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ABELIAN CONNECTION IN FEDOSOV DEFORMATION QUANTIZATION

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Abstract. General properties of an Abelian connection in Fedosov deformation quantization are investigated. The definition and the criterion of being a finite formal series for an Abelian connection are presented. Examples of finite and infinite Abelian connections are given.

1. Introduction

Deformation quantization of the phase space $\mathbb{R}^{2n}$ was invented in the middle of the previous century. Making use of the results obtained by Weyl [9], Wigner [10] and Groenewold [5] Moyal [7] presented quantum mechanics perceived as a statistical theory.

The first successful generalization of Moyal's results in case of a phase space different from $\mathbb{R}^{2n}$ appeared in 1977 when Bayen *et al.* [1] proposed an axiomatic version of the deformation quantization. In those articles quantum mechanics gained a new aspect – as a deformed version of the classical physics.

One of the realizations the quantization programme of Bayen *et al.* is the so called Fedosov deformation quantization [2, 3]. The Fedosov construction is algebraic and can be applied easily for example to solve the harmonic oscillator or to find momentum and position eigenvalues and Wigner eigenfunctions on a 2-D symplectic space with constant curvature tensor [4]. A great advantage of that method is the fact that computations may be done by computer programmes.

In Fedosov quantization we work with formal series. There is no general method to write these series in a compact form. Series of compact form appear for example when they contain finite number of terms. In that case the $*$-product of functions can be calculated exactly.

Fedosov deformation quantization is based on two recurrent equations. The first one is the formula defining an Abelian connection, the second – a relation introducing a series representing an observable. We deal only with the Abelian connection.

The text is based on our paper [8]. In all formulas in which summation limits are obvious we use Einstein summation convention.

2. Foundations of Fedosov Deformation Quantization

Let $(\mathcal{M}, \omega)$ be a $2n$-dimensional symplectic manifold. Since we work only with symplectic manifolds, in our paper we will denote a manifold $(\mathcal{M}, \omega)$ just by $\mathcal{M}$.

Definition 1. *A* **symplectic connection** Γ *on* $\mathcal{M}$ *is a torsion free connection locally satisfying conditions* $\omega_{ij;k} = 0,\ 1 \leq i, j, k \leq 2n$.

In Darboux coordinates the coefficients $\Gamma_{ijk} \stackrel{\mathrm{def}}{=} \Gamma^{l}_{jk}\omega_{li}$ are symmetric with respect to indices $\{i, j, k\}$.

Definition 2. *A* **symplectic manifold** $\mathcal{M}$ *equipped with a symplectic connection* Γ *is called a* **Fedosov manifold** $(\mathcal{M}, \Gamma)$.

Let $\hbar$ denote some positive parameter and $X^1_{\mathrm{p}}, \ldots, X^{2n}_{\mathrm{p}}$ components of an arbitrary vector $\mathbf{X}_{\mathrm{p}}$ belonging to the tangent space $T_{\mathrm{p}}\mathcal{M}$ at the point p. The components $X^1_{\mathrm{p}}, \ldots, X^{2n}_{\mathrm{p}}$ are written in the natural basis $\left(\frac{\partial}{\partial q^i}\right)_{\mathrm{p}}$ determined by the chart $(\mathcal{U}_z, \phi_z)$ such that $\mathrm{p} \in \mathcal{U}_z$. In the point p we introduce a formal series

$$a \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a_{k,i_1\ldots i_l} X^{i_1}_{\mathrm{p}} \ldots X^{i_l}_{\mathrm{p}}. \tag{1}$$

By $a_{k,i_1\ldots i_l}$ we denote components of a covariant tensor symmetric with respect to indices $\{i_1 \ldots, i_l\}$ taken in the basis $\mathrm{d}q^{i_1} \odot \ldots \odot \mathrm{d}q^{i_l}$.

The part of the series a standing at $\hbar^k$ and containing l components of the vector $\mathbf{X}_{\mathrm{p}}$ will be denoted by $a[k, l]$ so that $a = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a[k, l]$. The **degree** $\deg(a[k, l])$ of the component $a[k, l]$ is the sum $2k + l$. The degree of the series a is the maximal degree of its nonzero components $a[k, l]$.

Let $P^*_{\mathrm{p}}\mathcal{M}[[\hbar]]$ be the set of all elements a of the kind (1) at the point p.

Definition 3. *The product* $\circ : P^*_{\mathrm{p}}\mathcal{M}[[\hbar]] \times P^*_{\mathrm{p}}\mathcal{M}[[\hbar]] \to P^*_{\mathrm{p}}\mathcal{M}[[\hbar]]$ *of two ele-*

ments $a, b \in P^*_{\mathrm{p}}\mathcal{M}[[\hbar]]$ *is the mapping*

$$a \circ b \stackrel{\text{def}}{=} \sum_{t=0}^{\infty} \frac{1}{t!} \left(\frac{\mathrm{i}\hbar}{2}\right)^t \omega^{i_1 j_1} \dots \omega^{i_t j_t} \frac{\partial^t a}{\partial X^{i_1}_{\mathrm{p}} \dots \partial X^{i_t}_{\mathrm{p}}} \frac{\partial^t b}{\partial X^{j_1}_{\mathrm{p}} \dots \partial X^{j_t}_{\mathrm{p}}}. \quad (2)$$

The tensor ω^{ij} and the symplectic form ω_{jk} are related by $\omega^{ij}\omega_{jk} = \delta^i_k$. The pair $(P^*_{\mathrm{p}}\mathcal{M}[[\hbar]], \circ)$ is a noncommutative associative algebra called the **Weyl algebra**.

Definition 4. *A* **Weyl bundle** *is a triplet* $(\mathcal{P}^*\mathcal{M}[[\hbar]], \pi, \mathcal{M})$, *where* $\mathcal{P}^*\mathcal{M}[[\hbar]] \stackrel{\text{def}}{=} \bigcup_{\mathrm{p}\in\mathcal{M}}(P^*_{\mathrm{p}}\mathcal{M}[[\hbar]], \circ)$ *is a differentiable manifold called the total space,* $\mathcal{M}$ *is the base space and* $\pi : \mathcal{P}^*\mathcal{M}[[\hbar]] \to \mathcal{M}$ *the projection.*

A Weyl bundle is a vector bundle in which the typical fibre is also an algebra.

Definition 5. *An* m**-differential form** *with value in the Weyl bundle is a form written locally*

$$a = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \hbar^k a_{k,i_1\dots i_l,j_1\dots j_m}(q^1,\dots,q^{2n}) X^{i_1}\dots X^{i_l} \mathrm{d}q^{j_1} \wedge \dots \wedge \mathrm{d}q^{j_m} \quad (3)$$

where $0 \le m \le 2n$. *Now* $a_{k,i_1\dots i_l,j_1\dots j_m}(q^1,\dots,q^{2n})$ *are components of smooth tensor fields on* $\mathcal{M}$ *and* $C^\infty(\mathcal{TM}) \ni \mathbf{X} \stackrel{\text{locally}}{=} X^i \frac{\partial}{\partial q^i}$ *is a smooth vector field.*

Let Λ^m be a smooth field of m-forms on the symplectic manifold $\mathcal{M}$. Forms of the kind (3) are smooth sections of $\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda \stackrel{\text{def}}{=} \oplus_{m=0}^{2n}(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m)$. The **projection** $\sigma(a)$ of $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^0)$ means $a|_{\mathbf{X}=\mathbf{0}}$.
For simplicity we will omit the variables $(q^1,\dots,q^{2n})$.

Definition 6. *The* **commutator** *of the forms* a *and* b $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m_1})$, $b \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m_2})$ *is the form* $[a,b] \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m_1+m_2})$

$$[a,b] \stackrel{\text{def}}{=} a \circ b - (-1)^{m_1 \cdot m_2} b \circ a. \quad (4)$$

A form $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ is called **central**, if for every other form $b \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ the commutator $[a,b]$ vanishes. Only forms not containing X^i's are central.

Definition 7. *The antiderivation operator*

$$\delta : C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \to C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m+1})$$

is defined by $\delta a \stackrel{\text{def}}{=} \mathrm{d}q^k \wedge \frac{\partial a}{\partial X^k}$.

Definition 8. *The operator*

$$\delta^{-1} : C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \to C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m-1})$$

is

$$\delta^{-1}a = \begin{cases} \frac{1}{l+m}\, X^k \frac{\partial}{\partial q^k} \rfloor a & \text{for} \quad l+m>0 \\ 0 & \text{for} \quad l+m=0 \end{cases} \tag{5}$$

where l is the degree of a in X^i's, i.e., the number of X^i's.

The Hodge decomposition is described by the following formula.

Theorem 1. ([2, 3]) *For every $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$*

$$a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00} \tag{6}$$

where a_{00} is a smooth function on the symplectic manifold $\mathcal{M}$.

Definition 9. *The* **exterior covariant derivative** ∂_γ *of the form* $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m)$ *determined by a connection one-form* $\gamma \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1)$ *is the linear operator*

$$\partial_\gamma : C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \to C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m+1})$$

defined in a Darboux chart by the formula

$$\partial_\gamma a \stackrel{\text{def}}{=} \mathrm{d}a + \frac{1}{\mathrm{i}\hbar}[\gamma, a]. \tag{7}$$

In the case of a symplectic connection, we use the symbol Γ instead of γ and put $\Gamma = \frac{1}{2}\Gamma_{ijk}X^iX^j\mathrm{d}q^k$.

The curvature form R_γ of a connection one-form γ in a Darboux chart can be expressed by the formula

$$R_\gamma = \mathrm{d}\gamma + \frac{1}{2\mathrm{i}\hbar}[\gamma, \gamma] = \mathrm{d}\gamma + \frac{1}{\mathrm{i}\hbar}\gamma \circ \gamma. \tag{8}$$

A crucial role in the Fedosov deformation quantization is played by an Abelian connection $\tilde{\Gamma}$. By an **Abelian** connection we mean a connection $\tilde{\Gamma}$ whose curvature form $R_{\tilde{\Gamma}}$ is central so that $\partial_{\tilde{\Gamma}}(\partial_{\tilde{\Gamma}}a) = 0$ for every $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$.

The Abelian connection proposed by Fedosov is of the form

$$\tilde{\Gamma} = \omega_{ij}X^i\mathrm{d}q^j + \Gamma + r. \tag{9}$$

Its curvature is

$$R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1 j_2}\mathrm{d}q^{j_1} \wedge \mathrm{d}q^{j_2} + R_\Gamma - \delta r + \partial_\Gamma r + \frac{1}{\mathrm{i}\hbar} r \circ r. \tag{10}$$

The requirement that the curvature two-form $R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1 j_2}\mathrm{d}q^{j_1} \wedge \mathrm{d}q^{j_2}$ must be central means that r must satisfy the equation

$$\delta r = R_\Gamma + \partial_\Gamma r + \frac{1}{\mathrm{i}\hbar} r \circ r. \tag{11}$$

Theorem 2. ([2, 3]) *The equation* (11) *has a unique solution*

$$r = \delta^{-1} R_\Gamma + \delta^{-1}\left(\partial_\Gamma r + \frac{1}{\mathrm{i}\hbar} r \circ r\right) \tag{12}$$

fulfilling the additional conditions

$$\delta^{-1} r = 0, \qquad 3 \leq \deg(r). \tag{13}$$

We work only with the Abelian connection of the form (9) with the correction r defined by (12) and fulfilling (13).

Definition 10. $\mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{\Gamma}} \subset C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^0)$ *denotes the subalgebra consisting of flat sections, i.e., sections such that* $\partial_{\tilde{\Gamma}} a = 0$.

Theorem 3. ([2, 3]) *For any* $a_0 \in C^\infty(\mathcal{M})$ *there exists a unique smooth section* $a \in \mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{\Gamma}}$ *such that* $\sigma(a) = a_0$.

Applying the operator δ^{-1} it follows from the Hodge decomposition (6) that

$$a = a_0 + \delta^{-1}\Big(\partial_\Gamma a + \frac{1}{\mathrm{i}\hbar}[r, a]\Big). \tag{14}$$

Using the one-to-one correspondence between $\mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{\Gamma}}$ and $C^\infty(\mathcal{M})$ we introduce an associative star product '$*$' of functions $a_0, b_0 \in C^\infty(\mathcal{M})$

$$a_0 * b_0 \stackrel{\mathrm{def}}{=} \sigma(\sigma^{-1}(a_0) \circ \sigma^{-1}(b_0)). \tag{15}$$

The $*$-product (15) satisfies the axioms of a star product in deformation quantization and is interpreted as quantum multiplication of observables.

3. Properties of the Abelian Connection

Let $\mathcal{P}^*\mathcal{M}[[\hbar]]$ be a Weyl algebra bundle equipped with some connection determined by the one-form γ. We do not assume that γ is an Abelian or symplectic.

Proposition 1. *Every connection $\gamma \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1)$ such that $\delta\gamma = 0$ satisfies $\delta R_\gamma = 0$.*

A consequence of Proposition 1 and decomposition (6) is the following corollary.

Corollary 1. *If the connection form γ fulfills the condition $\delta\gamma = 0$ then its curvature $R_\gamma = \delta\delta^{-1}R_\gamma$.*

Let us apply the above corollary to the symplectic connection represented by the one-form Γ. Since the coefficients Γ_{ijk} are symmetric in indices $\{i, j, k\}$, we obtain that $\delta\Gamma = 0$. From Corollary 1 we conclude

Proposition 2. *Two symplectic curvature forms R_Γ and $R_{\Gamma'}$ defined by symplectic connections Γ and Γ' respectively, are equal if and only if $\delta^{-1}R_\Gamma = \delta^{-1}R_{\Gamma'}$.*

From Proposition 2 we see that the geometry of a symplectic space can be characterized by a tensor R_Γ or, equivalently, by a tensor $\delta^{-1}R_\Gamma$.

Let us consider the structure of equation (12). Its solution fulfilling conditions (13) can be found by the iteration method [2,3]. The component of r of the lowest degree is $\delta^{-1}R_\Gamma$ and $\deg(\delta^{-1}R_\Gamma) = 3$. From Proposition 2 we conclude that if $R_\Gamma \neq R_{\Gamma'}$ then the corrections r determined by connections Γ and Γ' respectively are different. From (12) we deduce that each component of r contains one or more X's. Moreover, the product $r \circ r$ generates only odd powers of $\hbar$. Therefore,

$$r = \delta^{-1}R_\Gamma + \sum_{z=4}^{\infty} \sum_{k=0}^{[\frac{z-1}{4}]} \hbar^{2k}\, r_m[2k, z-4k]\mathrm{d}q^m. \tag{16}$$

By $[\frac{z-1}{4}]$ we denote the maximal integer number not bigger than $\frac{z-1}{4}$.

In the case when $\deg(r) = d,\ \ d \in \mathbb{N}$ we say that r is a **finite formal series**. For $\deg(r) = \infty$ we deal with an infinite series.

If in an arbitrary chart the term $r_m[2k, z-4k]\mathrm{d}q^m$ for fixed k and z does not disappear, the same happens in any other chart. Moreover, at an arbitrary point $\mathrm{p} \in \mathcal{M}$ the fact that the series r is finite does not depend on the chart.

We are ready to present a necessary and sufficient condition for an Abelian connection to be a finite formal series. By $r[z]$ we will denote the component $r[z] \stackrel{\mathrm{def}}{=} \sum_{k=0}^{[\frac{z-1}{4}]} \hbar^{2k} r_m[2k, z-4k]\mathrm{d}q^m, \;\; 3 \leq z$ of r of the degree z.
We start from formulas defining $r[z], \; 3 \leq z$. As proved in [6] one has

$$r[3] = \delta^{-1} R_\Gamma$$

$$r[z] = \delta^{-1}\Big(\partial_\Gamma r[z-1] + \frac{1}{\mathrm{i}\hbar}\sum_{j=3}^{z-2} r[j] \circ r[z+1-j]\Big), \qquad 4 \leq z. \tag{17}$$

From Proposition 2 we see that for curvature $R_\Gamma \neq 0$ it must hold that $\delta^{-1} R_\Gamma \neq 0$. So on any nonflat Fedosov manifold $(\mathcal{M}, \Gamma)$ the term $r[3]$ is different from 0.

Assume that r is a finite formal series of the degree $m-1, \; 4 \leq m$. Hence, from (17) and applying the fact that according to Theorem 2 the series r is the only solution of equation (11) we see that (for details see [8])

$$\partial_\Gamma r[m-1] + \frac{1}{\mathrm{i}\hbar}\sum_{j=3}^{m-2} r[j] \circ r[m+1-j] = 0 \tag{18a}$$

$$\sum_{j=3}^{m-1} r[j] \circ r[m+2-j] = 0 \tag{18b}$$

$$\vdots$$

$$r[m-1] \circ r[m-1] = 0. \tag{18c}$$

Conversely, let the components $r[z]$, $3 \leq z \leq m-1$, where $4 \leq m$ of the Abelian correction r fulfill the system of equations (18a - 18c). Then, applying formula (17) to (18a) we see that $r[m] = 0$. Substituting this result and relation (18b) in (17) we obtain $r[m+1] = 0$. Repeating this procedure we find that $r[z] = 0$ for any $m \leq z$. Hence $\deg(r) \leq m-1$ so r is a finite formal series. To conclude,

Theorem 4. *An Abelian connection* $\tilde{\Gamma} = \omega_{ij} X^i \mathrm{d}q^j + \Gamma + r$ *of the symplectic curvature two-form* $R_\Gamma \neq 0$ *is a finite formal series if there exists a natural number* $4 \leq m$ *such that the components* $r[z], \;\; 3 \leq z \leq m-1$, *of r fulfill the system of equations* (18a–18c).

Hence, a sufficient condition for the series r to be infinite is that for every $3 \leq z$ the product $r[z] \circ r[z] \neq 0$.

Example 1. *Assume that in some Darboux chart* $(\mathcal{U}, \phi)$ *on* $\mathcal{M}$ *nonzero symplectic connection coefficients* $\Gamma_{l_1 l_2 l_3}(q^{l_4}, \ldots, q^{l_s}),\ 1 \leq l_1, \ldots, l_s \leq \dim \mathcal{M}$ *are only these which Poisson brackets* $\{q^{l_i}, q^{l_j}\}_P = 0,\ 1 \leq i, j \leq s$. *Such connection can be curved only if* $4 \leq \dim \mathcal{M}$. *In the considered case all the products* $r[z] \circ r[k]$, $3 \leq z, k$ *disappear. Applying* (17) *we see that* $r[z] = (\delta^{-1}\partial_\Gamma)^{z-3}\delta^{-1}R_\Gamma$. *From* Theorem 4 *for* $R_\Gamma \neq 0$ *a sufficient and necessary condition for* r *to be a finite series is that for some* z

$$(\partial_\Gamma \delta^{-1})^{z-3} R_\Gamma = 0.$$

The minimal number z *for which the above relation holds, is the degree of* r.

4. An Abelian Connection on a 2-D Phase Space

In this final section we prove that every Abelian connection on any curved 2-D Fedosov space $(\mathcal{M}, \Gamma)$ is an infinite formal series.

Proposition 3. *Let* $(\mathcal{M}, \Gamma)$ *be a* 2-*D Fedosov manifold and the two-form* $F \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^2)$ *contains only terms of the same degree and only even powers* $\hbar^{2k}$. *Then* $\delta^{-1}F \circ \delta^{-1}F = 0$ *iff* $F = 0$.

The proof of Proposition 3 is rather technical and can be found in [8].

As it has been said in the previous section, on a 2-D Fedosov manifold $(\mathcal{M}, \Gamma)$ the relation $R_\Gamma \neq 0$ yields $r[3] \neq 0$. Hence $r[3] = \delta^{-1}R_\Gamma$, from Proposition 3 we obtain that $r[3] \circ r[3] \neq 0$. Using Theorem 4 we conclude that there exists at least one nonzero component $r[z]$ of the correction r of degree $3 < z$. Remembering that (see (17)) $r[z] = \delta^{-1}(F[z-1])$, where $F[z-1] \stackrel{\mathrm{def}}{=} \partial_\Gamma r[z-1] + \frac{1}{\mathrm{i}\hbar}\sum_{j=3}^{z-2} r[j] \circ r[z+1-j]$, and applying Proposition 3 to $r[z]$, we see that $r[z] \circ r[z] \neq 0$. Hence, from Theorem 4 there exits $r[z_1] \neq 0$ such that $z < z_1$. The formula (17) for $r[z_1]$ plus Proposition 3 guarantees that $r[z_1] \circ r[z_1] \neq 0$. Hence, by Theorem 4 it must hold $z_1 < \deg(r)$. Following this pattern we arrive at the following

Theorem 5. *On* 2-*D phase space with nonvanishing symplectic curvature two-form* R_Γ *every Abelian connection is an infinite series.*

We stress that Theorem 5 holds for 2-D real symplectic manifolds with the correction r determined by formula (12) fulfilling (13).

5. Conclusions

The Fedosov quantization method is based on the recurrent formulas (12) and (14). The first one defines the correction r to the Abelian connection and the second defines a flat section $a \in \mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{\Gamma}}$ representing a quantum observable a_0. There is no general rule saying in which cases the flat section a or the $*$-product of functions $a_0 * b_0$ can be written in a compact form. Such situation happens for example when both iterations (12) and (14) generate finite formal series.

We have considered the question, when the Abelian connection on a Fedosov manifold $(\mathcal{M}, \Gamma)$ described in Theorem 2 is a finite formal series. We have found a system of equations determining a sufficient and necessary condition for r to be finite.

Then we have applied the result quoted above to the case of 2-D phase space with nonvanishing curvature. We have shown that the series r on such spaces is always infinite.

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