *A closed geodesic is not only a closed curve, but it is a periodic closed curve.*

In the case of a closed geodesic, there exist a real T and two integers k and k' such that for every s:

$$u(s+T) = u(s) + 2k'\pi, \qquad v(s+T) = v(s) + kp. \tag{4}$$

For more details about this point, we refer to Mourad [10].

Remark 3. *We can interpret k' as the number (positive or zero) of rotations about the axis of revolution made by the geodesic, see Alexander* [1].

In fact, from System (3), we have either

- for every s, $\dot{u}(s) = 0$, i.e., the function $s \mapsto u(s)$ is constant. The geodesic is a meridian: it cannot have any rotation about the axis of revolution. Hence, k' defined by (4) is obviously zero.
- or, for every s, $\dot{u}(s)$ is strictly positive. The polar angle u is a strictly increasing function of s, we deduce that $k' > 0$. Since u is continuous, then it takes all values between $u(0)$ and $u(T) = u(0) + 2k'\pi$, and it is bijective: the geodesic completes exactly k' rotations about the axis of revolution.

Let us give an interpretation to the number k. In the sequel we denote by m the minimum of the function f over $[0, p]$. It is also the minimum over $\mathbb{R}$. Suppose that m is attained for some v_0. Since f is of class C^1, then $f'(v_0) = 0$. We denote by m_{ϕ} the minimum of $s \mapsto f(v(s))$ along the closed geodesic (Γ, ϕ). In other words, this minimum is taken over $[0, T]$ or equivalently over $\mathbb{R}$. If m_{ϕ} is attained at s_0, then $f'(v(s_0))\dot{v}(s_0) = 0$. For the geometric understanding of the results to follow, it is useful to recall that $f(v)$ denotes the distance from the axis of revolution to the point whose parameters are (u, v) on the surface.

Lemma 4. *If there exists s_0 such that $\dot{v}(s_0) = 0$, then m_{ϕ} is attained at s_0, and the Clairaut's constant C satisfies $C = f(v(s_0)) = m_{\phi}$.*

Proof: By equations (3), at s_0 such that $\dot{v}(s_0) = 0$, we have $f(v(s_0)) = C$ where C is the Clairaut's constant associated with the geodesic. We have already seen that, from the geometric interpretation, $C \leq \min_s f(v(s))$. Therefore, $C = f(v(s_0)) = \min_s f(v(s)) = m_{\phi}$. ■

Proposition 5. *If $k = 0$, then either the geodesic is a parallel, or there exists a parallel that it cannot cross it. We say that the geodesic does not make any rotation about the merdian of the surface.*

Proof: Since $k = 0$, then $v(T) = v(0)$ and there exists $s_0 \in]0, T[$ such that $\dot{v}(s_0) = 0$. The geodesic is tangent at $\phi(s_0)$ to the parallel through $\phi(s_0)$. Therefore, we have either $f'(v(s_0)) = 0$ and the geodesic coïncides with this parallel, or $f'(v(s_0)) \neq 0$ and m cannot be attained at $v(s_0)$. However, by Lemma 4, m_ϕ is attained at s_0. Therefore, $m_\phi > m$. In other words, the distance to the axis from the geodesic stays strictly greater than its minimal value over a meridian or over the surface. Then there exists at least one parallel that the geodesic cannot cross. ■

Now let us consider the case $k \neq 0$.

Proposition 6. *The number k is nonzero if and only if, for every s, $\dot{v}(s) \neq 0$.*

Proof: First, suppose $k \neq 0$. Since the function $s \mapsto v(s)$ is continuous on $\mathbb{R}$, then it is surjective over $[v(0), v(T)] = [v(0), v(0) + kp]$. But $k \neq 0$, then we deduce that the function $s \mapsto f(v(s))$ is surjective from $\mathbb{R}$ onto $\operatorname{im} f$. Hence $m = m_\phi$.
Suppose that there exists s_0 such that $\dot{v}(s_0) = 0$. By Lemma 4, m_ϕ is attained at s_0. Since $m_\phi = m$, then $v \mapsto f(v)$ attains its minimum at $v(s_0)$ and necessarily we have $f'(v(s_0)) = 0$. The geodesic coincides with the parallel passing through $\phi(s_0)$. Therefore we get $k = 0$, which is in contradiction with the hypothesis. Thus, for every s, $\dot{v}(s) \neq 0$.
Conversely, if v is strictly monotonic, for instance strictly increasing, we get $v(T) > v(0)$, then $v(0) + kp > v(0)$, hence $k \neq 0$. ■

Remark 7. *If $k \neq 0$, the function $s \mapsto v(s)$ being strictly monotonic and continuous, is bijective from $[0, T]$ onto $[v(0), v(T)] = [v(0), v(0) + kp]$. On the interval $[0, T]$, the geodesic crosses k times each parallel. We say that the geodesic makes k windings about the surface, see Alexander* [1].

We suppose that k is nonzero. We define $\varphi : c \in [0, m[\mapsto \varphi(c) \in \mathbb{R}$ by

$$\varphi(c) = \int_0^p h(c, v)\,\mathrm{d}v \quad \text{where} \quad h(c, v) = \frac{c}{2\pi}\sqrt{\frac{f'^2(v) + g'^2(v)}{f^2(v)(f^2(v) - c^2)}}. \tag{5}$$

The expression (5) of the function φ is similar to that given by Alexander [1].

Theorem 8. *If the geodesic (Γ, ϕ) is periodic and makes at least one winding, then the Clairaut's constant C is strictly less than m and it is related to the number* *of rotations k' and the number of windings k by the relation*

$$\varphi(C) = \frac{k'}{|k|}. \tag{6}$$

Proof: Since the geodesic Γ makes at least one winding, then it crosses k times each parallel, so $k \neq 0$ and we have $m = m_{\Phi}$. However $C \leq m_{\Phi}$ then $C \leq m$. If $C = m$, then $C = m_{\Phi}$, so there exists s_0 such that $C = f(v(s_0))$. By system (3), we get $\dot{v}(s_0) = 0$, thus by Proposition 6 we obtain $k = 0$ which is impossible. Consequenlty, $C < m$.

On the other hand, by system (3), and since $\dot{v}$ keeps a constant sign which is that of k, we get, for every s

$$\dot{u}(s) = \frac{C}{f^2(v(s))} \quad \text{and} \quad \dot{v}(s) = \text{sign}(k)\sqrt{\frac{f^2(v(s)) - C^2}{f^2(v(s))\big(f'^2(v(s)) + g'^2(v(s))\big)}}.$$

Integrating the first equation from 0 to T, making the change of variables $v = v(s)$ which is possible since v is strictly monotonic, and using the periodicity of f and g, we obtain

$$\begin{aligned} 2k'\pi &= C\int_0^T \frac{\mathrm{d}s}{f^2(v(s))} = \text{sign}(k)C\int_{v(0)}^{v(T)} \sqrt{\frac{f'^2(v) + g'^2(v)}{f^2(v)\big(f^2(v) - C^2\big)}}\,\mathrm{d}v \\ &= \text{sign}(k)kC\int_0^p \sqrt{\frac{f'^2(v) + g'^2(v)}{f^2(v)\big(f^2(v) - C^2\big)}}\,\mathrm{d}v. \end{aligned} \tag{7}$$

In other words,

$$\varphi(C) = \frac{k'}{\text{sign}(k)k} = \frac{k'}{|k|}. \tag{8}$$

■

Let us now give the converse of the previous result.

Theorem 9. *If $C \in [0, m[$ is such that $\varphi(C) \in \mathbb{Q}$, then the geodesics that have C as their Clairaut's constant are periodic.*

Proof: Let $\varphi(C) = \dfrac{q'}{q}$ where q and $q' \in \mathbb{N}$ are relatively prime.
First we notice that $v(s)$ is strictly monotonic, because otherwise, there exists s_0 such that $\dot{v}(s_0) = 0$. However, by Lemma 4 easily extended to any geodesic, we

obtain $C = f(v(s_0)) = \min_s f(v(s)) \geq m$, which leads to a contradiction. To fix the ideas, we suppose that $v(s)$ is strictly increasing.
Let us prove that $v(s)$ is not bounded. We distinguish two cases:

- $u(s)$ is not bounded: then $u(s) \to +\infty$ whenever $s \to +\infty$. Let $n \in \mathbb{N}^*$ and s_1 be a positive value, then there exists a period $T_{n,1} > 0$ such that $u(s_1 + T_{n,1}) = u(s_1) + 2nq'\pi$ so $\displaystyle\int_{s_1}^{s_1+T_{n,1}} \dot{u}(s)\,\mathrm{d}s = 2nq'\pi$. Taking into account the system (3) and making the change of variables $v = v(s)$, we obtain
$$\int_{v(s_1)}^{v(s_1+T_{n,1})} h(C,v)\,\mathrm{d}v = nq'.$$
On the other hand, from $\varphi(C) = \dfrac{q'}{q}$ we get also $\displaystyle\int_0^p h(C,v)\,\mathrm{d}v = \frac{nq'}{nq}\cdot$ But $h(C,.)$ is of period p, then
$$\int_{v(s_1)}^{v(s_1)+nqp} h(C,v)\,\mathrm{d}v = nq\int_0^p h(C,v)\,\mathrm{d}v = nq'.$$
Consequently, $\displaystyle\int_{v(s_1)}^{v(s_1)+nqp} h(C,v)\,\mathrm{d}v = \int_{v(s_1)}^{v(s_1+T_{n,1})} h(C,v)\,\mathrm{d}v$, therefore we get $\displaystyle\int_{v(s_1+T_{n,1})}^{v(s_1)+nqp} h(C,v)\,\mathrm{d}v = 0$. Hence $v(s_1 + T_{n,1}) = v(s_1) + nqp$ since $h(c,v) > 0$.
Then, for s_1 fixed, and for every integer n, there exists $T_{n,1} > 0$ such that $v(s_1 + T_{n,1}) = v(s_1) + nqp$. Hence $v(s)$ is not bounded.

- $u(s)$ is bounded: then there exists $u_\infty \in \mathbb{R}$ such that $\lim\limits_{s\to+\infty} u(s) = u_\infty$. Let us proceed by contradiction. Suppose there exists $v_\infty \in \mathbb{R}$ such that $\lim\limits_{s\to+\infty} v(s) = v_\infty$. Then when $s \to +\infty$, $\mathbf{S}(u(s), v(s))$ converges to the point $\mathbf{S}(u_\infty, v_\infty)$. On the other hand, by system (3) we obtain that $\lim\limits_{s\to+\infty} \dot{u}(s) = C/f^2(v_\infty)$ and that $\dot{v}(s)$ converges to some value that we denote by $\dot{v}_\infty$ whenever $s \to +\infty$. Consequently, the point $\mathbf{S}(u_\infty, v_\infty)$ is a limit point of the geodesic Γ whose tangent at this point is given by the vector
$$\boldsymbol{\tau}_\infty = \begin{pmatrix} \dot{v}_\infty f'(v_\infty)\cos(u_\infty) - \dot{u}_\infty f(v_\infty)\sin(u_\infty) \\ \dot{v}_\infty f'(v_\infty)\sin(u_\infty) + \dot{u}_\infty f(v_\infty)\cos(u_\infty) \\ \dot{v}_\infty g'(v_\infty) \end{pmatrix}.$$

> However, there exists a geodesic, which is the continuity of Γ, starting from the point $\mathbf{S}(u_\infty, v_\infty)$ with tangent vector $\boldsymbol{\tau}_\infty$. Therefore the point $\mathbf{S}(u_\infty, v_\infty)$ is not a limit point of the geodesic Γ which leads to a contradiction. Consequently, $v(s)$ is not bounded.

Now let s_0 be any value, then there exists $T_0 > 0$ such that $v(s_0+T_0) = v(s_0) + qp$. We have: $\varphi(C) = \dfrac{q'}{q}$, i.e., $\displaystyle\int_0^p h(C,v)\,\mathrm{d}v = \frac{q'}{q}$. But f and g are periodic of period p, then $\displaystyle\int_{v(s_0)}^{v(s_0)+qp} h(C,v)\,\mathrm{d}v = q'$. Making the change of variables $v = v(s)$, and taking into account the system (3) we obtain: $\displaystyle\int_{s_0}^{s_0+T_0} \dot{u}(s)\,\mathrm{d}s = 2q'\pi$, then $u(s_0+T_0) = u(s_0) + 2q'\pi$.
Moreover, using the fact that $v(s_0+T_0) = v(s_0) + qp$, the system (3) gives $\dot{u}(s_0+T_0) = \dot{u}(s_0)$ and $\dot{v}(s_0+T_0) = \dot{v}(s_0)$. With these results we easily verify that the tangent vectors to the geodesic at $s = s_0$ and at $s = s_0 + T_0$ coincide. Hence the geodesic in question is periodic. ∎

Proposition 10. *The function $\varphi : c \in [0, m[\mapsto \varphi(c) \in \mathbb{R}$ is strictly increasing.*

Proof: It is obvious that φ is a continuous function on $[0, m[$ and differentiable on $]0, m[$ and we have:

$$\begin{aligned} \varphi'(c) &= \int_0^p \frac{\partial h(c,v)}{\partial c}\,\mathrm{d}v \\ \frac{\partial h(c,v)}{\partial c} &= \frac{1}{2\pi}\frac{f^2(v)}{f^2(v)-c^2}\sqrt{\frac{f'^2(v)+g'^2(v)}{f^2(v)(f^2(v)-c^2)}}. \end{aligned} \tag{9}$$

Since $\dfrac{\partial h(c,v)}{\partial c}$ is positive for every $c \in]0, m[$ so as $\varphi'(c)$. Hence the function φ is strictly increasing. ∎

Proposition 11. *If the function f is of class C^1 and twice differentiable, then*

$$\lim_{c\to m} \varphi(c) = +\infty. \tag{10}$$

Proof: Let $I(c) = \displaystyle\int_0^p \sqrt{\frac{f'^2(v)+g'^2(v)}{f^2(v)(f^2(v)-c^2)}}\,\mathrm{d}v.$
Since f' and g' are continuous over $\mathbb{R}$, p-periodic and ${f'}^2 + {g'}^2 \neq 0$, then there

exists $\gamma > 0$ such that $f^{'2} + g^{'2} \geq \gamma^2$. On the other hand f is continuous over $\mathbb{R}$ and p-periodic, then there exists M such that $0 < m \leq f(v) \leq M$ for every $v \in \mathbb{R}$. We deduce:

$$I(c) \geq \frac{\gamma}{M} \int_0^p \frac{1}{\sqrt{f^2(v) - c^2}} \, \mathrm{d}v, \quad \text{for all } c \in [0, m[. \tag{11}$$

Let us now study $J(c) = \displaystyle\int_0^p \frac{1}{\sqrt{f^2(v) - c^2}} \, \mathrm{d}v$. It is sufficient to study it for $c_n^2 = m^2 - \dfrac{1}{n}$. We have

$$J(c_n) = \int_0^p \frac{1}{\sqrt{f^2(v) - m^2 + \frac{1}{n}}} \, \mathrm{d}v.$$

It is clear that the sequence $n \mapsto \dfrac{1}{\sqrt{f^2(v) - m^2 + \frac{1}{n}}}$ is increasing for n (for v fixed). Then $J(c_n)$ is an increasing positive sequence. We distinguish two cases:

- either $J(c_n)$ is bounded above. Then $\psi(v) = \dfrac{1}{\sqrt{f^2(v) - m^2}}$ is integrable and $\lim\limits_{n \to +\infty} J(c_n) = \displaystyle\int_0^p \frac{1}{\sqrt{f^2(v) - m^2}} \, \mathrm{d}v$.
- or $\lim\limits_{n \to +\infty} J(c_n) = +\infty$.

Let us show that the first case cannot happen. Let $v_0 \in [0, p]$ such that f attains its minimum at v_0, we know that $f'(v_0) = 0$. Since f is of class C^1 and twice differentiable, then:

$$f(v) = f(v_0) + f'(v_0)(v - v_0) + \eta(v) \text{ with } \eta \in C^1 \text{ and } |\eta(v)| \leq k_\eta |v - v_0|^2.$$

So $f^2(v) = m^2 + 2m\eta(v) + \eta^2(v) = m^2 + \tilde{\eta}(v)$, with $0 \leq \tilde{\eta}(v) \leq k_{\tilde{\eta}} |v - v_0|^2$. Therefore we have: $\dfrac{1}{\sqrt{f^2(v) - m^2}} = \dfrac{1}{\tilde{\eta}(v)^{1/2}} \geq \dfrac{1}{k_{\tilde{\eta}}^{1/2} |v - v_0|}$ which is not integrable in the neighborhood of v_0. Consequently, we have $\lim\limits_{c \to m} I(c) = +\infty$ and $\lim\limits_{c \to m} \varphi(c) = +\infty$. ■

Corollary 12. $\operatorname{im} \varphi = \varphi([0, m[) = [0, +\infty[$.

Proof: It is sufficient to see that φ is continuous, increasing, $\varphi(0) = 0$ and $\lim\limits_{c \to m} \varphi(c) = +\infty$. ∎

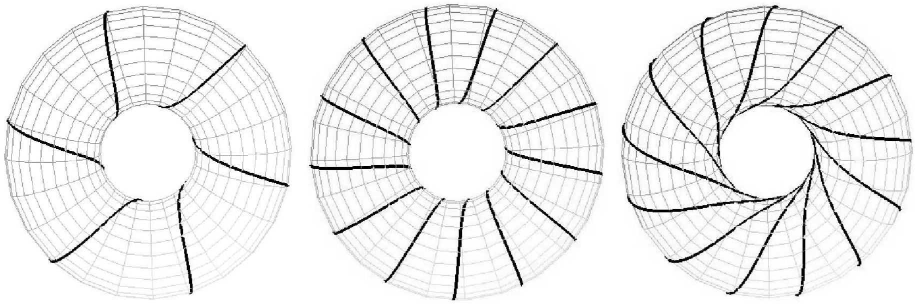

Figure 1. Examples of closed geodesics on a torus: left $(k = 6, k' = 1)$, middle $(k = 13, k' = 1)$, and right $(k = 13, k' = 5)$.

Remark 13. *The geodesic corresponding to a Clairaut's constant $C = m$ does not make any complete rotation about a meridian curve. Indeed it corresponds to the case $k = 0$. Therefore, if the geodesic is not a parallel, then it oscillates back and forth across a parallel.*

Proposition 14. *The integers k and k' of equation (4) are relatively prime.*

Proof: We notice that in equation (4), T is the period of the geodesic (Γ, ϕ), and k has the same signe as $\dot{v}$ that we assume to be positive.
Let q and $q' \in \mathbb{N}$ two integers relatively prime such that $\dfrac{k'}{k} = \dfrac{q'}{q}$. Then we have: $\varphi(C) = \dfrac{q'}{q}$. This implies by Theorem 9 that there exists $T_1 > 0$ such that:

$$u(s + T_1) = u(s) + 2q'\pi, \qquad v(s + T_1) = v(s) + qp. \tag{12}$$

We can easily verify that $\phi(s + T_1) = \phi(s)$, therefore we deduce that there exists an integer n such that $T_1 = nT$. Using equations (4) and (12), we obtain $q = nk$ et $q' = nk'$. However q and q' are relatively prime, this implies that $n = 1$, and consequently, k and k' are relatively prime. ∎

3. Validation

The problem of existence of a closed geodesic making at least one winding over a toroidal surface consists of finding a value of the Clairaut's constant C such that $\varphi(C) \in \mathbb{Q}$. Since $\mathrm{im}\ \varphi = [0, +\infty[$, there exist infinitely many values of C that have their images by φ in $\mathbb{Q}$. Consequently, on a toroidal surface, there exist infinitely many closed geodesics making at least one winding. Moreover, the number of rotations about the axis of revolution of the surface and the number of windings around the equator of the surface of a closed geodesic are relatively prime.

Algorithm: In order to compute a closed geodesic, we choose the initial values $u(0)$ and $v(0)$, and the number k' of rotations about the axis of revolution and the number k of windings about the equator that the geodesic will make. The Clairaut's constant is then obtained by solving equation (6). Using system (3), we compute the initial values $\dot{u}(0)$ and $\dot{v}(0)$ which in turn determine the tangent vector. Therefore, the equations of the geodesic can be obtained by solving system (3) numerically. We have implemented this algorithm in MAPLE, in particular, we have used the ordinary differential equation toolbox in order to solve the non-linear system whose unknowns are $(u(s), v(s))$ which lead to the equation of the geodesic.

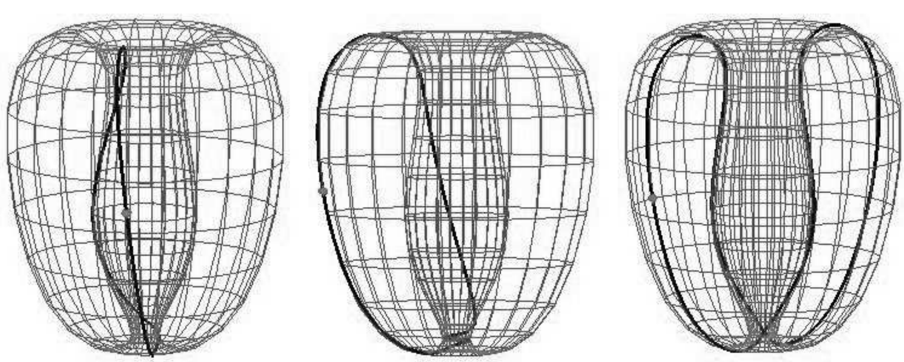

Figure 2. Closed geodesics on the model surface of the left ventricle: left $(k = 1, k' = 1)$, middle $(k = 1, k' = 2)$, and right $(k = 2, k' = 1)$. The bullets shown on the geodesics represent the starting point where the geodesics must close on themselves.

We have validated this result on two different examples. The first example corresponds to a torus. Fig. 1 shows three closed geodesics making different numbers of rotations k and k'.

The second example corresponds to a geometrical model of the left ventricle which can be considered as a toroïdal surface. The meridian curve corresponding to this model is a crescent-shaped curve (or like a bean seed) and it has been obtained using B-slpines. We illustrate in Fig. 2 three closed geodesics, with different rotation numbers, on the left ventricle model. In order to see the pattern of the geodesics on the surface of the internal cavity of the left ventricle model, we have made the surface transparent but we have left some vertical and horizontal contours in order to delimit the surface boundary. These closed geodesics are conceived as cardiac fibers running from the base (the top of the left ventricle) to the apex (the bottom of the left ventricle), so that they pass from the epicardium (external surface of the heart) into the endocardium (the surface of the cavity of the ventricles) through the apex, and they close on themselves after making a number of rotations. It has been mentionned in several papers that some of the fibers have the form of a figure eight, see for instance [12]. This description was given by researchers who tried to peel the fibers in order to discover their organization. We can see in Fig. 2 (middle) that the geodesic has the shape of a figure eight which is consistent with what was described in cardiac histology.

4. Geodesics and Cardiac Fibers

Since Streeter conjectured in 1979 that cardiac fibers of the left ventricle free wall run as geodesics on a nested set of toroïdal surfaces, it has not been possible to check this conjecture because of the lack of data about fiber directions until new sophiticated techniques of imaging became available. For instance, Jouk *et al* [5] have developed a new technique based on polarized light microscopy in order to measure the elevation and the azimuth angles of cardiac fibers in human fetal hearts. Since then, data about the orientation of fibers became available and we were able to start checking Streeter's conjecture using Jouk *et al* data. In [9], based on the Clairaut's constant in the case of surfaces of revolution, Mourad *et al* have checked the validity of the conjecture only for the left ventricle because the right ventricle has a complicated geometrical shape and the Clairaut's constant cannot be used anymore.

On the left of Fig. 3, some sections of the ventricular part of the heart from top

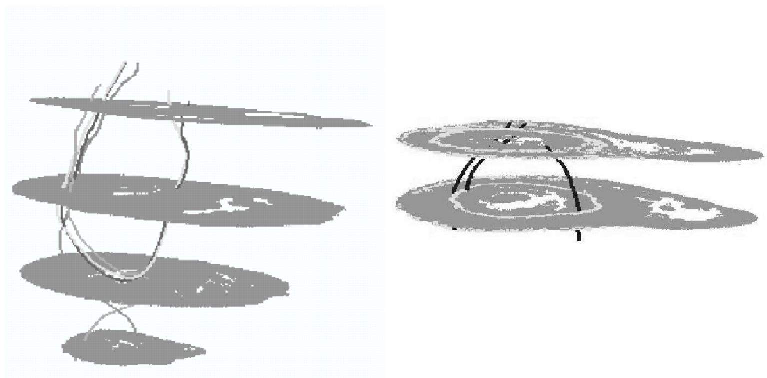

Figure 3. Left: Horizontal sections of the heart and some cardiac fibers in the left ventricle. Right: Cardiac fibers crossing the isolines of the Clairaut's constant in horizontal sections of the left ventricle (from [9] with permission).

(base) to bottom (apex) are shown along with some cardiac fibers in the left ventricle. We notice that these fibers do not close on themselves in order to get closed curves. This can be explained by the two sources of errors: first, we are using experimental data that may contain some noise, second we are using numerical methods to track fiber trajectories where the accumulation of the errors may lead to this problem. Moreover, with the measurements we have, at the top of the ventricular part (the base), the data are not complete because of technical limitations of the measurement process.

On the right of Fig. 3, we show two horizontal sections of the heart and three fibers in the left ventricle. In each section, we show the isolines of a given value of the Clairaut's constant and we see how the fibers cross these isolines from section to section. This means that along these fibers the Clairaut's constant is the same which backs up the idea of geodesics and is consistent with Streeter's conjecture. However, this is not sufficient to validate the conjecture and further study is necessary.

On the other hand, a mechanical explanation of the structure of geodesics in the heart is still to be done. Mathematical models of left ventricular wall mechanics have shown that the distribution of fiber strain during ejection is sensitive to the orientation of muscle fibers in the wall, see for instance [5]. In [11], it has been shown that the left ventricle structure is designed for maximum homogeneity of fiber strain during ejection. So this might be a physiological reason why geodesic fibers would be best. However, this requires more investigation of the mathematical properties of geodesics and their role in the uniform distribution of fiber strain during ejection.

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