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GENERALIZATION OF HADAMARD'S LAPLACE EIGENVALUE FORMULA TO DEFORMING MANIFOLDS

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Abstract. For a Euclidean domain with a moving boundary, Hadamard's formula relates the rate of change of the Laplace eigenvalues to the normal velocity of the boundary. We generalize Hadamard's formula to deforming Riemannian maniforlds with contour boundary moving in a compatible manner. Our analysis finds direct applications in the dynamics of fluid films. The spectrum of the surface Laplacian describes the frequencies of normal oscillations of the film's surface as well as tangential oscillations in thickness.

1. Introduction

How do the eigenvalues of the Laplace operator depend on the shape of the domain? This question was originally posed by Hadamard.

In 1908, having planted the seeds of what we now call the calculus of moving surfaces, Hadamard established an expression for the rate of change in Laplace eigenvalues for a deforming domain Ω – thereby collecting the first fruits of the new calculus [3]. This subject has been an area of active research. The reader will find excellent reviews in [4] and [12] along with many useful references therein.

Hadamard's formula applies to the Laplace operator on Euclidean domains with deforming boundaries. This goal of this paper is to establish analogous results for the surface Laplacian on deforming Riemannian manifolds with moving contour boundaries.

The surface Laplacian $\nabla_\alpha \nabla^\alpha$ finds a great number of applications. In [1], we present exact Hamiltonian equation for the dynamics of fluid films. We show that the equation that governs small normal oscillations about an equilibrium configuration is $u_{tt} = c^2 \left(\nabla_\alpha \nabla^\alpha - 2K\right) u$, where K is Gaussian curvature. Furthermore, small oscillations in thickness ρ are governed by the surface wave equation $\rho_{tt} = c^2 \nabla_\alpha \nabla^\alpha \rho$. These examples demonstrate applications of the surface Laplacian and illustrate the immediate relevance of its spectrum.

We begin by describing Hadamard's classical results [3]. Let S be the boundary of Ω and C be the normal velocity of S. Hadamard's result for Dirichlet boundary conditions reads (Hadamard's Formula)

$$\lambda'(\tau) = -\int_S C \; |\nabla\psi|^2 \, \mathrm{d}S \tag{1}$$

where λ is the eigenvalue, ψ is the corresponding eigenfunction and τ is a parametrization of the evolution. The normal velocity C, introduced by Hadamard, is the central object in the formalism of moving surfaces. It is constructed geometrically as the limit

$$C = \lim_{\Delta\tau\to 0} \frac{\Delta \mathrm{d}}{\Delta\tau} \tag{2}$$

where $\Delta\mathrm{d}$ is the normal distance between the surface at times τ and $\tau + \Delta\tau$. The quantity C is signed and depends on the choice of the normal.

To construct C algebraically, we must specify the evolution of the boundary

$$\mathbf{Z} = \mathbf{Z}(\tau, S) \tag{3}$$

where $\mathbf{Z}$ is a position vector in the ambient Euclidean space and S^α are the surface coordinates. We drop the tensor indices of function arguments and write $\mathbf{Z}(\tau, S)$ instead of $\mathbf{Z}(\tau, S^\alpha)$. Because the surface is moving, one must have a rule for constructing the surface coordinates S^α at every time τ. Given the evolution (3), C can be expressed as

$$C = \frac{\partial \mathbf{Z}(\tau, S)}{\partial\tau} \cdot \mathbf{N} \tag{4}$$

where $\mathbf{N}$ is the unit normal vector. By drawing a simple picture, one can be easily convinced that the algebraic and the geometric definitions of C in equations (2) and (4) are equivalent. A crucial implication is that $\partial\mathbf{Z}(\tau, S)/\partial\tau \cdot \mathbf{N}$ is independent of the choice of the surface coordinates (whereas $\partial\mathbf{Z}(\tau, S)/\partial\tau$ alone surely is not). Note that C is a scalar, but the term *velocity* is appropriate since the normal direction is implied. When one draws the velocity of the interface as a vector, one simply means $C\mathbf{N}$, as we do in Fig. 1.

Hadamard's result applies to three dimensional domains with surface boundaries, two dimensional domains with contour boundaries, and – with a little care – one dimensional segments with point boundaries.

Problems arise that require Hadamard's result to be generalized in a number of ways. One generalization involves Ω embedded in a stationary Riemannian manifold – for instance, a spherical shell – rather than a Euclidean space. In a very

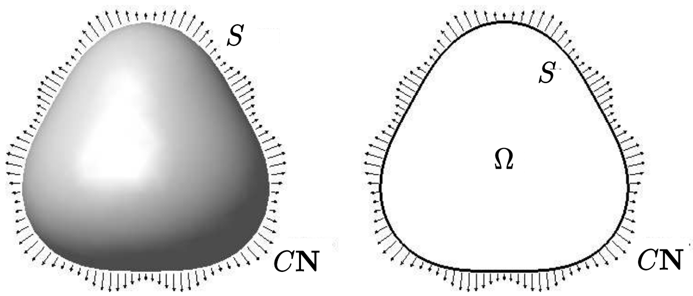

Figure 1. Hadamard classic formula applies to eigenvalues of the bulk Laplace operator in any dimension.

simplistic way, this could be a model for standing waves in oceans. A much more interesting and difficult problem, from the point of view of moving surfaces, is to allow the ambient manifold itself to deform while the boundary S evolves within the manifold as shown in Fig. 2.

In this paper, we focus on the second, more challenging, problem. The challenge is especially great when one intends to calculate the second derivative $\lambda''(\tau)$, which is beyond the scope of this paper. We first analyze a deforming closed surface (no boundary). We subsequently add an additional term associated with the motion of the contour boundary.

A change in notation will be helpful. We let S be the deforming domain and call its boundary γ. This notation is more convenient because the canonical choice of letters for a deforming surface in moving surfaces formulas is S rather than Ω.

2. Statement of the Problem

Consider a deforming closed surface $S(\tau)$. The Laplace eigenvalue problem on S consists of the bulk equation

$$\nabla_\alpha \nabla^\alpha \psi = -\lambda \psi \tag{5}$$

and the normalization condition

$$\int_S \psi^2 \mathrm{d}S = 1. \tag{6}$$

The covariant and contravariant derivatives ∇_α and ∇^α are defined with respect to the surface metric $S_{\alpha\beta}$. Our analysis relies heavily on tensor calculus and on

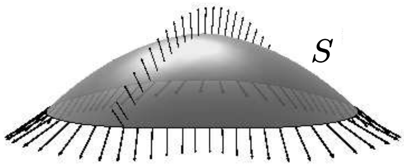

Figure 2. A generalization of Hadamard's problem: a deforming domain with a moving contour boundary.

the calculus of moving surfaces. We refer the reader to [7] and [8] for excellent sources on tensor calculus and to [1] and [2] for detailed overviews of the calculus of moving surfaces.

We consider simple eigenvalues λ. For these, ψ is unique save for the sign. The corresponding eigenvalue can be expressed in terms of the eigenfunction by the Rayleigh quotient (with unit denominator)

$$\lambda = \int_S \nabla_\alpha \psi \nabla^\alpha \psi \mathrm{d}S. \tag{7}$$

This expression provides a convenient starting point for our analysis.

We imagine that the manifold S deforms with velocity C. As S evolves, so do ψ, λ and all characteristics of the manifold S. They can therefore be treated as functions of τ.

To keep our notation as uncluttered as possible, we list function arguments only to highlight a particular dependence. We write

$$\lambda(\tau) = \int_S \nabla_\alpha \psi \nabla^\alpha \psi \mathrm{d}S \tag{8}$$

and it is to be remembered that S, ψ, and the metrics implicitly contained in ∇_α, ∇^α, and $\mathrm{d}S$ are functions of τ. Our goal is to evaluate $\lambda'(\tau)$.

A number of approaches have been applied to eigenvalue variations. One method in particular deserves a mention due to its attractive simplicity. It applies to smooth variations of domains whose eigenvalues are known. The method, used successfully by Migdal [9], (see also [6]) to estimate the spectrum of an electron trapped in a slightly ellipsoidal cavity, involves a change of variables that transfers the perturbation from the boundary to the differential operator. This approach, however, typically requires that perturbations are smooth and regular, whereas Hadamard's approach is less restrictive [5].

From a computational point of view, the level set method [10] has been used with success to eigenvalue optimization problems [11].

3. Analysis

We differentiate both sides of (8) with respect to τ. The closed-surface integral is differentiated according to the formula

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_S F\,\mathrm{d}S = \int_S \frac{\delta F}{\delta\tau}\mathrm{d}S - \int_S CB_\alpha^\alpha F\,\mathrm{d}S \tag{9}$$

where B_α^α is the trace of the curvature tensor B_β^α and $\delta/\delta\tau$ is the invariant surface derivative introduced by Hadamard. Hadamard's original definition applied to scalar fields. We use a generalized definition that is applicable to tensors, such as $\nabla_\alpha\psi$.

It is easy to show that the $\delta/\delta\tau$-derivative commutes with the covariant derivative ∇_α. For this reason, we rewrite equation (8) strictly in terms of covariant derivatives

$$\lambda(\tau) = \int_S S^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi\mathrm{d}S \tag{10}$$

where $S^{\alpha\beta}$ is the contravariant metric tensor [8], [7], [1].

An application of (9) to equation (10) yields

$$\lambda'(\tau) = \int_S\left(\frac{\delta S^{\alpha\beta}}{\delta\tau}\nabla_\alpha\psi\nabla_\beta\psi + 2S^{\alpha\beta}\nabla_\alpha\frac{\delta\psi}{\delta\tau}\nabla_\beta\psi - CB_\alpha^\alpha\nabla_\beta\psi\nabla^\beta\psi\right)\mathrm{d}S. \tag{11a}$$

In deriving equation (11a) we used the product rule and subsequently took advantage of the symmetry of $S^{\alpha\beta}$ to combine two terms. For the next step, recall that $\delta S^{\alpha\beta}/\delta\tau = 2CB^{\alpha\beta}$, [1] and use Gauss's theorem to transfer ∇_α from $\delta\psi/\delta\tau$ to $\nabla_\beta\psi$. There is no boundary term, since our integration domain has no boundary. We have

$$\lambda'(\tau) = \int_S\left(2CB^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi - 2S^{\alpha\beta}\frac{\delta\psi}{\delta\tau}\nabla_\alpha\nabla_\beta\psi - CB_\alpha^\alpha\nabla_\beta\psi\nabla^\beta\psi\right)\mathrm{d}S. \tag{11b}$$

Now recognize that $S^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi = \nabla_\alpha\nabla^\alpha\psi = -\lambda\psi$. Our expression therefore reads

$$\lambda'(\tau) = \int_S \left(2CB^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + 2\lambda\psi\frac{\delta\psi}{\delta\tau} - CB^\alpha_\alpha\nabla_\beta\psi\nabla^\beta\psi\right)\mathrm{d}S. \qquad (11c)$$

We continue our analysis on the term containing $\delta\psi/\delta\tau$. Consider the time derivative of the normalization condition (6)

$$2\int_S \psi\frac{\delta\psi}{\delta\tau}\mathrm{d}S - \int_S CB^\alpha_\alpha\psi^2\mathrm{d}S = 0. \qquad (12)$$

Substitute this identity in equation (11c) to obtain the final expression (main result without boundary)

$$\lambda'(\tau) = \int_S C\left(\lambda B^\alpha_\alpha\psi^2 + 2B^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi - B^\alpha_\alpha\nabla_\beta\psi\nabla^\beta\psi\right)\mathrm{d}S. \qquad (13)$$

This expression shares a number of important features with Hadamard's classic formula (1). First, $\lambda'(\tau)$ can be computed without reference to the eigenfunction perturbation $\delta\psi/\delta\tau$. In this, Hadamard's formula (1) and equation (13) are consistent with the standard eigenvalue perturbation scheme of linear algebra.

The correctness of the formula can be checked in one trivial case. On any closed surface, there is a simple eigenvalue $\lambda = 0$ with $\psi = 1/\sqrt{Area}$ as the corresponding eigenfunction. For this eigenvalue, the first term in the integrand vanishes because $\lambda = 0$, the other two vanish due to the gradients, and we have $\lambda'(\tau) = 0$.

For another insightful example, consider a surface that has a section that is instantaneously flat and assume that only that part of the surface is instantaneously deforming. Then equation (13) implies that $\lambda'(\tau) = 0$ since all curvature elements vanish. It is easy to see why this is correct. When a flat surface deforms, C and $-C$ result in mirror changes of shape and therefore lead to identical changes in eigenvalues. This is because mirror images have identical metrics – and eigenvalues depend on the metrics alone.

In the following section we use a nontrivial torus example to illustrate the correctness of formula (13).

Suppose now that the surface S has a moving contour γ. Let the velocity of the contour γ within S be c. The main formula can be adapted to this case by adding an Hadamard term for the contour

$$\lambda'(\tau) = \int_S C\left(\lambda B^\alpha_\alpha\psi^2 + 2B^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi - B^\alpha_\alpha\nabla_\beta\psi\nabla^\beta\psi\right)\mathrm{d}S \qquad (14)$$

the main result with boundary

$$-\int_{\gamma} c\nabla_{\alpha}\psi\nabla^{\alpha}\psi \mathrm{d}\gamma.$$

This expression represents a very broad generalization of Hadamard's formula (1). For a manifold whose interior is at rest, it reduces to a formula nearly identical to Hadamard's, except the Euclidian gradient is replaced by the surface gradient. For *flat* stationary manifolds, Hadamard's very formula is recovered.

We make one final point regarding the motion of the contour. Its evolution is often described by vector velocity field $\mathbf{C}_S$ in the ambient Euclidean space. Then the motions of the contour and surface S are compatible when

$$C = \mathbf{C}_S \cdot \mathbf{N}. \tag{15}$$

The velocity of the contour $\mathbf{c}$ within the surface S is obtained from $\mathbf{C}_S$ as the tangential projection

$$\mathbf{c} = \mathbf{C}_S - (\mathbf{C}_S \cdot \mathbf{N})\,\mathbf{N}. \tag{16}$$

4. Torus Demonstration

Consider a torus with radii R and r referred to coordinates θ and ϕ

$$\begin{aligned} x &= (R + r\cos\phi)\cos\theta \\ y &= (R + r\cos\phi)\sin\theta \\ z &= r\sin\phi. \end{aligned} \tag{17}$$

The metric tensors are given by

$$S_{\alpha\beta} = \begin{bmatrix} (R + r\cos\phi)^2 & \\ & r^2 \end{bmatrix} \quad \text{and} \quad S^{\alpha\beta} = \begin{bmatrix} (R + r\cos\phi)^{-2} & \\ & r^{-2} \end{bmatrix} \tag{18}$$

and therefore the area element $\mathrm{d}S$ is proportional to the square root of the determinant of $S_{\alpha\beta}$

$$\mathrm{d}S = r\,(R + r\cos\phi)\,\mathrm{d}\theta\mathrm{d}\phi. \tag{19}$$

The curvature tensors with respect to the outward normal are given by

$$B_{\alpha\beta} = \begin{bmatrix} -(R + r\cos\phi)\cos\phi & \\ & -r \end{bmatrix}$$

$$B^{\alpha}_{\beta} = \begin{bmatrix} -(R + r\cos\phi)^{-1}\cos\phi & \\ & -r^{-1} \end{bmatrix} \tag{20}$$

$$B^{\alpha\beta} = \begin{bmatrix} -(R + r\cos\phi)^{-3}\cos\phi & \\ & -r^{-3} \end{bmatrix}.$$

The trace of B^{α}_{β} is the mean curvature B^{α}_{α}

$$B^{\alpha}_{\alpha} = -\frac{\cos\phi}{R + r\cos\phi} - \frac{1}{r}. \tag{21}$$

We consider an evolution that comes from a uniformly expanding radius r

$$\begin{aligned} x &= (R + (r + V\tau)\cos\phi)\cos\theta \\ y &= (R + (r + V\tau)\cos\phi)\sin\theta \\ z &= (r + V\tau)\sin\phi. \end{aligned} \tag{22}$$

Then the velocity field C is uniform and $C = V$. This allows a slight simplification of equation (13), since an application of Gauss's theorem to the last term eliminates $\lambda B^{\alpha}_{\alpha}\psi^2$

$$\lambda'(\tau) = C\int_S \left(2B^{\alpha\beta}\nabla_\alpha\psi\nabla_\beta\psi + \psi\nabla^\beta B^\alpha_\alpha\nabla_\beta\psi\right)\mathrm{d}S. \tag{23}$$

Simple eigenvalues on the torus are functions only of the coordinate ϕ. Therefore, (23) reduces to

$$\lambda'(\tau) = C\int_S \left(2B^{\Phi\Phi}\psi_\phi^2 + \psi r^{-2}\left(B^\alpha_\alpha\right)_\phi\psi_\phi\right)\mathrm{d}S \tag{24}$$

which, in arithmetic form, reads ($\rho = R/r$)

$$\lambda'(\tau) = -\frac{2\pi C}{r}\int_0^{2\pi}\left(2\psi_\phi^2 - \frac{\rho\sin\phi}{(\rho+\cos\phi)^2}\psi\psi_\phi\right)(\rho + \cos\phi)\,\mathrm{d}\phi. \tag{25}$$

This expression evaluates numerically to $-2.00572986192\ C$. We also have the following numerical estimates for eigenvalues

$$\begin{aligned} \lambda_{R=1,r=1} &= 0.9767313134... \\ \lambda_{R=1,\ r=1+10^{-6}} &= 0.9767293077... \end{aligned} \tag{26}$$

These estimates imply a rate of change of

$$\frac{\lambda_{R=1,r=1+10^{-6}} - \lambda_{R=1,r=1}}{10^{-6}} = -2.00572686687... \tag{27}$$

which is convincingly close ($\approx 3 \times 10^{-6}$) to the value of integral (25).

5. Conclusion

We have derived a formula for the derivative of simple eigenvalues of the surface Laplacian. When the surface in question has a contour boundary, the formula is valid for Dirichlet boundary conditions. Equation (14) applies to deforming manifolds with moving boundaries. For flat manifolds, the formula reduces to Hadamard's formula. For closed manifolds, it simplifies to equation (13). The formulas were obtained in the framework of the formalism of moving surfaces.

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