JGSP **13** (2009) 75–88

JOURNAL OF
Geometry and Symmetry
in Physics

EXISTENCE, UNIQUENESS, AND ANGLE COMPUTATION FOR THE LOXODROME ON AN ELLIPSOID OF REVOLUTION

ABED ELHASHASH

Communicated by Gregory L. Naber

Abstract. We summarize a proof for the existence and uniqueness of the loxodrome joining two distinct points p_o and p_1 on an open half of an ellipsoid of revolution. We also compute the unique angle $\alpha \in [0, 2\pi)$ which the loxodrome makes with the meridians intersecting the loxodrome.

1. Introduction

A loxodrome on an ellipsoid of revolution is a curve that traverses all the meridians along its way at a constant angle. Since the earth is modeled as an ellipsoid of revolution, understanding loxodromes plays an important role in the science of navigation; see, e.g., [4–6, 9]. The existence and uniqueness of a loxodrome on an ellipsoid of revolution and a formula for its angle are known results; see, e.g., [4, 9].

Typically, the existence of a loxodrome on an ellipsoid of revolution is proved by constructing a one-to-one conformal map (is a continuously differentiable map that preserves angles between curves) Ψ from the open square $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}^2$ onto an open connected subset of the ellipsoid of revolution which contains the points p_o and p_1 that are meant to be joined by a loxodrome. The map Ψ is such that every vertical straight line segment in the open square $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is mapped onto a meridian of the ellipsoid of revolution. Thus, if $q_o = \Psi^{-1}(p_o)$ and $q_1 = \Psi^{-1}(p_1)$, then q_o and q_1 can be joined by a straight line segment in the open square $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (which makes a constant angle with all vertical straight line segments on its way) and the image of that straight line segment joining q_o to q_1 under Ψ will be a curve on the ellipsoid of revolution joining p_o to p_1 which makes a constant angle with all meridians on its way (because Ψ is conformal). Moreover, the constant angle that the loxodrome makes with all the meridians is typically computed by using the technique of "infinitesimals"; see, e.g., [5, 6].

Our main aim in this paper is to present a rigorous mathematical proof that avoids the use of conformal maps for the existence and uniqueness of a loxodrome joining two distinct points p_o and p_1 that lie on an ellipsoid of revolution and that differ by less than π radians (modulo 2π) in longitude and to obtain a formula for the constant angle that the loxodrome makes with the meridians without the use of "infinitesimals".

In Section 2, we define the ellipsoid of revolution $\mathcal{E} \subset \mathbb{R}^3$ to be considered throughout this paper and we introduce some definitions and notation that we use in later sections. Then, in Section 3, we parameterize $\mathcal{E}^+ := \mathcal{E} \cap \{(x, y, z) \in \mathbb{R}^3;\, x > 0\}$, an open *half* of the ellipsoid $\mathcal{E}$, by latitude and longitude using a diffeomorphism ψ. After that, we use the diffeomorphism ψ to parameterize unit tangent vectors pointing towards the *north* and the *east*, respectively, at any point on $\mathcal{E}^+$ in terms of the longitude and latitude of that given point. We use the latter parametrization to formulate the problem of finding a loxodrome joining two distinct points in $\mathcal{E}^+$ and we solve the problem in Section 4.

2. Definitions and Notation

Let a and b be real scalars such that $a \geq b > 0$ and consider the set $\mathcal{E} := \{(x, y, z) \in \mathbb{R}^3;\, \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1\}$. Then, $\mathcal{E}$ is an ellipsoid of revolution. Any point $p = (x, y, z) \in \mathcal{E}$ other than $(0, 0, b)$ and $(0, 0, -b)$ can be located by two real numbers: λ in the interval $(-\pi, \pi]$ and ϕ in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where $\lambda = \lambda(p)$ is the *longitude* of $p \in \mathcal{E}$, i.e., the counter-clockwise angle measured in the xy-plane from the positive x-axis to the orthogonal projection (onto the xy-plane) of the straight line segment joining the origin to p, and $\phi = \phi(p)$ is the *latitude* of a point $p \in \mathcal{E}$, i.e., the angle (of elevation) that the normal line to $\mathcal{E}$ through p makes with the xy-plane; refer to Fig. 1 for illustrations. In particular, if the point $p = (x, y, z) \in \mathcal{E}$ satisfies $x > 0$ then the longitude $\lambda = \lambda(p)$ can be chosen in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, we note that for such a point $p = (x, y, z) \in \mathcal{E}$ having $x > 0$ the longitude is positive if and only if $y > 0$ and that the latitude is positive if and only if $z > 0$. Define $\mathcal{E}^+ := \{(x, y, z) \in \mathcal{E};\, x > 0\}$. By a *meridian* in $\mathcal{E}^+$, we mean a curve in $\mathcal{E}^+$ whose points have the same longitude and which is oriented in such a way that it starts at $(0, 0, -b)$ and ends at $(0, 0, b)$. Any curve in $\mathcal{E}$ obtained by rotating a meridian in $\mathcal{E}^+$ around the z-axis shall also be called a meridian. By a *semi-circle of latitude* in $\mathcal{E}^+$, we mean a curve in $\mathcal{E}^+$ whose points have the same latitude and which is oriented in a way that it starts at the point with the negative y-component and ends at the point with the positive y-component. If p is a point in $\mathcal{E}$ other than

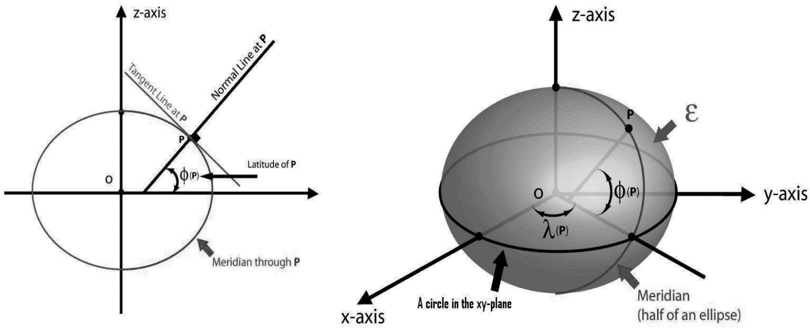

Figure 1. The longitude and latitude of a point p.

$(0, 0, b)$ and $(0, 0, -b)$ then we say that a non-zero tangent vector to the meridian at p points towards the *north* if its z-component is positive. If p is a point in $\mathcal{E}^+$ then we say that a non-zero tangent vector to the semi-circle of latitude at p points towards the *east* if its y-component is positive. A *loxodrome* in $\mathcal{E}$ is a curve that intersects all the meridians on its way at a constant angle.

Furthermore, the vectors that we mention in this paper are all real two-dimensional or three-dimensional column vectors, i.e., 3-by-1 or 2-by-1 matrices with real entries. Moreover, we will enclose the entries of a vector by a pair of brackets to distinguish it from a point. The superscript ^{T} will denote the transpose of a vector (treated as a matrix). For example, the vector $\mathbf{u} = [4\ 5\ 6]^T$ is the transpose of the 1-by-3 matrix $[4\ 5\ 6]$ and hence it is the 3-by-1 matrix whose (1,1)-entry is 4, (2,1)-entry is 5, and (3,1)-entry is 6.

If $\mathcal{M}, \mathcal{N} \subseteq \mathbb{R}^3$ are real submanifolds of $\mathbb{R}^3$ (see, e.g., [7, 8]), p is a point in $\mathcal{M}$, and $\psi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism (see, e.g., [7, 8]), then $\mathrm{d}\psi|_p$ will denote the derivative of ψ at the point p. If it is clear at which point the derivative is being taken then we just write $\mathrm{d}\psi$.

Remark 1. *We try to use the standard notation used by other authors in this subject. For instance, most of our notation is consistent with Pearson's notation* [5]. *Moreover, throughout this paper, we use the Greek letter* λ (*respectively, the Greek letter* ϕ) *sometimes to denote the longitude variable* (*respectively, the latitude variable*) *and some other times to denote a parametrization of that variable. To avoid confusion, the following convention is adopted*: *if the Greek letter* λ (*respectively, the Greek letter* ϕ) *is followed by a pair of parentheses enclosing a variable in* $\mathbb{R}$, *then it denotes the longitude variable* (*respectively, latitude vari-*

able) parameterized by the given real variable and if it is not followed by a pair of parentheses enclosing a variable then it simply denotes the longitude variable (respectively, the latitude variable).

For example, if $t \in \mathbb{R}$ then we have the following.

$$\begin{aligned}
\phi &= \textit{the latitude variable.}\\
\lambda &= \textit{the longitude variable.}\\
\phi(t) &= \textit{the latitude variable parametrized by a real variable } t.\\
\lambda(t) &= \textit{the longitude variable parametrized by a real variable } t.\\
\lambda(\phi(t)) &= \textit{the longitude variable parametrized by } \phi(t).\\
\lambda(\phi) &= \textit{the longitude variable parametrized by the latitude } \phi.
\end{aligned}$$

3. Parameterizing $\mathcal{E}^+$ and Stating the Problem

Since $\mathcal{E}$ is invariant under rotations around the z-axis and since we are interested in proving the existence and uniqueness of a loxodrome joining two distinct points whose difference in longitude is less than π radians (modulo 2π), it is enough to prove the existence and uniqueness of a loxodrome joining two distinct points in the subset $\mathcal{E}^+$ of the ellipsoid $\mathcal{E}$. To that end, we will do the following:

- we will parameterize $\mathcal{E}^+$ in terms of latitude and longitude,
- we will express the unit tangent vectors pointing towards the north and the east at a point on some curve η in $\mathcal{E}^+$ in terms of the latitude, the longitude, and the parameter of the curve η, and
- we will state the problem of finding the loxodrome in $\mathcal{E}^+$.

In the following lemma, we give a parametrization of $\mathcal{E}^+$ in terms of latitude and longitude. We postpone the proof of this lemma to the **Appendix** and we note that the parametrization presented in this lemma can be obtained from [5] or [9]. The details that we mention about the construction of this parametrization are presented for the sake of completeness.

Lemma 2. *Let J denote the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and let $e = \left((a^2 - b^2)/a^2\right)^{1/2}$ be the eccentricity of the ellipsoid $\mathcal{E}$. If the map $\psi : J \times J \to \mathcal{E}^+$ is defined by*

$$\psi(\phi, \lambda) = (R(\phi)\cos\lambda, R(\phi)\sin\lambda, R(\phi)(1 - e^2)\tan\phi) \tag{1}$$

where

$$R(\phi) = a\cos\phi(1 - e^2\sin^2\phi)^{-1/2} \tag{2}$$

then the map ψ is a diffeomorphism and its inverse diffeomorphism $\psi^{-1} : \mathcal{E}^+ \to J \times J$ is given by:

$$\psi^{-1}(x, y, z) = \left(\arctan\left(\frac{z}{(1-e^2)\sqrt{x^2+y^2}}\right), \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right). \tag{3}$$

Moreover, the first entry (*respectively, the second entry*) *of $\psi^{-1}(x, y, z)$ is the latitude* (*respectively, the longitude*) *of the point (x, y, z).*

The following lemma gives explicitly the entries of the unit tangent vector pointing towards the north (respectively, the east) at a point on some curve in $\mathcal{E}^+$ in terms of the latitude, the longitude, and the parameter of the curve. The proof of this lemma is postponed to the **Appendix**.

Lemma 3. *Let I denote a non-empty open interval and suppose that $\eta : I \to \mathcal{E}^+$ is a curve and that γ is the image of the curve η under the diffeomorphism ψ^{-1} defined in equation* (3). *If $\mathbf{N_t}$* (*respectively, $\mathbf{E_t}$*) *denotes the three-dimensional unit vector at $\eta(t)$ that depends smoothly on t and that points towards the north* (*respectively, the east*), *and $\gamma(t)$ is given by $\gamma(t) = (\phi(t), \lambda(t))$ for all $t \in I$, then the following statements are true*

- $\mathbf{N_t} = [-\sin\phi(t)\cos\lambda(t) \quad -\sin\phi(t)\sin\lambda(t) \quad \cos\phi(t)]^T$ *and*
- $\mathbf{E_t} = [-\sin\lambda(t) \quad \cos\lambda(t) \quad 0]^T$.

Moreover, if

$$r(\phi) := a(1-e^2)(1 - e^2\sin^2\phi)^{-3/2} \tag{4}$$

$R(\phi)$ is given by equation (2)*, and $\mathbf{n_t}$* (*respectively, $\mathbf{e_t}$*) *denotes the two-dimensional vector which is the image of $\mathbf{N_t}$* (*respectively, $\mathbf{E_t}$*) *under $\mathrm{d}\psi^{-1}|_{\eta(t)}$, then we have*

- $\mathbf{n_t} = \left[\frac{1}{r(\phi(t))} \quad 0\right]^T$ *and*
- $\mathbf{e_t} = \left[0 \quad \frac{1}{R(\phi(t))}\right]^T$.

Remark 4. *It is useful to keep in one's mind as one reads this article that the eccentricity e of the ellipsoid $\mathcal{E}$ is a real number in the interval $[0, 1)$ and that*

$R(\phi)$ (respectively, $r(\phi)$) as defined in equation (2) (respectively, equation (4)) is positive for all ϕ in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, we mention in passing that $R(\phi)$ is the radius of the semi-circle of latitude at a latitude equal to ϕ and that $r(\phi)$ is the radius of curvature of the meridian at any point whose latitude is ϕ; see, e.g., [5].

In what follows, if a function of one variable is followed by the superscript $'$ (prime) then this denotes the derivative with respect to the real variable t.

The problem of finding a loxodrome can be stated using the notation of Lemma 3 as follows:

Problem 1. *Given two distinct points p_o and p_1 in $\mathcal{E}^+$, we want to find a curve η in $\mathcal{E}^+$ defined on an open interval I that contains 0 and a real number α in the interval $[0, 2\pi)$ such that*:

- $\eta(0) = p_o$, $\eta(\epsilon) = p_1$ *for some positive scalar $\epsilon \in I$, and*
- $\eta'(t) = \cos\alpha \mathbf{N_t} + \sin\alpha \mathbf{E_t}$ *for all t in the closed interval* $[0, \epsilon]$.

4. Joining the Points

In this section, we will solve Problem 1. Moreover, we mention here that from this point onward we shall use the notation of Lemma 2, Lemma 3, and Problem 1. So, J will denote the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; e will denote the eccentricity of the ellipsoid $\mathcal{E}$ as given in Lemma 2; ψ is given by equation (1); $r(\phi)$ is given by equation (4); $R(\phi)$ is given by equation (2); and the tangent vectors $\mathbf{N_t}$, $\mathbf{E_t}$, $\mathbf{n_t}$, and $\mathbf{e_t}$ are as given in Lemma 3. We begin by considering the following problem:

Problem 2. *Let I be an open interval that contains 0 and let (ϕ_o, λ_o) and (ϕ_1, λ_1) be two distinct points in $J \times J \subset \mathbb{R}^2$. We want to find a curve γ in $J \times J$ having the form $\gamma(t) = (\phi(t), \lambda(t))$ for all t in I and a real number α in the interval* $[0, 2\pi)$ *such that*:

- $\gamma(0) = (\phi_o, \lambda_o)$, $\gamma(\epsilon) = (\phi_1, \lambda_1)$ *for some positive scalar $\epsilon \in I$, and*
- $\gamma'(t) = \begin{bmatrix} \frac{\cos\alpha}{r(\phi(t))} & \frac{\sin\alpha}{R(\phi(t))} \end{bmatrix}^T$ *for all t in the closed interval* $[0, \epsilon]$.

Proposition 5. *Problem 1 has a solution if and only if Problem 2 has a solution. Moreover, the solution of Problem 1 is unique if and only if the solution of Problem 2 is unique.*

Proof: Suppose that Problem 1 is solvable and consider Problem 2. If η is the solution curve for Problem 1 in the case when $p_o = \psi(\phi_o, \lambda_o)$ and $p_1 = \psi(\phi_1, \lambda_1)$ then we claim that $\gamma := \psi^{-1} \circ \eta$ is a solution for Problem 2. To see that, note that $\gamma(0) = \psi^{-1}(p_o) = (\phi_o, \lambda_o)$ and that $\gamma(\epsilon) = \psi^{-1}(p_1) = (\phi_1, \lambda_1)$. Moreover,

$$\begin{aligned}
\gamma'(t) = \mathrm{d}\psi^{-1}|_{\eta(t)}\eta'(t) &= \mathrm{d}\psi^{-1}|_{\eta(t)}\left[(\cos\alpha)\mathbf{N_t} + (\sin\alpha)\mathbf{E_t}\right] \\
&= (\cos\alpha)\mathrm{d}\psi^{-1}|_{\eta(t)}\mathbf{N_t} + (\sin\alpha)\mathrm{d}\psi^{-1}|_{\eta(t)}\mathbf{E_t} \\
&= (\cos\alpha)\mathbf{n_t} + (\sin\alpha)\mathbf{e_t} = \begin{bmatrix} \dfrac{\cos\alpha}{r(\phi(t))} & \dfrac{\sin\alpha}{R(\phi(t))} \end{bmatrix}^T
\end{aligned}$$

for all t in the closed interval $[0, \epsilon]$. Hence, Problem 2 is solvable.

Conversely, if Problem 2 is solvable then we consider Problem 1. If γ is the solution curve for Problem 2 in the case when $(\phi_o, \lambda_o) = \psi^{-1}(p_o)$ and $(\phi_1, \lambda_1) = \psi^{-1}(p_1)$ then by a similar argument we can show that $\eta := \psi \circ \gamma$ is a solution for Problem 1.

Moreover, since a solution of one problem is the image under a diffeomorphism of a solution of the other problem, it follows that a solution of one problem is unique if and only if a solution of the other problem is unique. ■

In the light of Proposition 5, in order to prove the existence of a unique solution for Problem 1, it is enough to prove the existence of a unique solution for Problem 2. Our strategy (which we follow in the proof of Theorem 8) is the following: show that there is a unique real number α in the interval $[0, 2\pi)$ for which there is a curve γ that solves Problem 2 and that this curve is unique. We mention some preliminary results before we prove our main theorem, Theorem 8.

The following lemma is a known result. Its proof and the relevant definitions can be found, e.g., in [1–3].

Lemma 6. *Let I be an open interval that contains t_o and let Ω be an open bounded and connected subset of $\mathbb{R}^n$ that contains the point $\mathbf{u_o}$. If $\mathbf{f} : I \times \Omega \to \mathbb{R}^n$ is a locally Lipschitz continuous vector-valued function (which holds, in particular, when $\mathbf{f}$ is a smooth function), then the initial value problem:*

- $\mathbf{u}(t_o) = \mathbf{u_o}$
- $\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t))$ *for all t in an open subinterval of I that contains t_o*

has a unique solution that can be extended to the boundary of Ω.

Lemma 7. *Let ϕ_o be a fixed real number in J and let α be a real number in the interval $[0, 2\pi)$. There is a parametrization of the latitude variable $\phi(t)$ defined for t in some open interval I that contains 0 such that $\phi(0) = \phi_o$, $\phi'(t) = \frac{\cos\alpha}{r(\phi(t))}$ for all t in I, and $\phi(t)$ assumes all values in J.*

Proof: Let λ_o be any real number in J and consider the following initial value problem:

- $\gamma(0) = (\phi(0), \lambda(0)) = (\phi_o, \lambda_o)$
- $\gamma'(t) = [\phi'(t) \;\; \lambda'(t)]^T = [\frac{\cos\alpha}{r(\phi(t))} \;\; 0]^T$ for all t in I.

Since the function $f(t, \phi, \lambda) = \left[\frac{\cos\alpha}{r(\phi)} \;\; 0\right]^T$ is a smooth function for all choices of α in the interval $[0, 2\pi)$, it follows by Lemma 6 that there is a unique curve γ that extends to the boundary of $J \times J$ and that has the form $\gamma(t) = (\phi(t), \lambda_o)$ for all t in I. The first entry of $\gamma(t)$, namely $\phi(t)$, is the desired parametrization of the latitude variable. ■

In the following theorem, the symbol $[g(\phi)]_{\phi=\phi_o}^{\phi=\phi_1}$ denotes $g(\phi_1) - g(\phi_o)$ if g is a real-valued function of the latitude variable ϕ.

Theorem 8. *There is a unique α in the interval $[0, 2\pi)$ for which there is a curve γ that solves Problem 2. Moreover, this curve γ is unique. Furthermore, if $\phi_o \neq \phi_1$ then*

$$\tan\alpha = \frac{\lambda_1 - \lambda_o}{\left[\ln\left(\left(\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right)\right]_{\phi=\phi_o}^{\phi=\phi_1}} \tag{5}$$

otherwise $\alpha = \frac{\pi}{2}$ (in the case when $\lambda_1 > \lambda_o$) or $\frac{3\pi}{2}$ (in the case when $\lambda_1 < \lambda_o$).

Proof: Let us begin by considering the case when $\phi_1 = \phi_o$. Since (ϕ_o, λ_o) and (ϕ_1, λ_1) are two distinct points in $J \times J$, we must have one of two subcases: i) $\lambda_1 > \lambda_o$ or ii) $\lambda_1 < \lambda_o$. Consider subcase i). By choosing $\alpha = \frac{\pi}{2}$ and $\epsilon = R(\phi_o)(\lambda_1 - \lambda_o)$, it is straightforward to show that the curve γ defined by $\gamma(t) = (\phi(t), \lambda(t)) := \left(\phi_o, \lambda_o + \frac{1}{R(\phi_o)}t\right)$ for all t in the interval $[0, \epsilon]$ is a solution for Problem 2 in subcase i). Moreover, we claim that $\frac{\pi}{2}$ is the unique value of α in the interval $[0, 2\pi)$ for which there is a solution for Problem 2 in subcase i). Suppose with the hope of getting a contradiction that this is not true. Then, there is a real number $\beta_o \neq \frac{\pi}{2}$ in the interval $[0, 2\pi)$ such that Problem 2 has a solution in subcase i) if we substitute β_o for α. Note that β_o can not be equal to $\frac{3\pi}{2}$ otherwise

$\phi(t) = \phi_o$ for all t in I (because $\phi'(t) = \frac{\cos\beta_o}{r(\phi(t))} = 0$ for all t in I) and $\lambda(t)$ would be a decreasing function of t (owing to the fact that $\lambda'(t) = \frac{\sin\beta_o}{R(\phi(t))} = \frac{-1}{R(\phi_o)} < 0$). Thus, the function $t \to \lambda(t)$ can not satisfy $\lambda(\epsilon) = \lambda_1 > \lambda_o = \lambda(0)$ for a positive scalar ϵ. Hence, β_o is neither $\frac{\pi}{2}$ nor $\frac{3\pi}{2}$, and as a result, $\cos\beta_o$ is either positive or negative. If $\cos\beta_o$ is positive, then, the function $\phi'(t) = \frac{\cos\beta_o}{r(\phi(t))} > 0$ for all t in I in spite that $0 = \phi_1 - \phi_o = \phi(\epsilon) - \phi(0) = \int_0^\epsilon \phi'(t)\mathrm{d}t$ where ϵ is a positive scalar, a contradiction. Similarly, if $\cos\beta_o$ is negative we get a contradiction. Therefore, $\frac{\pi}{2}$ is the unique value of α in the interval $[0, 2\pi)$ for which Problem 2 in subcase i) has a solution. Moreover, this solution is unique and its uniqueness follows from Lemma 6. Similarly, one can show that $\frac{3\pi}{2}$ is the unique value of α in the interval $[0, 2\pi)$ for which Problem 2 in subcase ii) has a solution and that this solution is unique.

Consider now the case when $\phi_1 \neq \phi_o$. Then, we have one of two subcases: 1) $\phi_1 > \phi_o$ or 2) $\phi_1 < \phi_o$. Suppose that subcase 1) is true. Then, choose α in the interval $[0, 2\pi)$ so that:

$$\tan\alpha = (\lambda_1 - \lambda_o)\left(\int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\mathrm{d}\phi\right)^{-1} \quad \text{and} \quad \cos\alpha > 0. \tag{6}$$

By Lemma 6, there is a unique curve $\phi \to \lambda(\phi)$ defined for all ϕ in J and which solves the following initial value problem:

- $\lambda(\phi_o) = \lambda_o$
- $\dfrac{\mathrm{d}\lambda}{\mathrm{d}\phi} = \dfrac{r(\phi)}{R(\phi)}\tan\alpha$ for all ϕ in J.

This curve is in effect a parametrization of the longitude variable by the latitude variable. Moreover, let $\phi(t)$ be the parametrization of the latitude variable guaranteed by Lemma 7. We claim that the curve γ defined by $\gamma(t) := (\phi(t), \lambda(\phi(t)))$ is the unique curve that solves Problem 2 in subcase 1). To see that, observe that $\gamma(0) = (\phi(0), \lambda(\phi(0))) = (\phi_o, \lambda(\phi_o)) = (\phi_o, \lambda_o)$. Moreover, since $\phi'(t) = \frac{\cos\alpha}{r(\phi(t))} > 0$, it follows that $\phi(t)$ is an increasing function of t and by Lemma 7 this function assumes all values in J. Thus, for a sufficiently large positive scalar ϵ we have $\phi(\epsilon) = \phi_1$. And therefore,

$$\begin{aligned}
\lambda(\phi(\epsilon)) = \lambda(\phi_1) &= \int_{\phi_o}^{\phi_1} \frac{\mathrm{d}\lambda}{\mathrm{d}\phi}\,\mathrm{d}\phi + \lambda(\phi_o) = \int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\tan\alpha\,\mathrm{d}\phi + \lambda_o \\
&= \tan\alpha \int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\,\mathrm{d}\phi + \lambda_o = (\lambda_1 - \lambda_o) + \lambda_o = \lambda_1.
\end{aligned}$$

Hence, $\gamma(\epsilon) = (\phi_1, \lambda_1)$. Furthermore,

$$\begin{aligned}\gamma'(t) = \left[\phi'(t) \ \ \frac{\mathrm{d}\lambda}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right]^T &= \left[\frac{\cos\alpha}{r(\phi(t))} \ \ \left(\frac{r(\phi(t))}{R(\phi(t))}\tan\alpha\right)\left(\frac{\cos\alpha}{r(\phi(t))}\right)\right]^T \\ &= \left[\frac{\cos\alpha}{r(\phi(t))} \ \ \frac{\sin\alpha}{R(\phi(t))}\right]^T.\end{aligned}$$

Thus, for the particular choice of α specified in (6) we showed that the curve γ is the unique solution for Problem 2 in subcase 1). Moreover, there is no other value of α for which Problem 2 in subcase 1) has a solution. To see that, suppose to the contrary that there is an $\tilde{\alpha}$ in the interval $[0, 2\pi)$ and a curve γ defined on an interval I that contains 0 such that:

- $\tilde{\alpha}$ is not equal to the value of α given in (6),
- $\gamma(0) = (\phi_o, \lambda_o)$, $\gamma(\epsilon) = (\phi_1, \lambda_1)$ for some positive scalar ϵ in I, and
- $\gamma'(t) = [\phi'(t) \ \lambda'(t)]^T = \left[\frac{\cos\tilde{\alpha}}{r(\phi(t))} \ \ \frac{\sin\tilde{\alpha}}{R(\phi(t))}\right]^T$ for all t in the closed interval $[0, \epsilon]$.

Since $\phi_1 > \phi_o$ and since $\phi'(t) = \frac{\cos\tilde{\alpha}}{r(\phi(t))}$, we must have $\cos\tilde{\alpha} > 0$ otherwise $\phi(t)$ would be a decreasing function of t. Moreover, note that $\frac{\lambda'(t)}{\phi'(t)} = \frac{r(\phi(t))}{R(\phi(t))}\tan\tilde{\alpha}$, and thus, $\lambda'(t) = \frac{r(\phi(t))}{R(\phi(t))}\phi'(t)\tan\tilde{\alpha}$. Hence,

$$\begin{aligned}\lambda_1 - \lambda_o = \lambda(\epsilon) - \lambda(0) &= \int_0^\epsilon \lambda'(t)\mathrm{d}t = \int_0^\epsilon \frac{r(\phi(t))}{R(\phi(t))}\phi'(t)\tan\tilde{\alpha}\ \mathrm{d}t \\ &= \tan\tilde{\alpha}\int_0^\epsilon \frac{r(\phi(t))}{R(\phi(t))}\phi'(t)\ \mathrm{d}t = \tan\tilde{\alpha}\int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\ \mathrm{d}\phi.\end{aligned}$$

And therefore,

$$\tan\tilde{\alpha} = (\lambda_1 - \lambda_o)\left(\int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\mathrm{d}\phi\right)^{-1}.$$

But this implies that $\tilde{\alpha}$ is equal to the value of α given in (6), a contradiction. Thus, there is a unique value of α in the interval $[0, 2\pi)$ given by (6) for which Problem 2 in subcase 1) has a solution. Moreover, this solution is unique by Lemma 6.

If subcase 2) holds then by a similar argument one can show that if we choose α in the interval $[0, 2\pi)$ so that:

$$\tan\alpha = (\lambda_1 - \lambda_o)\left(\int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)}\mathrm{d}\phi\right)^{-1} \quad \text{and} \quad \cos\alpha < 0 \tag{7}$$

then Problem 2 in subcase 2) has a solution and the value of α given in (7) is the only value of α in the interval $[0, 2\pi)$ such that Problem 2 in subcase 2) has a solution. Moreover, this solution is unique by Lemma 6.

We finally note that the form of $\tan\alpha$ given in (6) and (7) is the same as that given in (5). To see that, observe that

$$\begin{aligned}
\int_{\phi_o}^{\phi_1} \frac{r(\phi)}{R(\phi)} \mathrm{d}\phi &= \int_{\phi_o}^{\phi_1} \frac{(1-e^2)}{(1-e^2\sin^2\phi)\cos\phi} \mathrm{d}\phi \\
&= \int_{\phi_o}^{\phi_1} \frac{1}{\cos\phi} \mathrm{d}\phi + \int_{\phi_o}^{\phi_1} \frac{-\frac{e^2}{2}\cos\phi}{1-e\sin\phi} \mathrm{d}\phi + \int_{\phi_o}^{\phi_1} \frac{-\frac{e^2}{2}\cos\phi}{1+e\sin\phi} \mathrm{d}\phi \\
&= \left[\ln\left(\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right)\right]_{\phi=\phi_o}^{\phi=\phi_1} + \left[\frac{e}{2}\ln\left(1-e\sin\phi\right)\right]_{\phi=\phi_o}^{\phi=\phi_1} \\
&\quad + \left[-\frac{e}{2}\ln\left(1+e\sin\phi\right)\right]_{\phi=\phi_o}^{\phi=\phi_1} \\
&= \left[\ln\left(\left(\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right)\right]_{\phi=\phi_o}^{\phi=\phi_1}.
\end{aligned}$$

Hence, the form of $\tan\alpha$ given in (6) and (7) reduces to the form given in (5). ■

Corollary 9. *There is a unique α in the interval $[0, 2\pi)$ for which there is a curve η that solves* Problem 1. *Moreover, this curve η is unique. Furthermore, if $\psi^{-1}(p_o) = (\phi_o, \lambda_o)$ and $\psi^{-1}(p_1) = (\phi_1, \lambda_1)$, then α is as described in* Theorem 8, *i.e., if $\phi_o \neq \phi_1$ then* $\tan\alpha$ *is given by* (5) *otherwise $\alpha = \frac{\pi}{2}$ (when $\lambda_1 > \lambda_o$) or $\alpha = \frac{3\pi}{2}$ (when $\lambda_1 < \lambda_o$).*

Appendix

We present here the postponed proofs of some of the results

Proof of Lemma 2: Let us first prove that the first entry (respectively, the second entry) of $\psi^{-1}(x, y, z)$ is the latitude (respectively, the longitude) of the point (x, y, z). Let $p = (x, y, z)$ be any point in $\mathcal{E}^+$ and let ϕ and λ be its corresponding latitude and longitude, respectively. Since $\sin\lambda = \dfrac{y}{\sqrt{x^2+y^2}}$ (see Fig. 2), it follows that $\lambda = \arcsin\left(\dfrac{y}{\sqrt{x^2+y^2}}\right)$, i.e., the longitude of $p = (x, y, z)$ is the second entry of $\psi^{-1}(x, y, z)$.

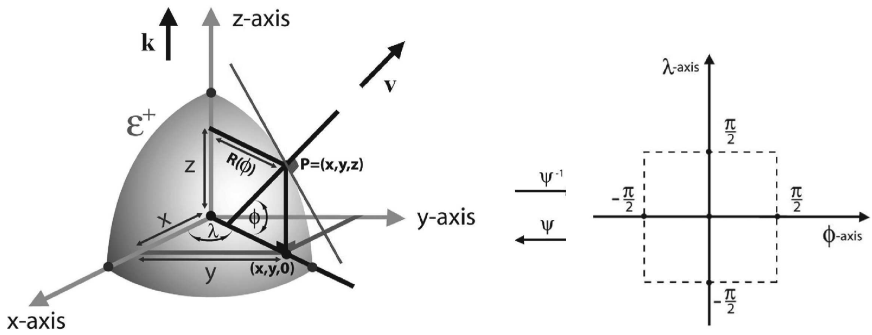

Figure 2. The diffeomorphism ψ and its inverse.

Moreover, the angle between the normal line to $\mathcal{E}^+$ at p and the z-axis is $\dfrac{\pi}{2} - \phi$; see Fig. 2. If $\mathbf{v}$ denotes the outward unit normal vector at p and $\mathbf{k}$ denotes the unit vector pointing in the direction of increasing values on the z-axis, then by taking the inner product of $\mathbf{v}$ with $\mathbf{k}$ one can find $\cos(\dfrac{\pi}{2} - \phi)$. Indeed, if we let $||\cdot||$ denote the Euclidean vector norm, let $\langle\ ,\ \rangle$ denote the Euclidean inner product, and let $\mathbf{w} = \begin{bmatrix} \dfrac{2x}{a^2} & \dfrac{2y}{a^2} & \dfrac{2z}{b^2} \end{bmatrix}^T$, then we have the following. The vector $\mathbf{v} = \dfrac{1}{||\mathbf{w}||}\mathbf{w}$, the vector $\mathbf{k} = [0 \ \ 0 \ \ 1]^T$, and $\sin\phi = \cos\left(\dfrac{\pi}{2} - \phi\right) = \langle \mathbf{k}, \mathbf{v}\rangle = \dfrac{z/b^2}{\sqrt{(x^2+y^2)/a^4 + z^2/b^4}}$. Since $\dfrac{x^2+y^2}{a^2} + \dfrac{z^2}{b^2} = 1$ and since $|\phi| < \dfrac{\pi}{2}$ (ϕ being the latitude of a point $p = (x, y, z)$ in $\mathcal{E}^+$), it follows that $\cos\phi = \dfrac{(x^2+y^2)^{1/2}/a^2}{\sqrt{(x^2+y^2)/a^4 + z^2/b^4}}$, and hence,

$$\tan\phi = \frac{a^2 z}{b^2(x^2+y^2)^{1/2}} = \frac{z}{(1-e^2)(x^2+y^2)^{1/2}}. \tag{8}$$

Therefore,

$$\phi = \arctan\left(\frac{z}{(1-e^2)\sqrt{x^2+y^2}}\right)$$

i.e., the longitude of $p = (x, y, z)$ is the first entry of $\psi^{-1}(x, y, z)$. Furthermore, we note that since $\tan\phi$ is given by (8) and since $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$, it follows that

$$\begin{aligned} R(\phi) &= a\cos\phi(1-e^2\sin^2\phi)^{-1/2} = a(1+(1-e^2)\tan^2\phi)^{-1/2} \\ &= a\left(1+\frac{a^2z^2}{b^2(x^2+y^2)}\right)^{-1/2} = \sqrt{x^2+y^2}. \end{aligned}$$

And thus, it becomes straightforward (with the aid of Fig. 2) to verify that $\psi\circ\psi^{-1}=\mathrm{Id}$ and $\psi^{-1}\circ\psi=\mathrm{Id}$ for the expressions for ψ and ψ^{-1} given in equations (1) and (3). Moreover, the smoothness of ψ and ψ^{-1} follow from the following facts about smooth functions of one variable: i) linear combinations, products, and compositions of smooth functions are smooth functions, ii) if f is a smooth function then so is $\frac{1}{f}$ on the set $\{x\in \text{Domain of } f\,;\ f(x)\neq 0\}$, and iii) if f is a smooth function then so is $\sqrt{f}$ on the set $\{x\in \text{Domain of } f\,;\ f(x)>0\}$. ■

Proof of Lemma 3: It is clear that the Euclidean norm of the vector $\mathbf{N_t}$ (respectively, $\mathbf{E_t}$) is 1. Moreover, since ψ is a diffeomorphism and since $(\phi(t),\lambda(t))=\gamma(t)=\psi\circ\eta(t)$, it follows that $\phi(t)$ and $\lambda(t)$ depend smoothly on the parameter t, and thus, the vector $\mathbf{N_t}$ (respectively, $\mathbf{E_t}$) depends smoothly on the parameter t. Note that the vector $\mathbf{N_t}$ (respectively, $\mathbf{E_t}$) points towards the north (respectively, the east) since its z-component (respectively, y-component) is positive.

Furthermore, differentiating $R(\phi)$ with respect to ϕ, one gets:

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}\phi} &= (-a\sin\phi)(1-e^2\sin\phi)^{-1/2} + ae^2\cos\phi\sin\phi(1-e^2\sin\phi)^{-3/2} \\ &= a\sin\phi(1-e^2\sin\phi)^{-3/2}\left[-(1-e^2\sin^2\phi)+e^2\cos\phi\right] \\ &= a\sin\phi(1-e^2\sin\phi)^{-3/2}\left[-1+e^2\right] = -r(\phi). \end{aligned}$$

And thus,

$$\mathrm{d}\psi = \begin{bmatrix} -r(\phi)\sin\phi\cos\lambda & -R(\phi)\sin\lambda \\ -r(\phi)\sin\phi\sin\lambda & -R(\phi)\cos\lambda \\ r(\phi)\cos\phi & 0 \end{bmatrix}$$

$$\mathrm{d}\psi\left[0\ \ \frac{1}{R(\phi)}\right]^T = [-\sin\lambda\ \ \cos\lambda\ \ 0]^T$$

and $\mathrm{d}\psi\left[\frac{1}{r(\phi)}\ \ 0\right]^T = [-\sin\phi\cos\lambda\ \ -\sin\phi\sin\lambda\ \ \cos\phi]^T$. In particular, when ϕ and λ are parameterized by a real variable t, the abovementioned matrix equalities hold. Hence, $\mathrm{d}\psi|_{\gamma(t)}\mathbf{e_t}=\mathbf{E_t}$ and $\mathrm{d}\psi|_{\gamma(t)}\mathbf{n_t}=\mathbf{N_t}$. Since $\mathrm{d}\psi^{-1}|_{\eta(t)}\circ\mathrm{d}\psi|_{\gamma(t)}$ $=\mathrm{Id}_{\mathbb{R}^2}$, it follows that the claim of this lemma about the form of $\mathbf{n_t}$ and $\mathbf{e_t}$ is true. ■

References

[1] Bourbaki N., *Elements of Mathematics*, Springer, New York, 2004.

[2] Coddington E. and Levinson N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[3] Cole R., *Theory of Ordinary Differential Equations*, Appleton Century Crofts, New York, 1968.

[4] Hofmann-Wellenhof B., Legat K., Wieser M. and Lichtenegger H., *Navigation*: *Principles of Positioning and Guidance*, Springer, New York, 2003.

[5] Pearson F., *Map Projections*: *Theory and Applications*, CRC Press, Boca Raton, 1990.

[6] Smart W., *Text-Book on Spherical Astronomy*, Cambridge University Press, Cambridge, 1947.

[7] Spivak M., *A Comprehensive Introduction to Differential Geometry*, vol. **1**, Publish or Perish Inc., Willmington, 1979.

[8] Warner F., *Foundations of Differentiable Manifolds and Lie Groups*, Springer, New York, 1983.

[9] Williams R., *Geometry of Navigation*, Horwood Publ., West Sussex, 1998.

Abed Elhashash
Department of Mathematics
Drexel University
3141 Chestnut Street
Philadelphia, PA 19104
USA
E-mail address: `aae36@drexel.edu`