A REMARK ON COMPACT MINIMAL SURFACES IN $S^5$ WITH NON-NEGATIVE GAUSSIAN CURVATURE

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Communicated by John Oprea

Abstract. The purpose of this paper is to show that a generalized Clifford immersion with non-negative Gaussian curvature has constant contact angle, thus extending previous results.

1. Introduction

In [4] we introduced the notion of contact angle, which can be considered as a new geometric invariant useful for investigating the geometry of immersed surfaces in $S^3$. Geometrically, the contact angle $\beta$ is the complementary angle between the contact distribution and the tangent space of the surface. Also in [4], we derived formulae for the Gaussian curvature and the Laplacian of an immersed minimal surface in $S^3$, and we gave a characterization of the Clifford Torus as the only minimal surface in $S^3$ with constant contact angle.

Recently, in [5], we constructed a family of minimal tori in $S^5$ with constant contact and holomorphic angles. These tori are parametrized by the following circle equation

\[ a^2 + \left( b - \frac{\cos \beta}{1 + \sin^2 \beta} \right)^2 = 2 \frac{\sin^4 \beta}{(1 + \sin^2 \beta)^2} \]  

(1)

where $a$ and $b$ are given in Section 3 (equation (9)). In particular, when $a = 0$, we recover the examples found by Kenmotsu [3]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$, we find a new family of minimal tori in $S^5$, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Also, in [5], when $\beta = \frac{\pi}{2}$, we give an alternative proof of this classification of a Theorem proved by Blair in [1], and Yamaguchi, Kon and Miyahara in [6] for Legendrian minimal surfaces in $S^5$ with constant Gaussian curvature.

The immersions that we investigate in this paper are those that satisfy the following conditions:
1) $S$ is compact
2) $\iota$ is a minimal immersion
3) $\alpha$ is constant on $S$, and
4) The principal curvatures of the immersion in the direction of $e_3$ are constant and correspond to the directions $e_1$ and $e_2$.

We will call **generalized Clifford immersion** as the immersions $\iota$ of $S$ into $S^5$ that verifies the conditions from 1) until 4).

As a consequence of the Gauss equation and using the above notation, supposing that $S$ has non-negative Gaussian curvature, we have proved the main result:

**Theorem 1.** Suppose that $S$ is a generalized Clifford immersion with non-negative Gaussian curvature ($K \geq 0$), then the contact angle $\beta$ must be constant.

### 2. Contact Angle For Immersed Surfaces In $S^5$

Consider in $\mathbb{C}^3$ the following objects:

- The Hermitian product: $(z, w) = \sum_{j=0}^{2} z^j \bar{w}^j$.
- The Inner product: $\langle z, w \rangle = \text{Re}(z, w)$.
- The Unit sphere: $S^5 = \{ z \in \mathbb{C}^3; (z, z) = 1 \}$.
- The Reeb vector field in $S^5$, given by: $\xi(z) = iz$.
- The Contact distribution in $S^5$, which is orthogonal to $\xi$

$$\Delta_\xi = \{ v \in T_z S^5; \langle \xi, v \rangle = 0 \}.$$  

Note that $\Delta$ is invariant by the complex structure of $\mathbb{C}^3$.

Let now $S$ be an orientable immersed surface in $S^5$.

**Definition 2.** The contact angle $\beta$ is the complementary angle between the contact distribution $\Delta$ and the tangent space $TS$ of the surface.

Let $(e_1, e_2)$ be a local orthonormal frame of $TS$, where $e_1 \in TS \cap \Delta$. Then $\cos \beta = \langle \xi, e_2 \rangle$. Finally, let $v$ be the unit vector in the direction of the orthogonal projection of $e_2$ on $\Delta$, defined by the following relation

$$e_2 = \sin \beta v + \cos \beta \xi$$  \hspace{1cm} (2)
Definition 3. The holomorphic angle $\alpha$ is the angle given by $\cos \alpha = \langle i e_1, v \rangle$. The holomorphic angle $\alpha$ is the analogue of the Kähler angle introduced by Chern and Wolfson in [2].

3. Equations for Gaussian Curvature and Laplacian of a Minimal Surface in $S^5$ with Constant Holomorphic Angle $\alpha$

In this section, we derive the equations for the Gaussian curvature and for the Laplacian of a minimal surface in $S^5$ in terms of the contact angle and the holomorphic angle.

Let $S$ be a minimal immersed Riemann surface in $S^5$ with constant holomorphic angle. Consider the normal vector fields

$$
e_3 = i \csc \alpha e_1 - \cot \alpha v$$
$$e_4 = \cot \alpha e_1 + i \csc \alpha v$$
$$e_5 = \csc \beta \xi - \cot \beta e_2$$

where $\beta \neq 0, \pi$ and $\alpha \neq 0, \pi$. Let $(e_j)_{1 \leq j \leq 5}$ be an adapted frame.

Using (2) and (3), we get

$$v = \sin \beta e_2 - \cos \beta e_5, \quad i v = \sin \alpha e_4 - \cos \alpha e_1, \quad \xi = \cos \beta e_2 + \sin \beta e_5.$$  \hspace{1cm} (4)

It follows from (3) and (4) that

$$i e_1 = \cos \alpha \sin \beta e_2 + \sin \alpha e_3 - \cos \alpha \cos \beta e_5$$
$$i e_2 = -\cos \beta z - \cos \alpha \sin \beta e_1 + \sin \alpha \sin \beta e_4.$$  \hspace{1cm} (5)

Let $(\theta^j)$ be the coframe of $(e_j)$. Connection forms $(\theta^j_k)$ are given by

$$D\theta^j = \theta^j_k e_k$$

and the second fundamental form with respect to this frame is given by

$$II^j = \theta^j_1 \theta^1 + \theta^j_2 \theta^2,$$  \hspace{1cm} (j = 3, ..., 5).

Using (5) and differentiating $v$ and $\xi$ on the surface $S$, we get

$$D\xi = -\cos \alpha \sin \beta \theta^2 e_1 + \cos \alpha \sin \beta \theta^1 e_2 + \sin \alpha \theta^1 e_3 + \sin \alpha \sin \beta \theta^2 e_4 - \cos \alpha \cos \beta \theta^4 e_5$$
$$De = (\sin \beta \theta^2 - \cos \beta \theta^1) e_1 + \cos \beta (d\beta - \theta^2 e_2 + (\sin \beta \theta^2 - \cos \beta \theta^1) e_3 + (\sin \beta \theta^2 - \cos \beta \theta^2) e_4 + \sin \beta (d\beta + \theta^2 e_5).$$  \hspace{1cm} (6)
Differentiating \( e_3, e_4 \) and \( e_5 \), we have

\[
\begin{align*}
\theta_1^1 &= -\theta_1^1 \\
\theta_2^1 &= \sin \beta \theta_1^1 - \cos \beta \sin \alpha \theta_1^1 \\
\theta_3^1 &= \csc \beta \theta_1^1 - \cot \alpha (\theta_1^1 + \csc \beta \theta_2^1) \\
\theta_4^1 &= \cot \beta \theta_1^1 - \csc \beta \sin \alpha \theta_1^1 \\
\theta_5^1 &= -\csc \beta \theta_2^1 + \sin \alpha \cot \beta \theta_1^1 \\
\theta_2^2 &= -\theta_2^2 \\
\theta_3^2 &= \csc \beta \theta_2^1 + \cot \alpha (\theta_1^1 + \csc \beta \theta_2^1) \\
\theta_4^2 &= \cot \beta \theta_2^1 - \sin \alpha \theta_2^1 \\
\theta_5^2 &= -\theta_5^2 \\
\theta_3^3 &= -\csc \beta \theta_1^1 + \csc \beta \sin \alpha \theta_1^1 \\
\theta_4^3 &= -\csc \beta \theta_2^1 + \sin \alpha \cot \beta \theta_2^1. \\
\end{align*}
\]

The conditions of minimality and of symmetry are equivalent to the following equations

\[
\theta_1^1 \wedge \theta_3^1 + \theta_2^1 \wedge \theta_4^1 = \theta_1^1 \wedge \theta_2^1 - \theta_2^1 \wedge \theta_2^1 = 0. 
\] (8)

On the surface \( S \), we consider

\[
\theta_1^1 = a \theta_1^1 + b \theta_2^1. 
\]

It follows from (8) that

\[
\begin{align*}
\theta_1^1 &= a \theta_1^1 + b \theta_2^1 \\
\theta_2^1 &= b \theta_1^1 - a \theta_2^1 \\
\theta_3^1 &= (b \csc \beta - \sin \alpha \cot \beta) \theta_1^1 - a \csc \beta \theta_2^1 \\
\theta_4^1 &= -a \csc \beta \theta_1^1 - (b \csc \beta - \sin \alpha \cot \beta) \theta_2^1 \\
\theta_5^1 &= d \beta \circ J - \cos \alpha \theta_2^1 \\
\theta_2^2 &= -d \beta - \cos \alpha \theta_2^1 \\
\theta_3^2 &= -\sec \beta d \beta \circ J + a \cot \alpha \cot \beta \theta_1^1 \\
&+ (b \cot \alpha \cot \beta - \cos \alpha \cot \beta \csc \beta + 2 \sec \beta \cos \alpha) \theta_2^1 \\
\theta_4^2 &= (b \cot \beta - \csc \beta \sin \alpha \theta_1^1 - a \cot \beta) \theta_2^1 \\
\theta_5^2 &= -a \cot \beta \csc \beta \theta_1^1 + (\sin \alpha (\cot ^2 \beta - 1) - b \csc \beta \cot \beta) \theta_2^1. \\
\end{align*}
\] (9)
We suppose that the second fundamental forms in the direction $e_3$ are constant. The purpose of this paper is to study the case $b = 0$. Therefore, we have 
\[ \theta_1^3 = a \theta_1^1. \]

It follows from (9) that 
\[
\begin{align*}
\theta_1^3 &= a \theta_1^1 \\
\theta_2^3 &= -a \theta_2^1 \\
\theta_1^4 &= -\sin \alpha \cot \beta \theta_1^2 - a \csc \beta \theta_2^1 \\
\theta_2^4 &= -a \csc \beta \theta_1^2 + \sin \alpha \cot \beta \theta_2^2 \\
\theta_1^5 &= d \beta \circ J - \cos \alpha \theta_1^2 \\
\theta_2^5 &= -d \beta - \cos \alpha \theta_1^1
\end{align*}
\]

where $J$ is the complex structure of $S$ is given by $Je_1 = e_2$ and $Je_2 = -e_1$. Moreover, normal connection forms are given by 
\[
\begin{align*}
\theta_1^3 &= \sec \beta \delta \delta \circ J + a \cot \alpha \cot^2 \beta \theta_1^3 \\
\theta_2^3 &= -\csc \beta \sin \alpha \theta_1^1 - a \cot \beta \theta_2^2 \\
\theta_2^2 &= -a \cot \beta \csc \beta \theta_1^2 + \sin \alpha \cot^2 \beta - 1 \theta_2^2
\end{align*}
\]

while the Gauss equation is equivalent to the equation 
\[ d \theta_2^1 + \theta_1^1 \wedge \theta_2^2 = \theta_1^3 \wedge \theta_2^3. \]

Therefore, using equations (10) and (12), we have 
\[ K = 1 - (1 + \csc^2 \beta) a^2 - |\nabla \beta| + 2 \cos \alpha \theta_2^1 = \sin^2 \alpha \cot^2 \beta \]

where $\beta_1$ and $\beta_2$ are defined by $\beta_1 = d \beta (e_1)$ and $\beta_2 = d \beta (e_2)$. Using (7) and the complex structure of $S$, we get 
\[ \theta_2^1 = \tan \beta \delta \delta \circ J - 2 \cos \alpha \theta_2^2. \]

Differentiating (13), we conclude that 
\[ d \theta_2^1 = -((1 + \tan^2 \beta) |\nabla \beta|^2 + \tan \beta \Delta \beta + 2 \cos \alpha (1 + 2 \tan^2 \beta) \beta_1 + 4 \tan^2 \beta \cos^2 \alpha) \theta_1^1 \wedge \theta_2^2. \]
where $\Delta = \text{tr} \nabla^2$ is the Laplacian of $S$. The Gaussian curvature is therefore given by

$$K = -(1 + \tan^2 \beta) \nabla^2 - \tan \alpha (1 + 2\tan^2 \beta) \beta_1 - 4\tan^2 \beta \cos^2 \alpha.$$  \hfill (14)

From (13) and (14), we obtain the following formula for the Laplacian of $S$

$$\tan \beta \Delta \beta = (1 + \csc^2 \beta) \alpha^2 + \sin^2 \alpha (1 - \tan^2 \beta) - \tan^2 \beta (|\nabla \beta| + 2 \cos \alpha \rho_1^2 - |\sin \alpha (1 - \cot^2 \beta)|^2).$$ \hfill (15)

4. Proof of Theorem 1

In this section, in order to compute Gauss-Codazzi-Ricci equations, we consider that the holomorphic angle $\alpha$ is constant, and suppose that the principal curvature in the direction of $e_3$ is constant, that is, $\alpha$ is constant. The following Codazzi-Ricci equations

$$d\theta_3^1 + \theta_1^2 \wedge \theta_2^1 + \theta_1^4 \wedge \theta_2^1 + \theta_1^5 \wedge \theta_2^1 = 0$$
$$d\theta_4^1 + \theta_1^2 \wedge \theta_2^1 + \theta_1^3 \wedge \theta_2^1 + \theta_1^5 \wedge \theta_2^1 = 0$$
$$d\theta_5^1 + \theta_1^2 \wedge \theta_2^1 + \theta_1^3 \wedge \theta_2^1 + \theta_1^4 \wedge \theta_2^1 = 0$$

simplify to

$$\beta_1 = -2 \cos \alpha.$$ \hfill (17)

Also using (17) in equation (14), we have

$$K = -(1 + \tan^2 \beta) \beta_1^2 - \tan \alpha \Delta \beta.$$ \hfill (18)
Therefore
\[ \tan \beta \Delta \beta = -K - (1 + \tan^2 \beta) \beta^2. \tag{19} \]

Now using the condition that $K \geq 0$ and the Hopf’s Lemma (for $0 < \beta < \pi / 2$), we get that the contact angle $\beta$ is constant, which prove Theorem 1. □

Theorem 1 of [5] states that any generalized Clifford immersion of constant contact and holomorphic angles is a flat torus. Combining this with Theorem 1 of this paper, we have the following

**Corollary 4.** Any generalized Clifford immersion of a compact Riemann surface with non-negative Gaussian curvature is a flat torus.

**ACKNOWLEDGEMENTS**

I would like to thank Professor Gary R. Jensen for his valuable encouragement and suggestions during this work. I would also like to thank the members of the Department of Mathematics at Washington University in Saint Louis (WUSL) for their hospitality, and the Brazilian National Research Council (CNPq) for the support. Also, I would like to thank the referee for helpful and valuable comments on the paper.

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