



## RIGGINGS OF LOCALLY COMPACT ABELIAN GROUPS

MANUEL GADELLA, FERNANDO GÓMEZ AND  
SUJEEV WICKRAMASEKARA

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**Abstract.** We obtain a set of generalized eigenvectors that provides a generalized spectral decomposition for a given unitary representation of a commutative, locally compact topological group. These generalized eigenvectors are functionals belonging to the dual space of a rigging on the space of square integrable functions on the character group. These riggings are obtained through suitable spectral measure spaces.

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### 1. Introduction

The purpose of the present paper is to take a first step towards a general formalism of unitary representations of groups and semigroups on rigged Hilbert spaces. To begin with, we want to introduce the theory corresponding to Abelian locally compact groups, leaving the more general nonabelian case as well as semigroups for a later work. We recall that a *rigged Hilbert space* or a rigging of a Hilbert space  $\mathcal{H}$  is a triplet of the form

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{1}$$

where  $\Phi$  is a locally convex space dense in  $\mathcal{H}$  with a topology stronger than that inherited from  $\mathcal{H}$  and  $\Phi^\times$  is the dual space of  $\Phi$ . In this paper, we shall always assume that  $\mathcal{H}$  is separable.

To each self adjoint operator  $A$  on  $\mathcal{H}$ , the von Neumann theorem [9] associates a *spectral measure space*. This is the quadruple  $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ , where  $\mathcal{H}$  is the Hilbert space on which  $A$  acts,  $\Lambda = \sigma(A)$  is the spectrum of  $A$ ,  $\mathcal{A}$  is the family of Borel sets in  $\Lambda$ , and  $P$  is the projection valued measure on  $\mathcal{A}$  determined by  $A$  through the von Neumann theorem. Obviously  $\Lambda \subset \mathbb{R}$ . A complete discussion on the relation between these concepts can be found in [3]. We say that the topological vector space  $(\Phi, \tau_\Phi)$  (vector space  $\Phi$  with the locally convex topology given by  $\tau_\Phi$ ) equips or rigs the spectral measure  $(\Lambda, \mathcal{A}, \mathcal{H}, P)$  if the following conditions hold:

- i. There exists a one-to-one linear mapping  $I : \Phi \mapsto \mathcal{H}$  with range dense in  $\mathcal{H}$ . We can assume that  $\Phi \subset \mathcal{H}$  is a dense subspace of  $\mathcal{H}$  and  $I$ , the canonical injection from  $\Phi$  into  $\mathcal{H}$ .
- ii. There exists a  $\sigma$ -finite measure  $\mu$  on  $(\Lambda, \mathcal{A})$ , a set  $\Lambda_0 \subset \Lambda$  with zero  $\mu$  measure and a family of vectors in  $\Phi^\times$  of the form

$$\{|\lambda k^\times\rangle \in \Phi^\times; \lambda \in \Lambda \setminus \Lambda_0, k \in \{1, 2, \dots, m\}\} \quad (2)$$

where  $m \in \{\infty, 1, 2, \dots\}$ , such that

$$(\phi, P(E)\varphi)_{\mathcal{H}} = \int_E \sum_{k=1}^m \langle \phi | \lambda k^\times \rangle \langle \varphi | \lambda k^\times \rangle^* d\mu(\lambda) \quad (3)$$

for all  $\phi, \varphi \in \Phi$ , for all  $E \in \mathcal{A}$ .

Each family of the form (2) satisfying (3) is called a *complete system of Dirac kets* of the spectral measure  $(\Lambda, \mathcal{A}, \mathcal{H}, P)$  in  $(\Phi, \tau_\Phi)$ . In this case, the triplet  $\Phi \subset \mathcal{H} \subset \Phi^\times$  is a rigged Hilbert space, which is called a *rigging* of  $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ .

Conversely, the von Neumann theorem asserts that a projection valued measure defined on the  $\sigma$ -algebra of Borel sets on a subset of the real line determines a self adjoint operator  $A$ . If  $(\Lambda, \mathcal{A}, \mathcal{H}, P)$ , where  $\Lambda \subset \mathbb{R}$ , is such a measure space, then for  $\varphi$  and  $\phi$  on a suitable dense domain, the self-adjoint operator  $A$  such that  $\Lambda = \sigma(A)$  is defined by

$$(\phi, A\varphi)_{\mathcal{H}} = \int_{\Lambda} \sum_{k=1}^m \lambda \langle \phi | \lambda k^\times \rangle \langle \varphi | \lambda k^\times \rangle^* d\mu(\lambda) \quad (4)$$

where  $\mu$ ,  $|\lambda k^\times\rangle$  and  $m$  are as defined in (3). Further, if  $f(\lambda)$  is a measurable complex valued function on  $\Lambda$ , then, for  $\phi, \varphi$  on a suitable dense domain, which is the whole of  $\mathcal{H}$  if  $f(\lambda)$  is bounded, the operator valued function  $f(A)$  is defined by

$$(\phi, f(A)\varphi)_{\mathcal{H}} = \int_{\Lambda} \sum_{k=1}^m f(\lambda) \langle \phi | \lambda k^\times \rangle \langle \varphi | \lambda k^\times \rangle^* d\mu(\lambda). \quad (5)$$

The functionals  $|\lambda k^\times\rangle \in \Phi^\times$  and the complex numbers  $f(\lambda)$  are the generalized eigenvectors and respective generalized eigenvalues of  $f(A)$  [1]. In particular, if  $f(\lambda) = e^{it\lambda}$ , where  $t \in \mathbb{R}$ , the set of operators  $e^{itA}$  forms a one parameter commutative group of unitary operators and  $\Phi \subset \mathcal{H} \subset \Phi^\times$  as defined above is a rigging for this group.

One can expect that similar riggings exist for unitary representations of arbitrary groups and semigroups and that the operators of the representations can be expanded in terms of generalized eigenvectors and eigenvalues as in (5). Riggings that make use of Hardy functions on a half plane exist for one parameter dynamical semigroups  $e^{-itH}$ ,  $t \leq 0$  and  $e^{-itH}$ ,  $t \geq 0$ , where  $H$  is the Hamiltonian [1].

In the present paper, we show that riggings along the above lines always exist for unitary representations of Abelian locally compact groups. In particular, let  $G$  be an Abelian locally compact group and  $\pi$ , a unitary representation of  $G$  on a separable Hilbert space  $\mathcal{H}$ . We will see that the Fourier transform on  $G$ , or equivalently, the Gel'fand transformation on the  $C^*$ -algebra  $L^1(G)$  allows us to represent  $\pi$  in terms of *generalized eigenfunctions* and riggings of  $\mathcal{H}$  in a manner similar to the description given in [3] for the action of a spectral measure.

## 2. Characters of Abelian Locally Compact Groups

Let  $G$  be a locally compact abelian group with Haar measure  $\mu$ . A character  $\chi$  of  $G$  is any continuous mapping from  $G$  into the set of complex numbers  $\mathbb{C}$  such that  $\chi(g_1g_2) = \chi(g_1)\chi(g_2)$  for all  $g_1, g_2 \in G$  and  $|\chi(g)| = 1$  for all  $g \in G$ , i.e., a character of  $G$  is a continuous homomorphism from  $G$  into the unit circle  $\mathbb{T}$ . The set of all the characters of  $G$  forms a group,  $\widehat{G}$ , which is often called the dual group of  $G$ . We shall use the notation  $\chi(g) := \langle g|\chi \rangle$ .

Let  $L^1(G)$  be the space of complex valued functions, integrable in the modulus with respect to the Haar measure  $\mu$  on  $G$ .  $L^1(G)$  is an abelian  $*$ -algebra, with the convolution product. The dual group  $\widehat{G}$  can be identified with the set of maximal ideals of  $L^1(G)$  [7]. When endowed with the Gel'fand topology,  $\widehat{G}$  is a compact Hausdorff space (see [8] page 268).

For any  $\chi \in \widehat{G}$ , we may define a linear functional  $\Lambda_\chi$  on  $L^1(G)$  by

$$\Lambda_\chi(f) = \int \langle g|\chi \rangle^* f(g) d\mu(g). \quad (6)$$

Let  $C(\widehat{G})$  be the space of complex continuous functions on  $\widehat{G}$  with the supremum norm topology. The Gel'fand-Fourier transform is the mapping  $\mathcal{F} : L^1(G) \mapsto C(\widehat{G})$  defined by

$$[\mathcal{F}f](\chi) = \widehat{f}(\chi) = \Lambda_\chi(f) = \int \langle g|\chi \rangle^* f(g) d\mu(g). \quad (7)$$

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Then (see [2] page 105), there is a unique spectral measure  $(\widehat{G}, \mathcal{B}, \mathcal{H}, P)$ , where  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel sets on

$\widehat{G}$ , such that for all  $g \in G$  and all  $f \in L^1(G)$ , we have

$$\pi(g) = \int_{\widehat{G}} \langle g|\chi \rangle dP(\chi), \quad \pi(f) = \int_{\widehat{G}} \Lambda_\chi(f) dP(\chi). \quad (8)$$

There is a one to one correspondence between unitary representations of  $G$  and non degenerate  $*$ -representations of  $L^1(G)$  as given by (7) and (8). Non degenerate means that  $\pi(f)v = \mathbf{0}$  for every  $f$  implies  $v = \mathbf{0}$ . The representation has also the property that  $\pi(f^*) = \pi^\dagger(f)$ , where  $f \mapsto f^*$  is the involution on  $L^1(G)$ , see [2].

## 2.1. Riggings of Functions of Characters

Let us consider the spectral measure space  $(\widehat{G}, \mathcal{B}, \mathcal{H}, P)$  introduced in the previous section. For simplicity in the discussion, we assume the existence of a cyclic vector  $u \in \mathcal{H}$ . This means that the subspace spanned by the vectors of the form  $P(E)u$  with  $E \in \mathcal{A}$  is dense in  $\mathcal{H}$ . The general case can be easily obtained as a finite or countable direct sum of cyclic subspaces of  $\mathcal{H}$ .

Then, the von Neumann decomposition theorem [9] establishes that being given the spectral measure space  $(\widehat{G}, \mathcal{B}, \mathcal{H}, P)$  and a positive measure  $\nu$  on  $(\widehat{G}, \mathcal{B})$  with maximal spectral type  $[P]$  (for a definition and properties of the spectral type, see [3, 9]), there exists a unitary mapping  $U : \mathcal{H} \mapsto L^2(\widehat{G}, d\mu)$ , such that  $\pi_\nu(g) := U\pi(g)U^{-1}$  is the multiplication by  $\langle g|\chi \rangle$  on  $L^2(\widehat{G}, d\nu)$

$$\pi_\nu(g)\phi(\chi) = U\pi(g)U^{-1}\phi(\chi) = \langle g|\chi \rangle \phi(\chi), \quad \text{for all } \phi(\chi) \in L^2(\widehat{G}, d\nu). \quad (9)$$

Since  $\pi_\nu(g)$  is a multiplication operator, it is easy to see that the Dirac delta type Radon measures  $\lambda(\chi)\delta_\chi$  form a complete system of Dirac kets for the spectral measure space  $(\widehat{G}, \mathcal{B}, \mathcal{H}, P)$  in the sense given by (3). For any  $f(\chi) \in \Phi$  these deltas satisfy

$$\int_{\widehat{G}} f(\chi) \delta_{\chi'} d\nu = f(\chi'). \quad (10)$$

Thus, a possible choice for  $\Phi$  is  $C(\widehat{G})$ , the space of continuous functions on  $\widehat{G}$  endowed with a topology  $\tau_\Phi$  stronger than both the topologies of the supremum and the  $\|\cdot\|_{L^2(\widehat{G}, d\nu)}$  norm. In this case, the dual  $\Phi^\times$  of  $\Phi$  includes the space of all Radon measures on  $(\widehat{G}, \mathcal{B})$ . We have the rigged Hilbert space  $\Phi \subset L^2(\widehat{G}) \subset \Phi^\times$ .

### 3. Positive Type Functions and Riggings

Next, we shall introduce another representation  $\pi_\phi$  of  $G$  linked to a function of positive type, that can be defined as follows: Let  $\phi(g) \in L^\infty(G)$ . We say that  $\phi(g)$  is a function of *positive type* if for any  $f(g) \in L^1(G)$ , we have that

$$\int_G \int_G f^*(g) f(gg') \phi(g') \, d\mu(g) \, d\mu(g') \geq 0 \quad (11)$$

where the star  $*$  denotes complex conjugation.

If  $\phi(g)$  is a function of positive type, then, the following positive Hermitian form on  $L^1(G)$

$$\langle h|f \rangle_\phi := \int_G \int_G h^*(g') f(g) \phi(g^{-1}g') \, d\mu(g') \, d\mu(g) \quad (12)$$

is semi-definite in the sense that it may exist non-zero functions  $f \in L^1(G)$  such that  $\langle f|f \rangle_\phi = 0$ . These functions form a subspace of  $L^1(G)$  that we denote by  $\mathcal{N}$ . Consider the factor space  $L^1(G)/\mathcal{N}$  and again denote by  $\langle \cdot | \cdot \rangle_\phi$  the scalar product induced on  $L^1(G)/\mathcal{N}$  by the Hermitian form (12). The completion of  $L^1(G)/\mathcal{N}$  by  $\langle \cdot | \cdot \rangle_\phi$  gives a Hilbert space usually denoted as  $\mathcal{H}_\phi$ . Then, for any  $g \in G$  and  $f(g) \in L^1(G)$ , we define

$$(L_g)f(g') := f(g^{-1}g'). \quad (13)$$

Note that  $L_g$  preserves the scalar product  $\langle \cdot | \cdot \rangle_\phi$

$$\begin{aligned} \langle L_g h | L_g f \rangle_\phi &= \int_G \int_G h^*(g^{-1}g') f(g^{-1}g'') \phi(g'^{-1}g'') \, d\mu(g') \, d\mu(g'') \\ &= \int_G \int_G h^*(g') f(g'') \phi((gg')^{-1}(gg'')) \, d\mu(g') \, d\mu(g'') = \langle h|f \rangle_\phi \end{aligned} \quad (14)$$

for all  $f(g) \in L^1(G)$ . This also shows that  $L_g\mathcal{N} \subset \mathcal{N}$  and therefore  $L_g$  induces a transformation on the factor space  $L^1(G)/\mathcal{N}$ , that we also denote as  $L_g$ , defined as

$$L_g(f(g') + \mathcal{N}) := f(g^{-1}g') + \mathcal{N} = L_g(f(g')) + \mathcal{N}. \quad (15)$$

By (14), we easily see that  $L_g$  preserves the scalar product on  $L^1(G)/\mathcal{N}$ . It is obviously invertible. Therefore, it can be uniquely extended into a unitary operator on  $\mathcal{H}_\phi$ . Then, if for each  $g \in G$  we write

$$\pi_\phi(g)f := L_g f, \quad \text{for all } f \in \mathcal{H}_\phi \quad (16)$$

then,  $\pi_\phi(g)$  determines a unitary representation of  $G$  on  $\mathcal{H}_\phi$ . The proof of this statement is straightforward.

The representation  $\pi_\phi(g)$  of  $G$  on  $\mathcal{H}_\phi$  can be lifted to a unitary representation of the group algebra  $L^1(G)$  on  $\mathcal{H}_\phi$  that we shall also denote as  $\pi_\phi$ . In this case, for all  $f \in L^1(G)$ , we have  $\pi_\phi(f)h := f * h$ . Here,  $*$  denotes convolution.

The existence of a cyclic vector  $\eta \in \mathcal{H}_\phi$  for the representation  $\pi_\phi$  is proven in [2]. Recall that  $\eta$  is cyclic vector if the subspace  $\{\pi_\phi(f)\eta, \text{ with } f \in L^1(G)\}$  is dense in  $\mathcal{H}_\phi$ . In addition, this result also gives the following formula that allows to find the function  $\phi(g)$  in terms of  $\eta$  and the unitary representation  $\pi_\phi$  of  $G$  on  $\mathcal{H}_\phi$

$$\phi(g) = \langle \eta | \pi_\phi(g) \eta \rangle. \quad (17)$$

Now, let us consider the unitary  $\pi_\nu$  representation of  $G$  given by (9) with cyclic vector  $\xi$  and define the following complex valued function on  $G$

$$\phi(g^{-1}g') := \langle \xi | \pi_\nu(g^{-1}g') \xi \rangle_{L^2(\widehat{G}, d\nu)} = \langle \pi_\nu(g) \xi | \pi_\nu(g') \xi \rangle_{L^2(\widehat{G}, d\nu)}. \quad (18)$$

Then, as shown in [2], Chapter 3

- i) the function  $\phi$  is of positive type in the sense of (11), and
- ii) the representation of  $G$  on  $\mathcal{H}_\phi$  given by  $\pi_\phi$ , where  $\phi$  is as (18) is equivalent to  $\pi_\nu$ .

Note that this result implies in particular that for this  $\phi$  as in (18)

$$\phi(g) = \langle \eta | \pi_\phi(g) \eta \rangle_\phi = \langle \xi | \pi_\nu(g) \xi \rangle_{L^2(\widehat{G}, d\nu)}, \quad \text{for all } g \in G. \quad (19)$$

According to (9) and (18), we have that

$$\phi(g^{-1}g') = \langle \pi_\nu(g) \xi | \pi_\nu(g') \xi \rangle_{L^2(\widehat{G}, d\nu)} = \int_{\widehat{G}} [ \langle g | \chi \rangle \xi(\chi) ]^* \langle g' | \chi \rangle \xi(\chi) d\nu(\chi). \quad (20)$$

If we carry this formula into (12) and apply the Fubini theorem of the change of the order of integration, we have for all  $f, h \in L^1(G)$

$$\begin{aligned} \langle f | h \rangle_\phi &= \int_{\widehat{G}} \left( \int_G [f(g) \langle g | \chi \rangle]^* d\mu(g) \right) \left( \int_G h(g') \langle g' | \chi \rangle d\mu(g') \right) |\xi(\chi)|^2 d\nu(\chi) \\ &= \int_{\widehat{G}} [\widehat{f}(\chi)]^* \widehat{h}(\chi) |\xi(\chi)|^2 d\nu(\chi) = \int_{\widehat{G}} \langle \widehat{f} | \chi \rangle \langle \chi | \widehat{h} \rangle |\xi(\chi)|^2 d\nu(\chi). \end{aligned} \quad (21)$$

This latter formula shows that the generalized eigenvalues  $F_\chi$  of  $\pi_\phi(g)$  are the following: if  $f \in \Phi := L^1(G) \cap L^2(G)$

$$|F_\chi\rangle \equiv F_\chi : f \mapsto |\eta(\chi)| \widehat{f^*}(\chi) = |\eta(\chi)| \int_G \langle g|\chi\rangle f^*(g) d\mu(g). \quad (22)$$

We endow  $\Phi$  with any topology stronger than the topologies  $L^1(G)$  and  $L^2(G)$ . For instance, we can choose a locally convex topology with the seminorms  $p_1(f) := \|f\|_{L^1(G)}$  and  $p_2(f) := \|f\|_{L^2(G)}$ , for all  $f \in \Phi$ . With this topology or another stronger one, the antilinear functional  $F_\chi$  is continuous. Then, if we use (8) in the scalar product on  $\mathcal{H}_\phi$ , we have

$$\begin{aligned} \langle f|\pi_\phi(g)h\rangle_\phi &= \int_{\widehat{G}} \langle g|\chi\rangle d\langle f|P(\chi)h\rangle_\phi = \int_{\widehat{G}} \langle g|\chi\rangle |\eta(\chi)|^2 \widehat{f^*}(\chi) \widehat{g}(\chi) d\nu(\chi) \\ &= \int_{\widehat{G}} \langle g|\chi\rangle \langle f|F_\chi\rangle \langle F_\chi|h\rangle d\nu(\chi). \end{aligned} \quad (23)$$

If we omit the arbitrary  $f, h \in \Phi$  in (23), we have the following spectral decomposition for  $\pi_\phi(g)$  for all  $g \in G$

$$\pi_\phi(g) = \int_{\widehat{G}} \langle \chi|g\rangle |F_\chi\rangle \langle F_\chi| d\nu(\chi). \quad (24)$$

Note that in the antidual space  $\Phi^\times$ , the generalized eigenvalue equation  $\pi_\phi(g)|F_\chi\rangle = \langle \chi|g\rangle |F_\chi\rangle$  is valid, where we use the same notation  $\pi_\phi(g)$  for the extensions of these unitary operators into  $\Phi^\times$ .

In conclusion, for each unitary representation of a locally compact Abelian topological group, we have found an equivalent representation and a rigged Hilbert space such that each of the unitary operators of the representation admits a generalized spectral decomposition in terms of generalized eigenvectors of them. The eigenvectors of the decomposition are labeled by the group characters only and their respective eigenvalues, complex numbers with modulus one, depend on both the corresponding character and the group element. The spectral decomposition and the corresponding rigging comes after the existence of a spectral measure space.

Note that the Abelian property is crucial in our derivation and in particular in the existence of the spectral measure space  $(\widehat{G}, \mathcal{B}, \mathcal{H}, P)$ , since then, the group

algebra is also Abelian and the Gel'fand theory applies. An extension of the present formalism to nonabelian locally compact groups will require an extension of the Gel'fand formalism that at least allows for a new and consistent definition of the Gel'fand Fourier transform (7), an essential feature of our construction.

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Manuel Gadella  
Departamento de Física Teórica  
Facultad de Ciencias  
c. Real de Burgos, s.n.  
47011 Valladolid  
SPAIN  
*E-mail address:* manuelgadella@yahoo.com.ar

Fernando Gómez-Cubillo  
Departamento de Análisis Matemático  
Facultad de Ciencias  
c. Real de Burgos, s.n.  
47011 Valladolid  
SPAIN  
*E-mail address:* fgcubill@am.uva.es

Sujeev Wickramasekara  
Department of Physics,  
Grinnell College, Grinnell, IA 50112  
USA  
*E-mail address:* WICKRAMA@Grinnell.EDU