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LIE GYROVECTOR SPACES

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Abstract. An arbitrary section of the canonical projection of a group onto the cosets modulo a subgroup is associated with a binary operation on the cosets. We provide sufficient conditions for obtaining a left loop, a left gyrogroup or a gyrocommutative gyrogroup in such a way. The non-positively curved sections in Lie groups allow a scalar multiplication, which turns them into quasi left Lie gyrovector spaces. The left invariant metrics on homogeneous spaces turn out to be compatible with the gyro-structure. For instance, their geodesics are gyro-lines; the associated distance to the origin is a gyro-homogeneous norm, satisfying gyrotriangle inequality; etc. The work establishes infinitesimal criteria for a homogeneous space to bear a left Lie gyrovector space or a Lie gyrovector space structure. It characterizes the Cartan gyrovector spaces and works out explicitly the example of the upper half-plane.

1. Introduction

Based on Einstein's velocity addition law and the relativistic Thomas precession, the second named author has developed in a series of articles (e.g., [13], [14], [15], [17], [5], [3], [4], etc.) and the monograph [16] the theory of gyrogroups and gyrovector spaces. It introduces the so called Thomas gyration, which measures the deviation of the addition of the relativistically admissible velocities from being associative. From mathematical point of view, one of the most important results of this theory is the proof of the fact that the gyro-semidirect product of a gyrogroup $(\mathcal{L}, \oplus)$ with a gyroautomorphism group $H \subset \mathrm{Aut}(\mathcal{L}, \oplus)$ is a group G (cf. Theorem 2.23 from [16]). Thus, the Thomas gyrations of a gyrogroup $(\mathcal{L}, \oplus)$ appear to be a sort of ”extension cocycles” of $\mathcal{L}$ with values in $\mathrm{Aut}(\mathcal{L}, \oplus)$. Therefore, Thomas gyrations techniques can be applicable for transmitting the classification of the finite simple groups H to finite groups $G \supset H$, in which H are of comparatively small index $[G : H]$. On the other hand, ideas, similar to the gyro-formalism have proved to be quite fruitful for studying affine connections on manifolds in the works of Sabinin [12], Nagy and Strambach [11], Kikkawa [9] and others.

Based on the identification $\mathcal{L} = G/H$ for any gyrogroup $(\mathcal{L}, \oplus)$, the present article studies the operations $\oplus_\sigma : (G/H) \times (G/H) \to G/H$, $(g_1 H) \oplus_\sigma (g_2 H) := \sigma(g_1 H)\sigma(g_2 H)H$ on the left coset space G/H, induced from the group multiplication in G via a section $\sigma : G/H \to G$ with $\sigma(H) = 1_G$. Our idea to work with sections emerge prior to the appearance of Nagy and Strambach's book [11]. Moreover, our Lemma 2 characterizes the sections $\sigma : G/H \to G$, associated with left loops $(G/H, \oplus_\sigma)$, while the starting point of [11] are the loops and the corresponding necessary and sufficient conditions on σ are quite different (cf. the remark after Lemma 2). Combining the notions of a left loop and a gyrovector space, studied in previous works of Ungar, Definition 6 introduces the term quasi left gyrovector space $(V, \oplus, \otimes)$. In the next Proposition 7 we establish that if G/H is a homogeneous space for a Lie group G and $\sigma : G/H \to G$ is a real analytic section, whose exponential map $\mathrm{Exp} : T^{\mathbb{R}}_{1_G}\sigma(G/H) \to \sigma(G/H)$ is a global diffeomorphism, then $\oplus_\sigma$ and $\otimes_\sigma$, induced from the multiplication by real numbers on the tangent space $T^{\mathbb{R}}_{1_G}\sigma(G/H)$ turn G/H into a quasi left Lie gyrovector space.

In Section 3 the invariant metrics on quasi left Lie gyrovector spaces are studied. It is proved that if the image of the section $\sigma : G/H \to G$ is closed under the inversion of elements of G, then a Riemannian metric on G/H is left G-invariant if and only if it is invariant under the adjoint action of H and under left $\oplus_\sigma$-translations (cf. Lemma 11). The central result of this section, Corollary 14 establishes that for quasi left Lie gyrovector spaces $(G/H, \oplus_\sigma, \otimes_\sigma)$ with complete simply connected $\sigma(G/H) \subset G$ of non-positive sectional curvature with respect to some G-invariant metric, the geodesics are exactly the gyro-lines and all Thomas gyrations are isometries.

Section 4 is devoted to left gyrogroups and left gyrovector spaces. It provides a simple specific example of two different sections τ, σ on one and a same coset space G_o/H_o, such that $(G_o/H_o, \oplus_\tau)$ is a group and $(G_o/H_o, \oplus_\sigma)$ is barely a left gyrogroup. Lemma 19 gives sufficient conditions for a section $\sigma : G/H \to G$ to be associated with a left gyrogroup $(G/H, \oplus_\sigma)$. Along the lines of Ungar's Theorem 2.23 from [16], Proposition 21 shows that any left gyrogroup $(\mathcal{L}, \oplus)$ is isomorphic to some $(G/H, \oplus_\sigma)$. The concluding Corollary 26 delivers sufficient condition for a non-positively curved analytic section $\sigma : G/H \to G$ of a homogeneous space G/H to be associated with a left Lie gyrovector space $(G/H, \oplus_\sigma, \otimes_\sigma)$.

Section 5 deals with gyrogroups and gyrovector spaces. Lemma 29 specifies sufficient conditions for a section $\sigma : G/H \to G$ to be associated with a gyrocommutative gyrogroup. A similar result for the Lie gyrovector spaces is Corollary 31. Corollary 32 verifies that the Cartan decomposition on the Lie algebra of a noncompact semisimple Lie group G provides a section, associated with a Lie gyrovector space. The last Corollary 33 characterizes the Cartan gyrovector spaces, proving sufficient infinitesimal conditions for a Lie gyrovector space $(G/H, \oplus_\sigma, \otimes_\sigma)$ of a non-positively curved real analytic section $\sigma : G/H \to G$ to arise from the Cartan decomposition on $\mathrm{Lie}\,(G)$.

The concluding Section 6 illustrates the results from the previous sections on the example of the upper half-plane $\mathcal{H} = SL(2, \mathbb{R})/SO(2)$. More precisely, Proposition 35 gives explicit formulae for the operations $\oplus_\sigma$ and $\otimes_\sigma$ of the Cartan gyrovector space $(\mathcal{H}, \oplus_\sigma, \otimes_\sigma)$, while Corollary 36 specifies the gyro-norm.

2. Left Loops and Quasi Left Lie Gyrovector Spaces

For an arbitrary group G and its normal subgroup $H \subset G$, the coset space G/H inherits the group structure of G. Conversely, arbitrary groups Γ and H are normal in their direct product $G = \Gamma \times H$ and Γ is isomorphic to the quotient group G/H. Before generalizing this well known situation, let us uncover the relevant construction in terms of a section of the canonical projection of G onto the set G/H of left cosets.

A section $\sigma : G/H \to G$ of $\pi : G \to G/H$ is a map with $\pi\sigma = \mathrm{Id}_{G/H}$ and $\sigma(H) = 1_G$. In the case of a normal subgroup $H \subset G$, an arbitrary section allows the group operation of G to be descended into a group operation in G/H,

$$(G/H) \times (G/H) \longrightarrow G/H$$
$$(g_1H)(g_2H) := \pi\left(\sigma(g_1H)\sigma(g_2H)\right) = \sigma(g_1H)\sigma(g_2H)H.$$

We claim that $(g_1H)(g_2H) = g_1g_2H$. More precisely, if $\sigma_i := \sigma(g_iH)$ then $\sigma_iH = \pi\sigma_i = \pi\sigma(g_iH) = g_iH$, i.e., $\sigma_i = g_ih_i$ for some $h_i \in H$, $i = 1, 2$. Thus,

$$(g_1H)(g_2H) = \sigma_1\sigma_2H = g_1g_2(g_2^{-1}h_1g_2)h_2H = g_1g_2H$$

since $g_2^{-1}h_1g_2 \in H$, so that $\left(g_2^{-1}h_1g_2\right)h_2 \in H$. Consequently,

$$(gH)H = gH = H(gH)$$
$$(gH)(g^{-1}H) = H = (g^{-1}H)(gH)$$
$$[(g_1H)(g_2H)](g_3H) = [(g_1g_2)g_3]H = [g_1(g_2g_3)]H = (g_1H)[(g_2H)(g_3H)].$$

In such a way, G/H turns out to be endowed with a group structure, independent of σ, and $\pi : G \to G/H$ appears to be a group homomorphism,

$$\pi(g_1 g_2) = g_1 g_2 H = (g_1 H)(g_2 H) = \pi(g_1)\pi(g_2).$$

More generally, even if H is not normal in G, an arbitrary section $\sigma : (G/H) \to G$ induces an operation

$$\oplus_\sigma : (G/H) \times (G/H) \longrightarrow G/H$$
$$(g_1 H) \oplus_\sigma (g_2 H) := \pi(\sigma(g_1 H)\sigma(g_2 H)) = \sigma(g_1 H)\sigma(g_2 H)H.$$

It is straightforward that

$$H \oplus_\sigma (gH) = \pi(\sigma(H)\sigma(gH)) = \pi(1\sigma(gH)) = \pi\sigma(gH) = gH$$
$$(gH) \oplus_\sigma H = \pi(\sigma(gH)\sigma(H)) = \pi(\sigma(gH)1) = \pi\sigma(gH) = gH$$

for $\forall gH \in G/H$, so that H is a two-sided neutral element with respect to $\oplus_\sigma$. Furthermore, $H \in G/H$ is unique with this property. Indeed, according to

$$\sigma(gH)H = \pi\sigma(gH) = gH \tag{1}$$

one can represent $\sigma(gH) = gh$ by some $h \in H$. Then

$$(g_1 H) \oplus_\sigma (g_2 H) = \sigma(g_1 H)\sigma(g_2 H)H = \sigma(g_1 H)g_2 h_2 H = \sigma(g_1 H)g_2 H$$

for $\forall g_1 H, g_2 H \in G/H$. If $(gH) \oplus_\sigma (g_o H) = gH$ for some $g_o H \in G/H$ and all $gH \in G/H$, then

$$gH = \sigma(gH)(g_o H) = ghg_o H$$

for some $h \in H$ implies that $H = hg_o H$, so that $g_o \in H$ and $g_o H = H$.

The equation $(aH) \oplus_\sigma (xH) = bH$ possesses a unique solution for arbitrary $aH, bH \in G/H$. Indeed,

$$(aH) \oplus_\sigma \{[\sigma(aH)]^{-1} bH\} = \sigma(aH)[\sigma(aH)]^{-1} bH = bH$$

so that $[\sigma(aH)]^{-1} bH$ is a solution. Arbitrary solutions $x_1 H, x_2 H$ satisfy the equalities

$$\sigma(aH)x_1 H = (aH) \oplus_\sigma (x_1 H) = bH = (aH) \oplus_\sigma (x_2 H) = \sigma(aH)x_2 H.$$

Following left multiplication by $[\sigma(aH)]^{-1} \in G$, we have $x_1 H = x_2 H$. Thus, for arbitrary $aH, bH \in G/H$ the equation $(aH) \oplus_\sigma (xH) = (bH)$ possesses a unique solution $[\sigma(aH)]^{-1} bH$.

Let us suppose, moreover, that $[\sigma(aH)]^{-1} \in \sigma\left(G/H\right)$ and define

$$\ominus_\sigma(aH) := [\sigma(aH)]^{-1}H \tag{2}$$

for the unique solution of $(aH) \oplus_\sigma (tH) = H$. Then the unique solution of $(aH) \oplus_\sigma (xH) = (bH)$ can be expressed as $\ominus_\sigma(aH) \oplus_\sigma (bH)$.

Definition 1. *A groupoid $(\mathcal{L}, \oplus)$ is a non-empty set $\mathcal{L}$ with a binary operation $\oplus : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. A left loop $(\mathcal{L}, \oplus)$ is a groupoid, possessing the following two properties:*

i) *there is a unique neutral element $\check{o} \in \mathcal{L}$, such that*

$$\check{o} \oplus x = x \oplus \check{o} = x \quad \textit{for } \forall x \in \mathcal{L} \tag{3}$$

ii) *for any $a, b \in \mathcal{L}$ the equation $a \oplus x = b$ has the unique solution $x = (\ominus a) \oplus b$, where $\ominus a$ is the unique solution of $a \oplus t = \check{o}$.*

Summarizing the above considerations, we have the following

Lemma 2. *Let G be a group, H be a subgroup of G and $\sigma : G/H \to G$ be a section of $\pi : G \to G/H$, i.e., $\pi\sigma = \mathrm{Id}_{G/H}$ and $\sigma(H) = 1_G$. If the image $S := \sigma\left(G/H\right)$ of σ is closed under inversion, $g \in S \Rightarrow g^{-1} \in S$, then the binary operation*

$$\begin{aligned} \oplus_\sigma : (G/H) \times (G/H) &\longrightarrow G/H \\ (aH) \oplus_\sigma (bH) &:= \sigma(aH)\sigma(bH)H = \sigma(aH)bH \end{aligned} \tag{4}$$

introduces a structure of a left loop on G/H.

A left loop $(\mathcal{L}, \oplus)$, in which the equation $x \oplus a = b$ has a unique solution is called a loop. In [11] Nagy and Strambach present necessary and sufficient conditions on a section $\sigma : G/H \to G$ of a set G/H of left cosets to be associated with a loop $\left(G/H, \oplus_\sigma\right)$. We have just observed that an arbitrary section $\sigma : G/H \to G$, whose image $\sigma\left(G/H\right)$ is closed under inversion, induces a left loop $\left(G/H, \oplus_\sigma\right)$. Nagy and Strambach establish that $(G/H, \oplus_\sigma)$ is a loop if and only if the image $\sigma\left(G/H\right)$ of the section generates G and acts transitively with trivial stabilizers on G/H.

Our Definition 1 of a left loop $(\mathcal{L}, \oplus)$ corresponds to Kreuzer and Wefelscheid's notion of a right loop with left inverse property, given in [10]. More precisely, they define that $(\mathcal{L}, \oplus)$ is a right loop if there is a unique two-sided neutral element

$\check{o} \in \mathcal{L}$ for $\oplus$ and the equation $a \oplus x = b$ has the unique solution $x \in \mathcal{L}$ for arbitrary $a, b \in \mathcal{L}$. The left inverse property asserts that for $\forall a \in \mathcal{L}$ there exists $\ominus a \in \mathcal{L}$ with $(\ominus a) \oplus (a \oplus b) = b$ for $\forall b \in \mathcal{L}$.

On an arbitrary left loop $(\mathcal{L}, \oplus)$, consider the left translations

$$L_a : \mathcal{L} \longrightarrow \mathcal{L}$$
$$L_a(x) := a \oplus x$$

by $a \in \mathcal{L}$. According to the property (ii) of Definition 1, all L_a are invertible and $L_a^{-1}(b) = L_{\ominus a}(b)$ for $\forall b \in \mathcal{L}$. Consequently,

$$a \oplus \{(\ominus a) \oplus x\} = L_a L_{\ominus a}(x) = L_a L_a^{-1}(x) = x \quad \text{for } \forall x \in \mathcal{L}.$$

Definition 3. *For any pair of elements a, b of a left loop $(\mathcal{L}, \oplus)$, the Thomas gyration* $\mathrm{gyr}\,[a, b]$ *is defined as the bijective map*

$$\mathrm{gyr}\,[a, b] = L_{\ominus(a \oplus b)} L_a L_b : \mathcal{L} \longrightarrow \mathcal{L}. \tag{5}$$

Kiechle's considerations in [7] imply that a left loop $(\mathcal{L}, \oplus)$ is a group if and only if its gyrations $\mathrm{gyr}\,[a, b] = \mathrm{Id}_{\mathcal{L}}$, are trivial for $\forall a, b \in \mathcal{L}$.

Lemma 4. a) *Let $(\mathcal{L}, \oplus)$ be a left loop with Thomas gyrations $\mathrm{gyr}\,[a, b] = L_{\ominus(a \oplus b)} L_a L_b$ for $a, b \in \mathcal{L}$. Then*

i) $a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]c$ *for $\forall a, b, c \in \mathcal{L}$ (left gyroassociative law);*

ii) $\mathrm{gyr}\,[a, \ominus a] = \mathrm{Id}_{\mathcal{L}}$ *for $\forall a \in \mathcal{L}$ (weak loop property);*

iii) *the unique right inverse $\ominus a$ of $a \in \mathcal{L}$ is the unique left inverse of a with respect to $\oplus$.*

b) *Let G be a group, $H \subset G$ be a subgroup and $\sigma : G/H \to G$ be a section of the canonical projection $\pi : G \to G/H$ with $[\sigma(xH)]^{-1} \in \sigma\,(G/H)$ for $\forall xH \in G/H$. Then for arbitrary $a = \sigma(aH)$ and $b = \sigma(bH)$ the Thomas gyration*

$$\mathrm{gyr}\,[aH, bH](xH) = \left(\mathrm{Ad}_{\,h(ab)}(x)\right) H \tag{6}$$

acts as a conjugation by

$$h(ab) := [\sigma(abH)]^{-1}\, ab \in H.$$

Proof: a) i) The associativity of the composition law for the bijections $\mathcal{L} \to \mathcal{L}$ implies the left gyroassociative law

$$(a \oplus b) \oplus \mathrm{gyr}\,[a, b]c = L_{a \oplus b} L_{\ominus(a \oplus b)} L_a L_b(c) = a \oplus (b \oplus c)$$

for arbitrary $a, b, c \in \mathcal{L}$.

ii) By Definition 3 of a Thomas gyration one has

$$\mathrm{gyr}\,[a, \ominus a] = L_{\ominus[a\oplus(\ominus a)]} L_a L_{\ominus a} = L_{\ominus \check{o}} L_a L_{\ominus a} = L_a L_a^{-1} = \mathrm{Id}_{\mathcal{L}}.$$

iii) Let a_1 be the unique right inverse of $\ominus a$. Then by i) and ii) we have

$$a = a \oplus \{(\ominus a) \oplus a_1\} = \{a \oplus (\ominus a)\} \oplus \mathrm{gyr}\,[a, \ominus a] a_1 = a_1.$$

Consequently, $(\ominus a) \oplus a = \check{o}$ and $\ominus a$ is a left inverse of a. Any other left inverse $a_2 \in \mathcal{L}$ of a satisfies $a_2 \oplus a = \check{o}$. According to the uniqueness of the solution of $a_2 \oplus x = \check{o}$ there follows $x = \ominus a_2 = a$. One more application of the left gyroassociative law and the weak loop property implies

$$a_2 = a_2 \oplus \{a \oplus (\ominus a)\} = (a_2 \oplus a) \oplus \mathrm{gyr}\,[a_2, a](\ominus a) = \mathrm{gyr}\,[a_2, \ominus a_2](\ominus a) = \ominus a$$

justifying the uniqueness $a_2 = \ominus a$ of the left inverse of an arbitrary $a \in \mathcal{L}$.

b) On the one hand,

$$\begin{aligned}\{(aH) \oplus_\sigma (bH)\} \oplus_\sigma \left\{\left[\mathrm{Ad}_{h(ab)}(x)\right] H\right\} &= (abH) \oplus_\sigma \left\{\left[\mathrm{Ad}_{h(ab)}(x)\right] H\right\} \\ &= \sigma(abH) \left[\mathrm{Ad}_{h(ab)}(x)\right] H.\end{aligned}$$

On the other hand,

$$\begin{aligned}(aH) \oplus_\sigma \{(bH) \oplus_\sigma (xH)\} &= (aH) \oplus_\sigma (bxH) = a(bx)H \\ &= (ab)xH = \sigma(abH)h(ab)xH \\ &= \sigma(abH)h(ab)x\,[h(ab)]^{-1} H \\ &= \sigma(abH) \left[\mathrm{Ad}_{h(ab)}(x)\right] H\end{aligned}$$

whereas

$$\{(aH) \oplus_\sigma (bH)\} \oplus_\sigma \left\{\left[\mathrm{Ad}_{h(ab)}(x)\right] H\right\} = (aH) \oplus_\sigma \{(bH) \oplus_\sigma (xH)\}.$$

Combining with the left gyroassociative law

$$\{(aH) \oplus_\sigma (bH)\} \oplus_\sigma \{\mathrm{gyr}\,[aH, bH](xH)\} = (aH) \oplus_\sigma \{(bH) \oplus_\sigma (xH)\}$$

and acting on the left by $L^{-1}_{(aH)\oplus_\sigma(bH)}$, one infers

$$\left[\mathrm{Ad}_{h(ab)}(x)\right] H = \mathrm{gyr}\,[aH, bH](xH) \qquad \text{Q.E.D.}$$

Definition 5. *If G is a connected Lie group, $H \subset G$ is a closed connected subgroup of G and $\sigma : G/H \to G$ is a real analytic section of $\pi : G \to G/H$ with $[\sigma(G/H)]^{-1} = \sigma(G/H)$, then $(G/H, \oplus_\sigma)$ is called a left Lie loop.*

Let G be a connected Lie group, $H \subset G$ be a closed connected subgroup and $\sigma : G/H \to G$ be a real analytic section of the canonical projection $\pi : G \to G/H$. Suppose that the exponential map $\mathrm{Exp} : \mathrm{Lie}(G) \to G$ restricts to a global diffeomorphism

$$\mathrm{Exp} : T^{\mathbb{R}}_{1_G}\sigma(G/H) \to \sigma(G/H). \tag{7}$$

Since $(\mathrm{d}\sigma)_{\breve{o}} : T^{\mathbb{R}}_{\breve{o}}(G/H) \to T^{\mathbb{R}}_{1_G}\sigma(G/H)$ is a linear isomorphism and π restricts to a diffeomorphism $\pi : \sigma(G/H) \to G/H$, inverting σ, the assumption is equivalent to the fact that

$$\pi \mathrm{Exp}(\mathrm{d}\sigma)_{\breve{o}} : T^{\mathbb{R}}_{\breve{o}}(G/H) \to G/H \tag{8}$$

is a diffeomorphism. For any $x \in \sigma(G/H)$ it is straightforward that

$$x^{-1} = \mathrm{Exp}(-\mathrm{Exp}^{-1}(x)) \in \sigma(G/H)$$

as far as the tangent space $T^{\mathbb{R}}_{1_G}\sigma(G/H)$ is invariant under multiplication by $-1 \in \mathbb{R}$. Therefore, $(G/H, \oplus_\sigma)$ is a left Lie loop.

Let us define a scalar multiplication by real numbers

$$\otimes_\sigma : \mathbb{R} \times (G/H) \longrightarrow G/H$$
$$t \otimes_\sigma (\mathrm{Exp}(u)H) := \mathrm{Exp}(tu)H \quad \text{for } \forall t \in \mathbb{R}, \forall u \in T^{\mathbb{R}}_{1_G}\sigma(G/H). \tag{9}$$

It is immediate that

$$\begin{aligned} 1 \otimes_\sigma (\mathrm{Exp}(u)H) &= \mathrm{Exp}(u)H \quad \text{for } \forall u \in T^{\mathbb{R}}_{1_G}\sigma(G/H) \\ (rs) \otimes_\sigma (\mathrm{Exp}(u)H) &= \mathrm{Exp}(rsu)H = r \otimes_\sigma (\mathrm{Exp}(su)H) \\ &= r \otimes_\sigma [s \otimes_\sigma (\mathrm{Exp}(u)H)] \end{aligned}$$

and

$$\begin{aligned} (rs) \otimes_\sigma (\mathrm{Exp}(u)H) &= \mathrm{Exp}(sru)H = s \otimes_\sigma (\mathrm{Exp}(ru)H) \\ &= s \otimes_\sigma [r \otimes_\sigma (\mathrm{Exp}(u)H)] \end{aligned}$$

for $\forall r, s \in \mathbb{R}$ and $\forall u \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$. Taking into account that $\mathrm{Exp}(v)\mathrm{Exp}(w) = \mathrm{Exp}(v + w)$ for arbitrary commuting $v, w \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$, $[v, w] = 0$, one observes that

$$\begin{aligned} &[r \otimes_\sigma (\mathrm{Exp}(u)H)] \oplus_\sigma [s \otimes_\sigma (\mathrm{Exp}(u)H)] = (\mathrm{Exp}(ru)H) \oplus_\sigma (\mathrm{Exp}(su)H) \\ &\quad = \mathrm{Exp}(ru)\mathrm{Exp}(su)H = \mathrm{Exp}(ru + su)H = \mathrm{Exp}((r + s)u)H \\ &\quad = (r + s) \otimes_\sigma (\mathrm{Exp}(u)H) \end{aligned}$$

for $\forall r, s \in \mathbb{R}$ and $\forall u \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$. Further, from Lemma 4 (b) one has

$$\mathrm{gyr}\,[\mathrm{Exp}\,(u)H, \mathrm{Exp}\,(v)H](\mathrm{Exp}\,(w)H) = \mathrm{Ad}\,_{h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v))}(\mathrm{Exp}\,(w))H$$

where

$$h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)) := [\sigma(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H)]^{-1}\,\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)$$

$\forall u, v, w \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$. In particular, for $r, s \in \mathbb{R}$, $u \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$, the commuting ru and su satisfy

$$\mathrm{Exp}\,(ru)\mathrm{Exp}\,(su) = \mathrm{Exp}\,((r+s)u) \in \sigma(G/H)$$

whereas

$$\sigma(\mathrm{Exp}\,(ru)\mathrm{Exp}\,(su)H) = \mathrm{Exp}\,(ru)\mathrm{Exp}\,(su) \;\text{ and }\; h(\mathrm{Exp}\,(ru)\mathrm{Exp}\,(su)) = 1.$$

Consequently,

$$\begin{aligned}\mathrm{gyr}\,[r\otimes_\sigma(\mathrm{Exp}\,(u)H), s\otimes_\sigma(\mathrm{Exp}\,(u)H)] &= \mathrm{gyr}\,[\mathrm{Exp}\,(ru)H, \mathrm{Exp}\,(su)H]\\ &= \mathrm{Ad}\,_{h(\mathrm{Exp}\,(ru)\mathrm{Exp}\,(su))} = \mathrm{Ad}\,_{1_G} = \mathrm{Id}\,_{G/H}.\end{aligned}$$

For arbitrary $u, v, w \in T^{\mathbb{R}}_{1_G}\sigma(G/H)$ and $t \in \mathbb{R}$, one has also

$$\begin{aligned}&\mathrm{gyr}\,[\mathrm{Exp}\,(u)H, \mathrm{Exp}\,(v)H]\{t\otimes_\sigma(\mathrm{Exp}\,(w)H)\}\\ &\quad= \left[\mathrm{Ad}\,_{h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v))}(\mathrm{Exp}\,(tw))\right]H = \mathrm{Exp}\,\left(t\mathrm{Ad}\,_{h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v))}(w)\right)H\\ &\quad= t\otimes_\sigma\left(\mathrm{Exp}\,\left(\mathrm{Ad}\,_{h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v))}(w)\right)H\right)\\ &\quad= t\otimes_\sigma\{\mathrm{Ad}\,_{h(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v))}(\mathrm{Exp}\,(w))H\}\\ &\quad= t\otimes_\sigma\{\mathrm{gyr}\,[\mathrm{Exp}\,(u)H, \mathrm{Exp}\,(v)H](\mathrm{Exp}\,(w)H)\}.\end{aligned}$$

In order to formulate the above considerations in a concise manner, we give the following

Definition 6. *A quasi left gyrovector space* $(V, \oplus, \otimes)$ *is a left loop* $(V, \oplus)$ *with a scalar multiplication*

$$\otimes : \mathbb{R}\times V \longrightarrow V$$

subject to the properties:

i) $1\otimes v = v$ *for* $\forall v \in V$

ii) $(rs)\otimes v = r\otimes(s\otimes v) = s\otimes(r\otimes v)$ *for* $\forall r, s \in \mathbb{R}, \forall v \in V$

iii) $(r+s)\otimes v = (r\otimes v)\oplus(s\otimes v)$ *for* $\forall r, s \in \mathbb{R}, \forall v \in V$

iv) $\operatorname{gyr}[r\otimes v, s\otimes v] = \operatorname{Id}_V$ *for* $\forall r,s\in\mathbb{R}, \forall v\in V$

v) $\operatorname{gyr}[a,b](r\otimes v) = r\otimes(\operatorname{gyr}[a,b]v)$ *for* $\forall a,b,v\in V$.

If $(V,\oplus)$ *is a left Lie loop and the scalar multiplication is a real analytic map, then* $(V,\oplus,\otimes)$ *is called a quasi left Lie gyrovector space.*

Proposition 7. *Let* G *be a connected Lie group with exponential map* $\operatorname{Exp}:\operatorname{Lie}(G)\to G$*,* $H\subset G$ *be a closed connected subgroup and* $\sigma: G/H\to G$ *be a real analytic section of* $\pi: G\to G/H$*, such that (8) is a global diffeomorphism. Then*

$$(\operatorname{Exp}(u)H)\oplus_\sigma(\operatorname{Exp}(v)H) := \operatorname{Exp}(u)\operatorname{Exp}(v)H \quad \textit{for } \forall u,v\in T^{\mathbb{R}}_{1_G}\sigma(G/H) \tag{10}$$

and (9) *defines a quasi left Lie gyrovector space* $(G/H,\oplus_\sigma,\otimes_\sigma)$.

The construction of a quasi left gyrovector space on an analytic left loop $(\mathcal{L},\oplus)$ is similar to Sabinin's left $\mathbb{R}$-odules from [12]. In his terminology, a left $\mathbb{R}$-odular structure on a smooth loop $(\mathcal{L},\oplus)$ is a scalar multiplication

$$\otimes:\mathbb{R}\times\mathcal{L}\longrightarrow\mathcal{L}$$

satisfying the properties i), ii) and iii) from Definition 6.

Comparing Definition 6 with the definition of an ordinary real vector space, one observes that our considerations omit the vector distributive law

$$r\otimes(a\oplus b) = (r\otimes a)\oplus(r\otimes b) \qquad \text{for } r\in\mathbb{R} \text{ and } a,b\in V.$$

The following Proposition 8 reveals that on a quasi left Lie gyrovector space, this property is a specific feature of the integral curves of commuting vector fields.

Proposition 8. *Let* G *be a connected Lie group with exponential map* $\operatorname{Exp}:\operatorname{Lie}(G)\to G$ *and faithful representation* $\rho: G\to GL(n,\mathbb{R})$*,* $H\subset G$ *be a closed connected subgroup and* $\sigma: G/H\to G$ *be a real analytic section of* $\pi: G\to G/H$*. Suppose that (8) is a global diffeomorphism and consider the operations* (10), (9)*. Then*

$$t\otimes_\sigma[(\operatorname{Exp}(u)H)\oplus_\sigma(\operatorname{Exp}(v)H)] = [t\otimes_\sigma(\operatorname{Exp}(u)H)]\oplus_\sigma[t\otimes_\sigma(\operatorname{Exp}(v)H)] \tag{11}$$

for $\forall t\in\mathbb{R}$ *if and only if* $u,v\in T^{\mathbb{R}}_{1_G}\sigma(G/H)$ *commute,* $[u,v]=0$.

Proof: The injective group homomorphism $\rho: G\to GL(n,\mathbb{R})$ induces an embedding of the Lie algebras $(\mathrm{d}\rho)_{1_G}:\operatorname{Lie}(G)\to\mathfrak{gl}(n,\mathbb{R})$. Arbitrary $u,v\in$

$T^{\mathbb{R}}_{1_G}\sigma(G/H)$ with $[u,v]=0$ are transformed into commuting matrices $U := (\mathrm{d}\rho)_{1_G}u$, $V := (\mathrm{d}\rho)_{1_G}v \in \mathfrak{g}l(n,\mathbb{R})$. Therefore

$$\begin{aligned}
\mathrm{Exp}\,(tu)\mathrm{Exp}\,(tv) = \rho^{-1}\left(\mathrm{Exp}\,(tU)\mathrm{Exp}\,(tV)\right)&\\
=&\rho^{-1}\left(\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}U^k\right)\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}V^l\right)\right)\\
=&\rho^{-1}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}U^iV^{k-i}\right)\right)\\
=&\rho^{-1}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}(U+V)^k\right)\\
=&\rho^{-1}\mathrm{Exp}\,\left(\mathrm{d}\rho\right)_{1_G}[t(u+v)] = \mathrm{Exp}\,(t(u+v))
\end{aligned}$$

for all $t\in\mathbb{R}$. That allows to derive that

$$\begin{aligned}
&[t\otimes_\sigma(\mathrm{Exp}\,(u)H)]\oplus_\sigma[t\otimes_\sigma(\mathrm{Exp}\,(v)H)] = \mathrm{Exp}\,(tu)\mathrm{Exp}\,(tv)H\\
&\quad=\mathrm{Exp}\,(t(u+v))H = t\otimes_\sigma(\mathrm{Exp}\,(u+v)H)\\
&\quad=t\otimes_\sigma(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H) = t\otimes_\sigma\left[(\mathrm{Exp}\,(u)H)\oplus_\sigma(\mathrm{Exp}\,(v)H)\right].
\end{aligned}$$

Conversely, suppose that (11). If $w := \mathrm{Exp}^{-1}\sigma(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H)$ then $\mathrm{Exp}\,(w)H = \mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H$. Denoting $U := (\mathrm{d}\rho)_{1_G}u$, $V := (\mathrm{d}\rho)_{1_G}v$, $W := (\mathrm{d}\rho)_{1_G}(w)$, one can express the assumption in the form

$$\mathrm{Exp}\,(tW)A = \rho\left(\mathrm{Exp}\,(tw)a\right) = \rho\left(\mathrm{Exp}\,(tu)\mathrm{Exp}\,(tv)\right) = \mathrm{Exp}\,(tU)\mathrm{Exp}\,(tV)$$

for $\forall t\in\mathbb{R}$ and some fixed $a\in H$, $A := \rho(a)\in GL(n,\mathbb{R})$. Since the exponential map of $\mathfrak{g}l(n,\mathbb{R})$ is given by the exponential series, one concludes that

$$\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}W^k\right)A = \left(\sum_{l=0}^{\infty}\frac{t^l}{l!}U^l\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}V^m\right)\qquad\text{for }\forall t\in\mathbb{R}.$$

In particular, at $t=0$ there follows $A = I_n$. Then, by comparing the derivatives on both sides at $t=0$, we have

$$\begin{aligned}
W =&\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}W^k\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}U^l\right)\Big|_{t=0}\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}V^m\right)\Big|_{t=0}\\
&+\left(\sum_{l=0}^{\infty}\frac{t^l}{l!}U^l\right)\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}V^m\right)\Big|_{t=0} = U+V.
\end{aligned}$$

Consequently,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}(U+V)^k = \left(\sum_{l=0}^{\infty} \frac{t^l}{l!}U^l\right)\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}V^m\right) \qquad \text{for } \forall t \in \mathbb{R}.$$

Equating the second derivatives at $t=0$, one concludes that

$$\begin{aligned} U^2+UV+VU+V^2 = (U+V)^2 &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}(U+V)^k\right)\Big|_{t=0} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{l=0}^{\infty} \frac{t^l}{l!}U^l\right)\Big|_{t=0}\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}V^m\right)\Big|_{t=0} \\ &+ 2\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{l=0}^{\infty} \frac{t^l}{l!}U^l\right)\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}V^m\right)\Big|_{t=0} \\ &+ \left(\sum_{l=0}^{\infty} \frac{t^l}{l!}U^l\right)\Big|_{t=0}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}V^m\right)\Big|_{t=0} \\ &= U^2+2UV+V^2 \end{aligned}$$

whereas

$$VU = UV.$$

As a result,

$$[U,V] = \Big[(\mathrm{d}\rho)_{1_G}\, u, (\mathrm{d}\rho)_{1_G}\, v\Big] = (\mathrm{d}\rho)_{1_G}\, [u,v] = 0.$$

Due to the injectiveness of $(\mathrm{d}\rho)_{1_G} : \mathrm{Lie}\,(G) \to \mathfrak{g}l(n,\mathbb{R})$, there follows $[u,v]=0$, Q.E.D.

3. Left Invariant Metrics on Quasi Left Gyrovector Spaces

Definition 9. *If $f : M \to N$ is a smooth map of manifolds and g is a Riemannian metric on N then the metric f^*g, given by*

$$(f^*g)\,(u_p,v_p) := g_{f(p)}\,((\mathrm{d}f)_p u_p, (\mathrm{d}f)_p v_p) \qquad \textit{for } \forall u_p, v_p \in T_p^{\mathbb{R}}M \textit{ and } \forall p \in M$$

is called the pull-back of g by f.

Definition 10. *A Riemannian metric g on a manifold M is invariant under a diffeomorphism $f : M \to M$ if the pull-back $f^*g = g$ coincides with g.*

A Riemannian metric g is invariant with respect to a group G of diffeomorphisms of M if g is invariant under any element of G.

A Riemannian metric g on a left Lie loop $(G/H, \oplus_\sigma)$ is said to be left $\oplus_\sigma$-invariant if g is invariant under the left translations $L_{aH} : G/H \to G/H$, $L_{aH}(xH) = (aH) \oplus_\sigma (xH)$ for $\forall aH \in G/H$.

Let G be a connected Lie group, $H \subset G$ be a closed connected subgroup, $\sigma : G/H \to G$ be an analytic section of $\pi : G \to G/H$ and g be a Riemannian metric on G/H. If g is invariant under left G-multiplications on G/H, then g is left $\oplus_\sigma$-invariant and $\mathrm{Ad}\,(H)$-invariant. Indeed,

$$L_{aH}(xH) = \sigma(aH)xH$$

acts as a left multiplication by $\sigma(aH) \in G$ and

$$\mathrm{Ad}_{\,h}(xH) := \mathrm{Ad}_{\,h}(x)H = hxh^{-1}H = hxH$$

reduces to a left multiplication by $h \in H$. Conversely, $\pi\sigma = \mathrm{Id}_{G/H}$ implies that $\sigma(xH)H = xH$, whereas $h_x := [\sigma(xH)]^{-1}\, x \in H$ for $\forall x \in G$. If a Riemannian metric g on G/H is left $\oplus_\sigma$-invariant and $\mathrm{Ad}\,(H)$-invariant then g is invariant under the left multiplication by $\sigma(xH)$ and h_x. Consequently, g is invariant under the left multiplication by an arbitrary $x = \sigma(xH)h_x \in G$. Thus, we have proved the following

Lemma 11. *Let $(G/H, \oplus_\sigma)$ be a left Lie loop, associated with an analytic section $\sigma : G/H \to G$ of $\pi : G \to G/H$ with $[\sigma\,(G/H)]^{-1} = \sigma\,(G/H)$ and g be a Riemannian metric on G/H. Then g is left G-invariant if and only if g is left $\oplus_\sigma$-invariant and $\mathrm{Ad}\,(H)$-invariant.*

Proposition 12. *Let G be a connected Lie group with exponential map $\mathrm{Exp} : \mathrm{Lie}\,(G) \to G$ and $M \subset G$ be a complete, simply connected, real analytic submanifold through 1_G.*

Then the following are equivalent:

i) $\mathrm{Exp}\, : T^{\mathbb{R}}_{1_G} M \to M$ *is a global analytic diffeomorphism;*

ii) M *has non-positive sectional curvatures with respect to any left G-invariant metric g on G;*

iii) M *has non-positive sectional curvatures with respect to some left G-invariant metric g on G.*

Proof: Towards the proof of i) $\Rightarrow$ ii), let us suppose that $\mathrm{Exp} : T^{\mathbb{R}}_{1_G}M \to M$ is a global diffeomorphism and for some left G-invariant metric g on G there exists a point $p \in M$ and tangent vectors $u_p, w_p \in T^{\mathbb{R}}_p M$, such that the sectional curvature

$$K\left(\mathrm{Span}_{\mathbb{R}}(u_p, w_p)\right) = \frac{g_p(R_p(u_p, w_p)w_p, u_p)}{Area(u_p \vee w_p)} > 0.$$

Here R stands for the curvature tensor $R : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ of the Levi-Civita connection of g, acting on the analytic vector fields $\mathcal{V}$ on G, R_p is the restriction of R at $p \in M$, and $u_p \vee w_p$ denotes the parallelogram, spanned by u_p, w_p. Let $\lambda_p^{-1} = \lambda_{p^{-1}} : G \to G$ be the left multiplication by $p^{-1} \in G$ and $\left(\mathrm{d}\lambda_p^{-1}\right)_p : T^{\mathbb{R}}_p G \to T^{\mathbb{R}}_{1_G} G$ be its differential at p. Consider the totally geodesic surface

$$\Sigma = \Sigma(u_p, w_p) := \{\mathrm{Exp}\,[x(\mathrm{d}\lambda_p^{-1})_p u_p + y(\mathrm{d}\lambda_p^{-1})_p w_p] \;\; ; \;\; x, y \in \mathbb{R}\} \subset M \subset G.$$

Its tangent bundle is left G-invariant and

$$T^{\mathbb{R}}_p \Sigma = (\mathrm{d}\lambda_p)_{1_G} T^{\mathbb{R}}_{1_G}\Sigma = (\mathrm{d}\lambda_p)_{1_G}\,\mathrm{Span}_{\mathbb{R}}\{(\mathrm{d}\lambda_p^{-1})_p u_p, (\mathrm{d}\lambda_p^{-1})_p w_p\}$$
$$= \mathrm{Span}_{\mathbb{R}}\{u_p, w_p\}.$$

Let U, V be the left G-invariant analytic vector fields with $U_p = u_p$, $W_p = w_p$. Then U, W are parallel vector fields, generating the tangent bundle $T^{\mathbb{R}}\Sigma \to \Sigma$ at all the points of Σ. The areas of the parallelograms $U_t \vee W_t$, $t \in \Sigma$, as well as the corresponding values $R_t(U_t, W_t)W_t$ of the curvature tensor and the metric $g_t(R_t(U_t, W_t)W_t, U_t)$ are constant. Consequently, Σ has constant sectional (i.e., Gaussian) curvature. As a diffeomorphic image of $T^{\mathbb{R}}_{1_G}\Sigma \simeq \mathbb{R}^2$, the manifold Σ is contractible. Thus, the surface Σ with constant positive sectional curvatures is simply connected and, therefore, isometric to the sphere S^2. In particular, S^2 turns out to be topologically trivial, which is an absurd (e.g., $\pi_2(S^2) = \mathbb{Z}$).

Concerning iii) $\Rightarrow$ i), let g be a left G-invariant metric on G and $1_G \in M \subset G$ be a complete, simply connected submanifold, whose sectional curvatures with respect to g are non-positive. Then according to Cartan-Hadamard Theorem (cf. [1]), the exponential maps $\exp_x : T^{\mathbb{R}}_x M \to M$ at all the points $x \in M$ are diffeomorphisms. In particular, $\exp_{1_G} = \mathrm{Exp} : T^{\mathbb{R}}_{1_G}M \to M$ is a global diffeomorphism, Q.E.D.

Definition 13. *Let G be a connected Lie group and $H \subset G$ be a closed connected subgroup. The analytic section $\sigma : G/H \to G$ of $\pi : G \to G/H$ is said to be non-positively curved if its image $\sigma(G/H) \subset G$ is a complete simply connected manifold of non-positive sectional curvature with respect to some (and therefore all) G-invariant metrics on G.*

According to Proposition 12, an analytic section $\sigma : G/H \to G$ is non-positively curved exactly when the exponential map $\mathrm{Exp} : \mathrm{Lie}\,(G) \to G$ of G restricts to a global analytic diffeomorphism (7). In Section 1 we have already explained that it is equivalent to (8) being a diffeomorphism. Thus, an arbitrary non-positively curved analytic section $\sigma : G/H \to G$ is associated with a quasi left Lie gyrovector space $(G/H, \oplus_\sigma, \otimes_\sigma)$ (cf. Proposition 7).

Corollary 14. *Let $(G/H, \oplus_\sigma, \otimes_\sigma)$ be a quasi left Lie gyrovector space, associated with a non-positively curved real analytic section $\sigma : G/H \to G$ and let g be a left G-invariant metric on G/H. Then:*

i) *the g-geodesics $\gamma_{a,b}(t)$ through $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$ coincide with the gyro-lines*

$$\gamma_{a,b}(t) = a \oplus_\sigma \{t \otimes_\sigma (\ominus_\sigma a \oplus_\sigma b)\}, \quad t \in \mathbb{R} \tag{12}$$

ii) *the Thomas gyrations $\mathrm{gyr}\,[a, b]$ are isometries of g for $\forall a, b \in G/H$.*

Proof: i) According to Theorem IV.3.3 (iii) from Helgason's book [6], for an arbitrary $u \in \mathrm{Lie}\,(G)$ the g-geodesic from $\check{o} = H \in G/H$, tangent to $(\mathrm{d}\pi)_{1_G}(u) \in T^{\mathbb{R}}_{\check{o}}(G/H)$ is $\mathrm{Exp}\,(tu)H$, where $t \in \mathbb{R}$. In particular, for $\forall a, b \in G/H$ and $u := \mathrm{Exp}^{-1}\sigma(\ominus_\sigma a \oplus_\sigma b) \in T^{\mathbb{R}}_{1_G}\sigma\,(G/H)$ the real analytic curves $t \otimes_\sigma (\ominus_\sigma a \oplus_\sigma b) = t \otimes_\sigma (\mathrm{Exp}\,(u)H) = \mathrm{Exp}\,(tu)H$, $\forall t \in \mathbb{R}$ are g-geodesics. Further, Lemma 11 reveals that the metric g is left $\oplus_\sigma$-invariant. Therefore, the left translations $L_a : G/H \to G/H$, $L_a(x) = a \oplus_\sigma x$ are isometries for g and transform the geodesics $t \otimes_\sigma (\ominus_\sigma a \oplus_\sigma b)$ into the geodesics (12) through $\gamma_{a,b}(0) = a \oplus_\sigma \check{o} = a$ and $\gamma_{a,b}(1) = a \oplus_\sigma (\ominus_\sigma a \oplus_\sigma b) = b$.

Conversely, if $\gamma_{a,b} : \mathbb{R} \to G/H$ is a g-geodesic through $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$ then $\mu(t) := L_{\ominus_\sigma a}(\gamma_{a,b}(t))$ is a g-geodesic through $\mu(0) = \check{o}$ and $\mu(1) = \ominus_\sigma a \oplus_\sigma b$. As far as the metric g on G/H is complete and non-positively curved, the geodesic $\mu(t)$ through $\mu(0) = \check{o}$ and $\mu(1) = \ominus_\sigma a \oplus_\sigma b$ is unique. Thus, $\mu(t) = \ominus_\sigma a \otimes_\sigma \gamma_{a,b}(t) = \mathrm{Exp}\,(t\mathrm{Exp}^{-1}\sigma(\ominus_\sigma a \oplus_\sigma b))H = t \otimes_\sigma (\ominus_\sigma a \oplus_\sigma b)$, whereas (12).

ii) Lemma 4 b) has established that for $\forall a, b \in G/H$ the Thomas gyrations $\mathrm{gyr}\,[a, b]$ act as conjugations by $h_{a,b} := [\sigma(a \oplus_\sigma b)]^{-1}\,\sigma(a)\sigma(b) \in H$. On the other hand, by Lemma 11, the left G-invariant metric g on G/H is $\mathrm{Ad}\,(H)$-invariant. Therefore, the H-conjugations and, in particular, the gyrations $\mathrm{gyr}\,[a, b]$ are isometries for g, Q.E.D.

Corollary 15. *Suppose that G/H is a homogeneous space with left G-invariant metric g and $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a quasi left Lie gyrovector space, associated with* *a non-positively curved analytic section* $\sigma : G/H \to G$ *of* $\pi : G \to G/H$. *Let*

$$||x|| := \left[g_{\check{o}}((\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x), (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x))\right]^{\frac{1}{2}} \quad \textit{for } \forall x \in G/H. \quad (13)$$

Then:

i) *the distance function* d *of* g *satisfies*

$$d(x,y) = || \ominus_\sigma x \oplus_\sigma y|| \quad \textit{for } \forall x, y \in G/H$$

ii) $||x|| \geq 0$ *with* $||x|| = 0$ *if and only if* $x = \check{o}$

iii) $||t \otimes_\sigma x|| = |t|||x||$ *for* $\forall t \in \mathbb{R}, \forall x \in G/H$

iv) $||x \oplus_\sigma y|| \leq ||x|| + ||y||$ *for* $\forall x, y \in G/H$

v) $||\mathrm{Exp}\,\left(\mathrm{Exp}^{-1}\sigma(x) + \mathrm{Exp}^{-1}\sigma(y)\right) H|| \leq ||x|| + ||y||$ *for* $\forall x, y \in G/H$.

Proof: i) According to Lemma 11, the left translations $L_{\ominus_\sigma x} : G/H \to G/H$, $L_{\ominus_\sigma x}(y) = \ominus_\sigma x \oplus_\sigma y$ are isometries for the left G-invariant metric g. Therefore,

$$d(x,y) = d(\check{o}, \ominus_\sigma x \oplus_\sigma y) \quad \text{for } \forall x, y \in G/H$$

and it suffices to justify the equality

$$d(\check{o}, x) = ||x|| \quad \text{for } \forall x \in G/H.$$

To this end, let us recall from Corollary 14 i) that

$$\gamma(t) = t \otimes_\sigma x = \mathrm{Exp}\,(t\mathrm{Exp}^{-1}\sigma(x))H$$

is the unique geodesic from $\gamma(0) = \check{o}$ to $\gamma(1) = x$. The distance $d(\check{o}, x)$ equals the length of the geodesic segment $\gamma(t)$ for $t \in [0,1]$. By the definition of a geodesic, the tangent vector field $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$ is parallel along itself, so that the lengths

$$\begin{aligned} g_{\gamma(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right) &= g_{\check{o}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|_{t=0}, \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|_{t=0}\right) \\ &= g_{\check{o}}\left((\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x), (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x)\right) \end{aligned}$$

are constant for all $t \in [0,1]$. Consequently,

$$\begin{aligned} d(\check{o}, x) &= \int_0^1 \left[g_{\gamma(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right)\right]^{\frac{1}{2}} \mathrm{d}t \\ &= \left[g_{\check{o}}\left((\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x), (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x)\right)\right]^{\frac{1}{2}} \int_0^1 \mathrm{d}t = ||x||. \end{aligned}$$

ii) By the definition of a Riemannian metric g, its restriction $g_{\check{o}}$ to the tangent space at the origin is a positive definite symmetric bilinear form

$$g_{\check{o}} : T_{\check{o}}^{\mathbb{R}}(G/H) \times T_{\check{o}}^{\mathbb{R}}(G/H) \longrightarrow \mathbb{R}.$$

Therefore $g_{\check{o}}(\xi,\xi) \geq 0$ for $\forall \xi \in T_{\check{o}}^{\mathbb{R}}(G/H)$ and $g_{\check{o}}(\xi,\xi) = 0$ only when $\xi = 0$. Putting $\xi := (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x)$ for an arbitrary $x \in G/H$, one gets $||x|| \geq 0$ with $||x|| = 0$ if and only if $x = \sigma^{-1}\mathrm{Exp}\,(\mathrm{d}\sigma)_{\check{o}}\xi = \sigma^{-1}\mathrm{Exp}\,(\mathrm{d}\sigma)_{\check{o}}0 = \sigma^{-1}\mathrm{Exp}\,(0) = \sigma^{-1}(1_G) = \check{o}$.

iii) For arbitrary $t \in \mathbb{R}$ and $x \in G/H$ one has $t \otimes_\sigma x = \mathrm{Exp}\,(t\mathrm{Exp}^{-1}\sigma(x))H$. Since $g_{\check{o}}(\ ,\ )$ is bilinear, one concludes that

$$\begin{aligned}||t \otimes_\sigma x|| &= \left[g_{\check{o}}(t(\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x), t(\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x))\right]^{\frac{1}{2}} \\ &= \left[t^2 g_{\check{o}}((\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x), (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x))\right]^{\frac{1}{2}} = |t|||x||.\end{aligned}$$

iv) The triangle inequality for the distance provides

$$d(\ominus_\sigma x, y) \leq d(\ominus_\sigma x, \check{o}) + d(\check{o}, y) \quad \text{for } \forall x, y \in G/H.$$

According to i) and $\ominus_\sigma x = (-1) \otimes_\sigma x$, one can express $d(\ominus_\sigma x, y) = ||x \oplus_\sigma y||$, $d(\ominus_\sigma x, \check{o}) = ||x||$, $d(\check{o}, y) = ||y||$. Thus, the aforementioned triangle inequality takes the form

$$||x \oplus_\sigma y|| \leq ||x|| + ||y||.$$

v) The triangle inequality in the Euclidean inner product vector space $\left(T_{\check{o}}^{\mathbb{R}}(G/H), g_{\check{o}}\right)$ states that

$$[g_{\check{o}}(\xi+\eta, \xi+\eta)]^{\frac{1}{2}} \leq [g_{\check{o}}(\xi,\xi)]^{\frac{1}{2}} + [g_{\check{o}}(\eta,\eta)]^{\frac{1}{2}}$$

for arbitrary $\xi, \eta \in T_{\check{o}}^{\mathbb{R}}(G/H)$.

If $\xi := (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(x)$ and $\eta := (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma(y)$ then

$$\begin{aligned}&\big[g_{\check{o}}\big((\mathrm{d}\pi)_{1_G}(\mathrm{Exp}^{-1}\sigma(x) \\ &\quad + \mathrm{Exp}^{-1}\sigma(y)), (\mathrm{d}\pi)_{1_G}(\mathrm{Exp}^{-1}\sigma(x) + \mathrm{Exp}^{-1}\sigma(y))\big)\big]^{\frac{1}{2}} \leq ||x|| + ||y||.\end{aligned}$$

Applying $\sigma\pi\mathrm{Exp}\,(\zeta) = \sigma\pi\sigma(\mathrm{Exp}\,(\zeta)H) = \sigma(\mathrm{Exp}\,(\zeta)H) = \mathrm{Exp}\,(\zeta)$ to the tangent vector $\zeta = \mathrm{Exp}^{-1}\sigma(x) + \mathrm{Exp}^{-1}\sigma(y)$, one expresses

$$\begin{aligned}&(\mathrm{d}\pi)_{1_G}(\mathrm{Exp}^{-1}\sigma(x) + \mathrm{Exp}^{-1}\sigma(y)) \\ &\qquad = (\mathrm{d}\pi)_{1_G}\mathrm{Exp}^{-1}\sigma\left[\mathrm{Exp}\,(\mathrm{Exp}^{-1}\sigma(x) + \mathrm{Exp}^{-1}\sigma(y))H\right]\end{aligned}$$

for arbitrary $x, y \in G/H$, Q.E.D.

Let us conclude the section, observing that the straightforward application of Corollary I.13.2 from Helgason's book [6] yields the following:

Corollary 16. *Let $\sigma : G/H \to G$ be a non-positively curved analytic section of $\pi : G \to G/H$, associated with a quasi left Lie gyrovector space $(G/H, \oplus_\sigma, \otimes_\sigma)$ and g be a left G-invariant metric on G/H with distance function $d : (G/H) \times (G/H) \to G/H$. Define the norm of $x \in G/H$ by (13) and put*

$$\sphericalangle(x, y) := \arccos \frac{g_{\check{o}}((\mathrm{d}\pi)_{1_G} \mathrm{Exp}^{-1} \sigma(x), (\mathrm{d}\pi)_{1_G} \mathrm{Exp}^{-1} \sigma(y))}{||x|| ||y||}$$

for the angle between the geodesic rays from $\check{o}$ through $x \in G/H$ and $y \in G/H$. Then for arbitrary $a, b, c \in G/H$ there holds

i) $d^2(a, b) \geq d^2(a, c) + d^2(b, c) - 2d(a, c)d(b, c) \cos \sphericalangle (\ominus_\sigma c \oplus_\sigma a, \ominus_\sigma c \oplus_\sigma b)$

ii) $\sphericalangle (\ominus_\sigma a \oplus_\sigma b, \ominus_\sigma a \oplus_\sigma c) + \sphericalangle (\ominus_\sigma b \oplus_\sigma c, \ominus_\sigma b \oplus_\sigma a)$
$+ \sphericalangle (\ominus_\sigma c \oplus_\sigma a, \ominus_\sigma c \oplus_\sigma b) \leq \pi.$

4. Left Gyrogroups and Left Lie Gyrovector Spaces

Definition 17. *A left loop $(\mathcal{L}, \oplus)$, subject to the gyro-automorphism property*

$$\mathrm{gyr}\,[a, b](x \oplus y) = (\mathrm{gyr}\,[a, b]x) \oplus (\mathrm{gyr}\,[a, b]y) \quad \textit{for } \forall a, b, x, y \in \mathcal{L} \tag{14}$$

is called a left gyrogroup.

The left gyrogroups are introduced and studied by A. Ungar in a series of articles, starting with [13], where they are initially called weakly associative groups.

The following example provides two different sections τ, σ of the same space G_o/H_o of left cosets, such that $(G_o/H_o, \oplus_\tau)$ is a group and $(G_o/H_o, \oplus_\sigma)$ is a non-group left gyrogroup. More precisely, let $G_o = \mathrm{Sym}\,(3)$ be the symmetric group, acting on the set $\{1, 2, 3\}$. Denote by $(i_1, \ldots, i_k)$ the cycle, transforming i_1 in i_2, i_2 in i_3, etc., i_{k-1} in i_k and i_k in i_1. Then fix the cyclic subgroup $H_o := \langle (1, 2) \rangle \subset \mathrm{Sym}\,(3)$ of order 2. For

$$\tau_1 := (1, 2, 3), \qquad \tau_2 := (1, 3, 2) = (1, 2, 3)^2$$
$$\sigma_1 := (2, 3), \qquad \sigma_2 := (1, 3)$$

there are disjoint decompositions into unions of left cosets

$$G_o = H_o \cup \tau_1 H_o \cup \tau_2 H_o = H_o \cup \sigma_1 H_o \cup \sigma_2 H_o$$

with $\tau_i H_o = \sigma_i H_o$ for $i = 1, 2$. For convenience, introduce $\tau_0 = \sigma_0 := \mathrm{Id}_{\{1,2,3\}}$ and define the sections

$$\begin{gathered} \tau : G_o/H_o \longrightarrow G_o \\ \tau(\tau_i H_o) := \tau_i, \quad i = 0, 1, 2 \end{gathered}$$

and

$$\begin{gathered} \sigma : G_o/H_o \longrightarrow G_o \\ \sigma(\sigma_i H_o) := \sigma_i, \quad i = 0, 1, 2. \end{gathered}$$

Since the image $\tau(G_o/H_o) = \left\{ \tau_1^i ;\, i = 0, 1, 2 \right\}$ of τ is the alternative group A_3, consisting of the even permutations of $1, 2, 3$, the operation

$$\begin{gathered} \oplus_\tau : (G_o/H_o) \times (G_o/H_o) \longrightarrow G_o/H_o \\ (\tau_i H_o) \oplus_\tau (\tau_j H_o) := \tau_i \tau_j H_o = \tau_1^{i+j} H_o \end{gathered}$$

turns G_o/H_o into a cyclic group of order 3.
The image $\sigma(G_o/H_o) = \left\{ \mathrm{Id}_{\{1,2,3\}}, (2,3), (1,3) \right\}$ of σ is closed under inversion, as far as $(i,j)^{-1} = (i,j)$ for any transposition (i,j). Therefore,

$$\begin{gathered} \oplus_\sigma : (G_o/H_o) \times (G_o/H_o) \longrightarrow G_o/H_o \\ (\sigma_i H_o) \oplus_\sigma (\sigma_j H_o) := \sigma_i \sigma_j H_o \end{gathered}$$

is a left loop operation on G_o/H_o. In order to examine the truth of (14), note the equalities $\mathrm{gyr}\,[H_o, \sigma_i H_o] = \mathrm{gyr}\,[\sigma_i H_o, H_o] = \mathrm{Id}_{\{1,2,3\}}$ and $\mathrm{gyr}\,[\sigma_i H_o, \sigma_i H_o] = \mathrm{Id}_{\{1,2,3\}}$ for all $0 \leq i \leq 2$. It suffices to study the action of $\mathrm{gyr}\,[\sigma_1 H_o, \sigma_2 H_o] = \mathrm{Ad}_{h(\sigma_1\sigma_2)}$ and $\mathrm{gyr}\,[\sigma_2 H_o, \sigma_1 H_o] = \mathrm{Ad}_{h(\sigma_2\sigma_1)}$. Making use of $\sigma_1\sigma_2 = \sigma_2(1,2)$, $\sigma_2\sigma_1 = \sigma_1(1,2)$, one obtains that

$$\begin{aligned} h(\sigma_1\sigma_2) &:= [\sigma(\sigma_1\sigma_2 H_o)]^{-1}\, \sigma_1\sigma_2 = \sigma_2^{-1}\sigma_2(1,2) = (1,2) \\ h(\sigma_2\sigma_1) &:= [\sigma(\sigma_2\sigma_1 H_o)]^{-1}\, \sigma_2\sigma_1 = \sigma_1^{-1}\sigma_1(1,2) = (1,2). \end{aligned}$$

According to $\mathrm{Ad}_{(1,2)}\sigma_0 = \sigma_0$, $\mathrm{Ad}_{(1,2)}\sigma_1 = \sigma_2$, $\mathrm{Ad}_{(1,2)}\sigma_2 = \sigma_1$, one can write $\mathrm{Ad}_{(1,2)}\sigma_{\overline{i}} = \sigma_{-\overline{i}}$ for the congruence classes $\overline{i}$, $-\overline{i}$ modulo 3. On the other hand, observe that if $\sigma(\sigma_{\overline{k}}\sigma_{\overline{l}} H_o) = \sigma_{\overline{m}}$ then $\sigma(\sigma_{-\overline{k}}\sigma_{-\overline{l}} H_o) = \sigma_{-\overline{m}}$. This is clear when $\overline{k} = \overline{0}$ or $\overline{l} = \overline{0}$, as well as in the case of $\overline{k} = \overline{l}$. For $(\overline{k}, \overline{l}) = (\overline{1}, \overline{2})$ or $(\overline{k}, \overline{l}) = (\overline{2}, \overline{1})$

one has $\sigma(\sigma_{\overline{k}}\sigma_{\overline{l}}H_o) = \sigma_{\overline{l}}$ and $\sigma(\sigma_{-\overline{k}}\sigma_{-\overline{l}}H_o) = \sigma(\sigma_{\overline{l}}\sigma_{\overline{k}}H_o) = \sigma_{\overline{k}} = \sigma_{-\overline{l}}$. As a result, if $\sigma\left((\sigma_{\overline{k}}H_o) \oplus_\sigma (\sigma_{\overline{l}}H_o)\right) = \sigma_{\overline{m}}H_o$ then

$$\begin{aligned}\mathrm{Ad}_{(1,2)}\left\{(\sigma_{\overline{k}}H_o) \oplus_\sigma (\sigma_{\overline{l}}H_o)\right\} &= \mathrm{Ad}_{(1,2)}(\sigma_{\overline{m}})H_o = \sigma_{-\overline{m}}H_o \\ &= \sigma_{-\overline{k}}\sigma_{-\overline{l}}H_o = \left(\sigma_{-\overline{k}}H_o\right) \oplus_\sigma \left(\sigma_{-\overline{l}}H_o\right) \\ &= \left\{\mathrm{Ad}_{(1,2)}(\sigma_{\overline{k}})H_o\right\} \oplus_\sigma \left\{\mathrm{Ad}_{(1,2)}(\sigma_{\overline{l}})H_o\right\}\end{aligned}$$

for $\forall \overline{k}, \overline{l} \in \{\overline{0}, \overline{1}, \overline{2}\}$. Verifying the gyro-automorphism property (14), we establish that $(G_o/H_o, \oplus_\sigma)$ is a left gyrogroup.

Definition 18. *The bijections $\mathcal{L} \to \mathcal{L}$ of a set $\mathcal{L}$ form a group $B = B(\mathcal{L})$ with respect to the composition.*

The automorphism group $\mathrm{Aut}(\mathcal{L}, \oplus)$ of a groupoid $(\mathcal{L}, \oplus)$ consists of the bijections $\varphi \in B(\mathcal{L})$, preserving the operation $\oplus$, i.e.,

$$\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b) \quad \text{for } \forall a, b \in \mathcal{L}.$$

Thus, a left gyrogroup is a left loop $(\mathcal{L}, \oplus)$, whose gyrations $\mathrm{gyr}\,[a, b]$ are $\oplus$-automorphisms for $\forall a, b \in \mathcal{L}$.

Lemma 19. *Let G be a group, $H \subset G$ be a subgroup and $\sigma : G/H \to G$ be a section of $\pi : G \to G/H$. Suppose that $S := \sigma(G/H)$ is closed under inversion,*

$$S^{-1} = S \tag{15}$$

and the discrepancies

$$d^h(x) := \mathrm{Ad}_{h^{-1}}\left\{[\sigma(\mathrm{Ad}_h(x)H)]^{-1}\,\mathrm{Ad}_h(x)\right\} \tag{16}$$

belong to $\cap_{g\in G}\left(gHg^{-1}\right) = \cap_{y\in S}\left(yHy^{-1}\right)$ for $\forall x \in S, \forall h \in H$. Then $(G/H, \oplus_\sigma)$ is a left gyrogroup with respect to the induced operation (4).

Proof: According to Lemma 2, $(G/H, \oplus_\sigma)$ is a left loop, provided (15).

For arbitrary $a = \sigma(aH)$, $b = \sigma(bH)$, $x = \sigma(xH)$, and $y \in G$, there holds

$$\begin{aligned}\mathrm{gyr}\,[aH, bH]\{(xH) \oplus_\sigma (yH)\} &= \mathrm{gyr}\,[aH, bH](xyH) = \left[\mathrm{Ad}_{h(ab)}(xy)\right] H \\ &= \mathrm{Ad}_{h(ab)}(x)\mathrm{Ad}_{h(ab)}(y)H\end{aligned}$$

where $h(ab) := [\sigma(abH)]^{-1}\, ab \in H$. On the other hand,

$$\begin{aligned}&\{\mathrm{gyr}\,[aH, bH](xH)\} \oplus_\sigma \{\mathrm{gyr}\,[aH, bH](yH)\} \\ &\qquad = \left\{\left[\mathrm{Ad}_{h(ab)}(x)\right] H\right\} \oplus_\sigma \left\{\left[\mathrm{Ad}_{h(ab)}(y)\right] H\right\} \\ &\qquad = \sigma\left(\mathrm{Ad}_{h(ab)}(x)\right) \mathrm{Ad}_{h(ab)}(y)H.\end{aligned}$$

Therefore, the gyro-automorphism property (14) is satisfied if and only if $\left[\sigma\left(\mathrm{Ad}_{h(ab)}(x)H\right)\right]^{-1}\mathrm{Ad}_{h(ab)}(x)$ belongs to the stabilizer

$$\mathrm{Stab}\left(\mathrm{Ad}_{h(ab)}(y)H\right) := \left\{g\in G \ ; \ g\mathrm{Ad}_{h(ab)}(y)H = \mathrm{Ad}_{h(ab)}(y)H\right\}.$$

Since

$$\mathrm{Stab}\left(\mathrm{Ad}_{h(ab)}(y)H\right) = \mathrm{Ad}_{h(ab)}(y)H\left[\mathrm{Ad}_{h(ab)}(y)\right]^{-1} = \mathrm{Ad}_{h(ab)}\mathrm{Ad}_{y}H$$

the aforementioned condition is equivalent to $d^{h(ab)}(x)\in \mathrm{Ad}_{y}(H)$ for all $y\in G$. Thus, $d^{h(ab)}(x)\in \cap_{y\in G}\left(yHy^{-1}\right)$ for $\forall a,b,\in G/H$ is necessary and sufficient for $(G/H,\oplus_\sigma)$ to be a left gyrogroup. Since an arbitrary $y\in G$ can be written in the form $y=\sigma(yH)h_y$ for some $h_y\in H$ and $yHy^{-1}=\sigma(yH)H\left[\sigma(yH)\right]^{-1}$, we also have $\cap_{y\in G}\left(yHy^{-1}\right)=\cap_{y\in S}\left(yHy^{-1}\right)$, Q.E.D.

Here is an example of a left loop $(G_1/H_1,\oplus_\sigma)$, which is not a left gyrogroup. Let $G_1 := A_4$ be the alternative group, consisting of the even permutations of $1,2,3,4$ and $H_1 := \langle(1,2,3)\rangle$ be its cyclic subgroup of order 3, generated by the cycle $(1,2,3)$. One can represent as a disjoint union

$$G_1 = H_1\cup(2,3,4)H_1\cup(2,4,3)H_1\cup(1,4)(2,3)H_1$$

and define the section

$$\sigma : G_1/H_1 \longrightarrow \sigma\left(G_1/H_1\right) = \left\{\mathrm{Id}_{\{1,2,3,4\}},(2,3,4),(2,4,3),(1,4)(2,3)\right\}.$$

As far as $[(1,4)(2,3)]^{-1}=(1,4)(2,3)$, $(2,3,4)^{-1}=(2,4,3)$, the set $\sigma\left(G_1/H_1\right)$ is closed under inversion and $(G_1/H_1,\oplus_\sigma)$ is a left loop. Under a multiplication from left to right, note that $(2,3,4)(1,4)(2,3)=(1,4,3)$ and

$$h := \left[\sigma((1,4,3)H_1)\right]^{-1}(1,4,3) = \left[(1,4)(2,3)\right]^{-1}(1,4,3) = (1,3,2)$$

so that the gyration

$$\mathrm{gyr}\left[(2,3,4)H_1,(1,4)(2,3)H_1\right] = \mathrm{Ad}_{(1,3,2)}$$

acts as a conjugation by $(1,3,2)\in H_1$. On the one hand,

$$\begin{aligned}\mathrm{gyr}&\left[(2,3,4)H_1,(1,4)(2,3)H_1\right]\left\{((2,3,4)H_1)\oplus_\sigma((2,3,4)H_1)\right\}\\ &= \mathrm{Ad}_{(1,3,2)}((2,4,3))H_1 = (1,3,4)H_1 = (2,3,4)H_1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\{\text{gyr}\,[(2,3,4)H_1,\ &(1,4)(2,3)H_1](2,3,4)H_1\}\\ &\oplus_\sigma \{\text{gyr}\,[(2,3,4)H_1,(1,4)(2,3)H_1](2,3,4)H_1\}\\ &= \left\{\text{Ad}_{(1,3,2)}((2,3,4))H_1\right\} \oplus_\sigma \left\{\text{Ad}_{(1,3,2)}((2,3,4))H_1\right\}\\ &= ((1,4,3)H_1) \oplus_\sigma ((1,4,3)H_1)\\ &= ((1,4)(2,3)H_1) \oplus_\sigma ((1,4)(2,3)H_1) = H_1.\end{aligned}$$

Therefore, the gyro-automorphism law (14) is violated and $(G_1/H_1, \oplus_\sigma)$ is not a left gyrogroup.

In an interesting paper [2] on left gyrogroups, Feder studies the following question: Suppose that T is a subset of a finite group G, $1_G \in T$, and for $\forall a, ax, ay \in T$ there exists z from the commutator of the group, generated by x, y, such that $x \odot_a y = xyz \in T$. The problem is to obtain sufficient conditions for $\left(a^{-1}T, \odot_a\right)$ to be left gyrogroups for $\forall a \in T$.

Definition 20. *The groupoids* $(\mathcal{L}_1, \oplus_1)$ *and* $(\mathcal{L}_2, \oplus_2)$ *are isomorphic if there is a bijective map*

$$\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$$

with

$$\varphi(x \oplus_1 y) = \varphi(x) \oplus_2 \varphi(y) \quad \textit{for } \forall x, y \in \mathcal{L}_1.$$

The following result is proved by Ungar in [15]. We provide here the argument for the sake of completeness.

Proposition 21. *For any left gyrogroup* $(\mathcal{L}, \oplus)$ *there exists a group* G, *a subgroup* $H \subset G$ *and a section* $\sigma : G/H \to G$ *of* $\pi : G \to G/H$ *with*

$$[\sigma\,(G/H)]^{-1} = \sigma\,(G/H) \quad \textit{and} \quad h\sigma\,(G/H)\,h^{-1} \subseteq \sigma\,(G/H) \quad \textit{for } \forall h \in H$$

such that $(\mathcal{L}, \oplus)$ *is isomorphic to* $(G/H, \oplus_\sigma)$ *.*

Proof: Let $(\mathcal{L}, \oplus)$ be a left loop and H_o be any subgroup of the group $\text{Aut}(\mathcal{L}, \oplus)$ of the $\oplus$-automorphisms of $\mathcal{L}$, that contains all the gyrations $\text{gyr}\,[a, b]$, $a, b \in \mathcal{L}$. On the set $G := \mathcal{L} \times H_o$ consider the operation

$$(x, \alpha) \circ (y, \beta) := (x \oplus \alpha(y), \text{gyr}\,[x, \alpha(y)]\alpha\beta). \tag{17}$$

One checks straightforwardly that $1_G := (\check{o}, \mathrm{Id}_{\mathcal{L}})$ is a two-sided neutral element for $\circ$, i.e.,

$$(x,\alpha)\circ(\check{o},\mathrm{Id}_{\mathcal{L}}) = (x\oplus\alpha(\check{o}), \mathrm{gyr}\,[x,\alpha(\check{o})]\alpha) = (x\oplus\check{o}, \mathrm{gyr}\,[x,\check{o}]\alpha) = (x,\alpha)$$
$$(\check{o},\mathrm{Id}_{\mathcal{L}})\circ(x,\alpha) = (\check{o}\oplus x, \mathrm{gyr}\,[\check{o},x]\alpha) = (x,\alpha)$$

bearing in mind that $\alpha(\check{o}) = \check{o}$ for $\forall\alpha \in H_o = \mathrm{Aut}(\mathcal{L},\oplus)$ and $\mathrm{gyr}\,[x,\check{o}] = \mathrm{gyr}\,[\check{o},x] = \mathrm{Id}_{\mathcal{L}}$. Making use of $\mathrm{gyr}\,[x,\ominus x] = \mathrm{Id}_{\mathcal{L}}$ for $\forall x \in \mathcal{L}$, one verifies that

$$(x,\alpha)\circ\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right) = (x\oplus(\ominus x), \mathrm{gyr}\,[x,\ominus x]\mathrm{Id}_{\mathcal{L}}) = (\check{o},\mathrm{Id}_{\mathcal{L}})$$
$$\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)\circ(x,\alpha)$$
$$= \left(\left(\ominus\alpha^{-1}(x)\right)\oplus\alpha^{-1}(x), \mathrm{gyr}\,[\ominus\alpha^{-1}(x),\alpha^{-1}(x)]\mathrm{Id}_{\mathcal{L}}\right) = (\check{o},\mathrm{Id}_{\mathcal{L}}).$$

In other words,

$$(x,\alpha)^{-1} = \left(\alpha^{-1}(\ominus x),\alpha^{-1}\right) \qquad (18)$$

is a two-sided inverse of $(x,\alpha)\in G$.

The associativity of $\circ$ will be derived by constructing an injective homomorphism

$$\varphi : (G,\circ) \longrightarrow (B,.)$$

in the group $(B,.)$ of the bijections $\mathcal{L}\to\mathcal{L}$. Namely, for $\forall(x,\alpha)\in G = \mathcal{L}\times H_o$ let us define

$$\varphi(x,\alpha) : \mathcal{L}\longrightarrow\mathcal{L}$$
$$\varphi(x,\alpha)(y) := x\oplus\alpha(y).$$

According to

$$\varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)\varphi(x,\alpha)(y) = \varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)(x\oplus\alpha(y))$$
$$= \ominus\alpha^{-1}(x)\oplus\left\{\alpha^{-1}(x)\oplus y\right\} = y$$
$$\varphi(x,\alpha)\varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)(y) = \varphi(x,\alpha)\left(\ominus\alpha^{-1}(x)\oplus\alpha^{-1}(y)\right)$$
$$= x\oplus[\ominus x\oplus y] = y$$

all $\varphi(x,\alpha)$ are invertible and $[\varphi(x,\alpha)]^{-1} = \varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)$. Therefore $\varphi(x,\alpha) \in B$.

Towards the verification of the injectiveness of φ, let us suppose that $\varphi(x,\alpha) = \varphi(y,\beta)$ for some $(x,\alpha),(y,\beta)\in G$. Then

$$\check{o} = \mathrm{Id}_{\mathcal{L}}(\check{o}) = [\varphi(x,\alpha)]^{-1}\varphi(y,\beta)(\check{o}) = \varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)(y\oplus\beta(\check{o}))$$
$$= \varphi\left(\alpha^{-1}(\ominus x),\alpha^{-1}\right)(y) = \alpha^{-1}(\ominus x)\oplus\alpha^{-1}(y) = \alpha^{-1}(\ominus x\oplus y)$$

implies $\ominus x \oplus y = \alpha(\check{o}) = \check{o}$, so that $x = y$. Further, for $\forall z \in \mathcal{L}$ the identities

$$\begin{aligned} z = \mathrm{Id}_{\mathcal{L}}(z) &= [\varphi(x,\alpha)]^{-1}\varphi(x,\beta)(z) = \varphi\left(\alpha^{-1}(\ominus x), \alpha^{-1}\right)(x \oplus \beta(z)) \\ &= \alpha^{-1}(\ominus x) \oplus \alpha^{-1}(x \oplus \beta(z)) = \alpha^{-1}[\ominus x \oplus (x \oplus \beta(z))] = \alpha^{-1}\beta(z) \end{aligned}$$

reveal that $\alpha^{-1}\beta = \mathrm{Id}_{\mathcal{L}}$, i.e., $\alpha = \beta$ and $\varphi : G \to B$ is injective.

Next, φ is claimed to be a homomorphism with respect to the binary operation $\circ$ of G and the group multiplication in B. Namely,

$$\varphi((x,\alpha) \circ (y,\beta)) = \varphi(x,\alpha)\varphi(y,\beta) \quad \text{for } \forall (x,\alpha),(y,\beta) \in G = \mathcal{L} \times H_o.$$

For arbitrary $z \in \mathcal{L}$, let us observe that

$$\begin{aligned} \varphi((x,\alpha)\circ(y,\beta))(z) &= \varphi(x \oplus \alpha(y), \mathrm{gyr}\,[x,\alpha(y)]\alpha\beta)(z) \\ &= (x \oplus \alpha(y)) \oplus (\mathrm{gyr}\,[x,\alpha(y)]\alpha\beta(z)). \end{aligned}$$

Then the left gyroassociative law implies

$$(x \oplus \alpha(y)) \oplus (\mathrm{gyr}\,[x,\alpha(y)]\alpha\beta(z)) = x \oplus [\alpha(y) \oplus \alpha\beta(z)].$$

Consequently,

$$\varphi((x,\alpha)\circ(y,\beta))(z) = x \oplus \alpha\{y \oplus \beta(z)\} = \varphi(x,\alpha)(y \oplus \beta(z)) = \varphi(x,\alpha)\varphi(y,\beta)(z).$$

Now, the associative law for the group multiplication in B provides

$$\varphi((g_1 \circ g_2)\circ g_3) = \{\varphi(g_1)\varphi(g_2)\}\varphi(g_3) = \varphi(g_1)\{\varphi(g_2)\varphi(g_3)\} = \varphi(g_1 \circ (g_2 \circ g_3))$$

for $\forall g_1, g_2, g_3 \in G$. Putting together with the injectiveness of φ, one derives the associative law

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

establishing that $(G, \circ)$ is a group.

One can identify H_o with $H := \{(\check{o},\alpha);\ \alpha \in H_o\} \subset G$ and observe that $(\check{o},\alpha)^{-1} = \left(\alpha^{-1}(\ominus\check{o}), \alpha^{-1}\right) = \left(\alpha^{-1}(\check{o}), \alpha^{-1}\right) = \left(\check{o}, \alpha^{-1}\right)$, $(\check{o},\alpha)\circ(\check{o},\beta) = (\check{o} \oplus \alpha(\check{o}), \mathrm{gyr}\,[\check{o},\alpha(\check{o})]\alpha\beta) = (\check{o},\alpha\beta)$. Therefore H is a subgroup of G. Taking into account that

$$\begin{aligned} (x,\alpha)\circ H = (x,\alpha)\circ(\check{o},\alpha^{-1})\circ H &= \left(x \oplus \alpha(\check{o}), \mathrm{gyr}\,[x,\alpha(\check{o})]\alpha\alpha^{-1}\right)\circ H \\ &= (x, \mathrm{gyr}\,[x,\check{o}])\circ H = (x, \mathrm{Id}_{\mathcal{L}})\circ H \end{aligned}$$

for $\forall (x,\alpha) \in G = \mathcal{L} \times H_o$, one represents

$$G/H = \{(x, \mathrm{Id}_{\mathcal{L}})\circ H \ ; \ x \in \mathcal{L}\}.$$

Further, $(x, \mathrm{Id}_{\mathcal{L}}) \circ H = (y, \mathrm{Id}_{\mathcal{L}})$ is equivalent to

$$(x, \mathrm{Id}_{\mathcal{L}}) \circ (\check{o}, \alpha) = (x, \mathrm{gyr}\,[x, \check{o}]\alpha) = (x, \alpha) = (y, \mathrm{Id}_{\mathcal{L}})$$

for some $\alpha \in H_o$. Thus, $(x, \mathrm{Id}_{\mathcal{L}}) \circ H \neq (y, \mathrm{Id}_{\mathcal{L}}) \circ H$ for $x \neq y$. The injection

$$\sigma : G/H = \{(x, \mathrm{Id}_{\mathcal{L}}) \circ H \ \ ; \ \ x \in \mathcal{L}\} \longrightarrow G = \mathcal{L} \times H_o,$$
$$\sigma\left((x, \mathrm{Id}_{\mathcal{L}}) \circ H\right) = (x, \mathrm{Id}_{\mathcal{L}})$$

is a section of the canonical projection

$$\pi : G = \mathcal{L} \times H_o \longrightarrow G/H$$
$$\pi(x, \alpha) = (x, \mathrm{Id}_{\mathcal{L}}) \circ H$$

as far as $\sigma(H) = \sigma((\check{o}, \mathrm{Id}_{\mathcal{L}}) \circ H) = (\check{o}, \mathrm{Id}_{\mathcal{L}}) = 1_G$ and $\pi\sigma\left((x, \mathrm{Id}_{\mathcal{L}}) \circ H\right) = \pi(x, \mathrm{Id}_{\mathcal{L}}) = (x, \mathrm{Id}_{\mathcal{L}}) \circ H$ for $\forall (x, \mathrm{Id}_{\mathcal{L}}) \circ H \in G/H$. Moreover, $(x, \mathrm{Id}_{\mathcal{L}})^{-1} = (\ominus x, \mathrm{Id}_{\mathcal{L}})$ for $\forall x \in \mathcal{L}$ reveals that $\sigma\left(G/H\right) = \{(x, \mathrm{Id}_{\mathcal{L}})\ ;\ x \in \mathcal{L}\}$ is closed under inversion. One checks straightforwardly that

$$(\check{o}, \alpha) \circ (x, \mathrm{Id}_{\mathcal{L}}) \circ (\check{o}, \alpha)^{-1} = (\alpha(x), \mathrm{gyr}\,[\check{o}, \alpha(x)]\alpha) \circ (\check{o}, \alpha^{-1}) = (\alpha(x), \mathrm{Id}_{\mathcal{L}})$$

for $\forall x \in \mathcal{L}, \forall \alpha \in H_o$. Therefore, $\sigma\left(\mathrm{Ad}_h(s)H\right) = \mathrm{Ad}_h(s)$ and the discrepancies $d^h(s) = \mathrm{Ad}_{h^{-1}}(1_G) = 1_G$ for $\forall h \in H, \forall s \in \sigma\left(G/H\right)$. According to Lemma 19, the operation

$$\begin{aligned} \oplus_\sigma : (G/H) \times (G/H) &\longrightarrow G/H \\ ((a, \mathrm{Id}_{\mathcal{L}}) \circ H) \oplus_\sigma ((b, \mathrm{Id}_{\mathcal{L}}) \circ H) &= (a, \mathrm{Id}_{\mathcal{L}}) \circ (b, \mathrm{Id}_{\mathcal{L}}) \circ H \\ &= (a \oplus b, \mathrm{gyr}\,[a, b]) \circ H \\ &= (a \oplus b, \mathrm{Id}_{\mathcal{L}}) \circ H. \end{aligned}$$

turns G/H into a left gyrogroup.

The bijective map

$$\Psi : \mathcal{L} \longrightarrow G/H$$
$$\Psi(a) := (a, \mathrm{Id}_{\mathcal{L}}) \circ H$$

is an isomorphism of $(\mathcal{L}, \oplus)$ onto $\left(G/H, \oplus_\sigma\right)$, because

$$\begin{aligned} \Psi(a \oplus b) &= (a \oplus b, \mathrm{Id}_{\mathcal{L}}) \circ H \\ &= ((a, \mathrm{Id}_{\mathcal{L}}) \circ H) \oplus_\sigma ((b, \mathrm{Id}_{\mathcal{L}}) \circ H) = \Psi(a) \oplus_\sigma \Psi(b) \end{aligned}$$

for $\forall a, b \in \mathcal{L}$, Q.E.D.

Definition 22. *If G is a connected Lie group, $H \subset G$ is a closed connected subgroup of G and $\sigma : G/H \to G$ is a real analytic section of $\pi : G \to G/H$, inducing a left gyrogroup operation $\oplus_\sigma : (G/H) \times (G/H) \to G/H$, then $(G/H, \oplus_\sigma)$ is called a left Lie gyrogroup.*

Definition 23. *A left gyrogroup $(\mathcal{L}, \oplus)$ is said to be analytic if its underlying set $\mathcal{L}$ is a real analytic manifold and its operations*

$$\begin{array}{ll} \oplus : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}, & \ominus : \mathcal{L} \longrightarrow \mathcal{L} \\ (a, b) \mapsto a \oplus b, & a \mapsto \ominus a \end{array}$$

are real analytic maps.

Corollary 24. i) *Any left Lie gyrogroup $(G/H, \oplus_\sigma)$ is an analytic left gyrogroup.*
ii) *Any analytic left gyrogroup $(\mathcal{L}, \oplus)$ is analytically isomorphic to a left Lie gyrogroup $(G/H, \oplus_\sigma)$.*

Proof: i) The quotient G/H of a connected Lie group G by a closed connected subgroup $H \subset G$ is an analytic manifold. The operation (4) depends analytically on $aH, bH \in G/H$, as far as σ and the group multiplication in G are analytic. The analyticity of the multiplication and inversion in the Lie group G, implies the analyticity of $(gH) \mapsto \ominus_\sigma(gH) = g^{-1}H$.
ii) The group $\mathrm{Aut}^\omega(\mathcal{L}, \oplus)$ of the analytic automorphisms of $(\mathcal{L}, \oplus)$ is a Lie group as a closed subgroup of the group of the analytic diffeomorphisms $\mathcal{L} \to \mathcal{L}$. The left translations $L_a : \mathcal{L} \to \mathcal{L}$ ($a \in \mathcal{L}$), are analytic diffeomorphisms, so that the gyrations $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}^\omega(\mathcal{L}, \oplus)$ for $\forall a, b \in \mathcal{L}$. Repeating verbally the proof of Proposition 21, one constructs the group $\mathcal{G} := \mathcal{L} \times \mathcal{H}_0$ for an arbitrary subgroup $\mathcal{H}_0$ of $\mathrm{Aut}^\omega(\mathcal{L}, \oplus)$, containing all the gyrations $\mathrm{gyr}\,[a, b] \in \mathcal{H}_0$, $\forall a, b \in \mathcal{L}$. The operation (17) and the inversion (18) are analytic in all arguments. Therefore $\mathcal{G}$ is a Lie group and $\mathcal{H} := \{(\check{o}, \alpha);\ \alpha \in \mathcal{H}_0\}$ is a closed subgroup of $\mathcal{G}$. Further, $\sigma : \mathcal{G}/\mathcal{H} \to \mathcal{G}$, $\sigma((x, \alpha) \circ \mathcal{H}) = (x, \mathrm{Id}_{\mathcal{L}})$ is an analytic section of $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ and

$$\Psi : \mathcal{L} \longrightarrow \mathcal{G}/\mathcal{H},$$
$$\Psi(x) := (x, \mathrm{Id}_{\mathcal{L}}) \circ \mathcal{H}$$

turns to be an analytic isomorphism of $(\mathcal{L}, \oplus)$ with the left Lie gyrogroup $(G/H, \oplus_\sigma)$, Q.E.D.

Definition 25. *If $(V, \oplus, \otimes)$ is a quasi left gyrovector space and $(V, \oplus)$ is a left gyrogroup then $(V, \oplus, \otimes)$ is called a left gyrovector space.*

Corollary 26. *Let G be a connected Lie group, $H \subset G$ be a closed connected subgroup and $\sigma : G/H \to G$ be a non-positively curved analytic section of $\pi : G \to G/H$ with*

$$\left[\mathrm{Lie}\,(H), T^{\mathbb{R}}_{1_G}\sigma\,(G/H)\right] \subseteq T^{\mathbb{R}}_{1_G}\sigma\,(G/H)\,.$$

Then $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a left Lie gyrovector space with respect to the operations (10) and (9).

Proof: According to Proposition 7, $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a quasi left Lie gyrovector space.

Note that any $h \in H$ is of the form $h = \mathrm{Exp}\,(\xi)$ for some (not necessarily unique) $\xi \in \mathrm{Lie}\,(H)$. Since σ is non-positively curved, for any $x \in S := \sigma\,(G/H)$ there exists a unique $u := \mathrm{Exp}^{-1}(x) \in T^{\mathbb{R}}_{1_G}S$ with $x = \mathrm{Exp}\,(u)$. By assumption, $T^{\mathbb{R}}_{1_G}S$ is $\mathrm{ad}\,_\xi$-invariant for $\forall \xi \in \mathrm{Lie}\,(H)$. In particular, $\mathrm{ad}^k_\xi(u) \in T^{\mathbb{R}}_{1_G}S$ for $\forall k \in \mathbb{N}$ and $\forall u \in T^{\mathbb{R}}_{1_G}S$. Consequently,

$$\begin{aligned}\mathrm{Ad}\,_h(x) &= \mathrm{Ad}\,_h(\mathrm{Exp}\,(u)) = \mathrm{Exp}\,(\mathrm{Ad}\,_h(u)) = \mathrm{Exp}\,\left(\mathrm{Ad}\,_{\mathrm{Exp}\,(\xi)}(u)\right)\\ &= \mathrm{Exp}\,(\exp(\mathrm{ad}\,_\xi)(u)) = \mathrm{Exp}\,\left(\sum_{k=0}^{\infty}\frac{\mathrm{ad}\,^k_\xi(u)}{k!}\right) \in \mathrm{Exp}\,\left(T^{\mathbb{R}}_{1_G}S\right) = S.\end{aligned}$$

In other words, $\sigma(\mathrm{Ad}\,_h(x)) = \mathrm{Ad}\,_h(x)$, and the discrepancies $d^h(x)$, defined by (16) equal 1_G for $\forall h \in H$, $\forall x \in S$. Applying Lemma 19, one concludes that $(G/H, \oplus_\sigma)$ is a left gyrogroup, so that $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a left Lie gyrovector space, Q.E.D.

5. Gyrocommutative Gyrogroups, Gyrovector Spaces, Cartan Gyrovector Spaces

Definition 27. *A gyrogroup (respectively, a Lie gyrogroup or an analytic gyrogroup) $(\mathcal{L}, \oplus)$ is a left gyrogroup (respectively, a left Lie gyrogroup or an analytic left gyrogroup), which possesses the left loop property*

$$\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[a \oplus b, b] \quad \textit{for } \forall a, b \in \mathcal{L}. \tag{19}$$

Let us observe that the left gyrogroup $(G_o/H_o = \mathrm{Sym}\,(3)/\langle(1,2)\rangle, \oplus_\sigma)$, discussed below Definition 17 is not a gyrogroup. On the one hand, for $\sigma_1 = (2,3)$, $\sigma_2 = (1,3)$, one has $\mathrm{gyr}\,[\sigma_1 H_o, \sigma_2 H_o] = \mathrm{Ad}\,_{h(\sigma_1\sigma_2)} = \mathrm{Ad}\,_{(1,2)}$. On the other hand, $(\sigma_1 H_o) \oplus_\sigma (\sigma_2 H_o) = \sigma_1\sigma_2 H_o = \sigma_2 H_o$, so that $\mathrm{gyr}\,[(\sigma_1 H_o \oplus_\sigma (\sigma_2 H_o), \sigma_2 H_o] = \mathrm{gyr}\,[\sigma_2 H_o, \sigma_2 H_o] = \mathrm{Id}_{\{1,2,3\}}$. Consequently,

$$\mathrm{gyr}\,[\sigma_1 H_o, \sigma_2 H_o] \neq \mathrm{gyr}\,[(\sigma_1 H_o) \oplus_\sigma (\sigma_2 H_o), \sigma_2 H]$$

and $(G_o/H_o, \oplus_\sigma)$ is not a gyrogroup.

Definition 28. *A gyrogroup (respectively, a Lie gyrogroup or an analytic gyrogroup) $(\mathcal{L}, \oplus)$ is said to be gyrocommutative if it satisfies the gyrocommutative law*

$$a \oplus b = \mathrm{gyr}\,[a,b](b \oplus a) \quad \textit{for } \forall a, b \in \mathcal{L}. \tag{20}$$

A (gyrocommutative) gyrogroup is a natural extension of the (commutative) group notion. It first arose in the study of Einstein addition of relativistically admissible velocities [14], where it was recognized that Einstein addition is a gyrocommutative gyrogroup operation, in full analogy with the common vector addition of Newtonian velocities, which is a commutative group operation.

For examples of finite and infinite non-gyrocommutative gyrogroups, we refer the reader to Foguel and Ungar's article [4].

Lemma 29. *Let G be a group, $H \subset G$ be a subgroup and $\sigma : G/H \to G$ be a section of $\pi : G \to G/H$ with image $S := \sigma(G/H)$. Suppose the following conditions hold:*

i) $S = S^{-1}$ *is closed under inversion*

ii) *the discrepancies (16) belong to* $\cap_{g\in G}\left(gHg^{-1}\right)$

iii) $\sigma(x^{-1}y^{-1}H) = [\sigma(xyH)]^{-1}$ *for* $\forall x, y \in S$

iv) $xyx \in S$ *for* $\forall x, y \in S$ *(twisted group property – cf.* [3]*).*

Then $(G/H, \oplus_\sigma)$ is a gyrocommutative gyrogroup.

Proof: According to Lemma 19, i) and ii) suffice for $(G/H, \oplus_\sigma)$ to be a left gyrogroup. We claim that i) and iii) imply the automorphic inverse property

$$\ominus_\sigma \{(xH) \oplus_\sigma (yH)\} = \{\ominus_\sigma (xH)\} \oplus_\sigma \{\ominus_\sigma (yH)\} \tag{21}$$

for arbitrary $x = \sigma(xH)$, $y = \sigma(yH)$. Indeed, if S is closed under inversion then

$$\ominus_\sigma (\sigma(gH)H) = [\sigma(gH)]^{-1} H \quad \text{for } gH \in G/H$$

and

$$\ominus_\sigma \{(xH) \oplus_\sigma (yH)\} = \ominus_\sigma (xyH) = \ominus_\sigma [\sigma(xyH)H] = [\sigma(xyH)]^{-1} H$$

equals

$$\{\ominus_\sigma(xH)\} \oplus_\sigma \{\ominus_\sigma(yH)\} = (x^{-1}H) \oplus_\sigma (y^{-1}H) = x^{-1}y^{-1}H$$
$$= \sigma\left(x^{-1}y^{-1}H\right) H$$

provided that $[\sigma(xyH)]^{-1} = \sigma(x^{-1}y^{-1}H)$.

In an arbitrary left gyrogroup $(\mathcal{L}, \oplus)$, the automorphic inverse property (21) is known to force the gyrocommutative law (20) by Theorem 2.39 from Ungar's book [16]. For the sake of completeness we present the proof. Let $G_o := \mathcal{L} \times \mathrm{Aut}(\mathcal{L}, \oplus)$ be the gyro-semidirect product of $\mathcal{L}$ with its gyro-automorphism group $\mathrm{Aut}(\mathcal{L}, \oplus)$. Recall from the proof of Proposition 21 the group operation (17) and the inverse (18). Then the equality

$$[(x, \alpha) \circ (y, \beta)]^{-1} = (y, \beta)^{-1} \circ (x, \alpha)^{-1}$$

implies

$$(\ominus\beta^{-1}\alpha^{-1}(\mathrm{gyr}\,[x, \alpha(y)])^{-1}(x \oplus \alpha(y)), \beta^{-1}\alpha^{-1}(\mathrm{gyr}\,[x, \alpha(y)])^{-1})$$
$$= \left(\ominus\beta^{-1}(y), \beta^{-1}\right) \circ \left(\ominus\alpha^{-1}(x), \alpha^{-1}\right)$$
$$= (\ominus\beta^{-1}\alpha^{-1}(\alpha(y) \oplus x), \mathrm{gyr}\,[\ominus\beta^{-1}(y), \ominus\beta^{-1}\alpha^{-1}(x)]\beta^{-1}\alpha^{-1}).$$

By comparison of the corresponding entries, one obtains

$$(\mathrm{gyr}\,[x, z])^{-1}(x \oplus z) = z \oplus x \quad \text{and} \tag{22}$$
$$\beta^{-1}\alpha^{-1}(\mathrm{gyr}\,[x, z])^{-1} = \mathrm{gyr}\,[\ominus\beta^{-1}\alpha^{-1}(z), \ominus\beta^{-1}\alpha^{-1}(x)]\beta^{-1}\alpha^{-1} \tag{23}$$

for $z = \alpha(y)$. Since for arbitrary $a, b, c \in \mathcal{L}$, $\gamma \in \mathrm{Aut}(\mathcal{L}, \oplus)$, there holds

$$\gamma(\mathrm{gyr}\,[a, b]c) = \gamma L_{\ominus(a\oplus b)} L_a L_b(c) = L_{\ominus(\gamma(a)\oplus\gamma(b))}\gamma(L_a L_b(c))$$
$$= L_{\ominus(\gamma(a)\oplus\gamma(b))} L_{\gamma(a)} L_{\gamma(b)}\gamma(c) = \mathrm{gyr}\,[\gamma(a), \gamma(b)]\gamma(c)$$

(23) implies the identity

$$(\mathrm{gyr}\,[x, z])^{-1} = \mathrm{gyr}\,[\ominus z, \ominus x] \quad \text{for } \forall x, z \in \mathcal{L}. \tag{24}$$

Further, the automorphic inverse property and the left gyroassociative law provide

$$\{\ominus(a \oplus b)\} \oplus \mathrm{gyr}\,[a, b](\ominus c) = \ominus\{(a \oplus b) \oplus \mathrm{gyr}\,[a, b]c\} = \ominus\{a \oplus (b \oplus c)\}$$
$$= (\ominus a) \oplus \{(\ominus b) \oplus (\ominus c)\} = \{(\ominus a) \oplus (\ominus b)\} \oplus \mathrm{gyr}\,[\ominus a, \ominus b](\ominus c)$$
$$= \{\ominus(a \oplus b)\} \oplus \mathrm{gyr}\,[\ominus a, \ominus b](\ominus c)$$

whereas

$$\mathrm{gyr}\,[a,b] = \mathrm{gyr}\,[\ominus a, \ominus b] \quad \text{for } \forall a,b \in \mathcal{L}. \tag{25}$$

Putting together (22), (24) and (25), one derives the gyrocommutative law

$$\mathrm{gyr}\,[z,x](x \oplus z) = z \oplus x \quad \text{for } \forall x,z \in \mathcal{L}$$

thus obtaining the result of Ungar's Theorem 2.39 from [16].

We will derive the left loop property from the automorphic inverse property and the assumption (iv). For arbitrary $x, y \in S$ recall that $yx = \sigma(yxH)h(yx)$ and express

$$x(yx) = x\sigma(yxH)h(yx) = \sigma(x\sigma(yxH)H)h(x\sigma(yxH))h(yx) \in S.$$

Therefore $h(x\sigma(yxH)) = [h(yx)]^{-1}$. The presence of the automorphic inverse property implies

$$(\mathrm{gyr}\,[yH, xH])^{-1} = \mathrm{gyr}\,[xH, yH]$$

by combining (24) with (25). Since $\mathrm{gyr}\,[aH, bH] = \mathrm{Ad}_{h(ab)}$ for $\forall a, b \in S$, there follows

$$\begin{aligned}\mathrm{gyr}\,[xH,(yH)\oplus_\sigma(xH)] &= \mathrm{gyr}\,[xH,\sigma(yxH)H] = \mathrm{Ad}_{\,h(x\sigma(yxH))} = \mathrm{Ad}_{\,[h(yx)]^{-1}} \\ &= \left[\mathrm{Ad}_{\,h(yx)}\right]^{-1} = (\mathrm{gyr}\,[yH,xH])^{-1} = \mathrm{gyr}\,[xH,yH],\end{aligned}$$

whereas

$$\begin{aligned}\mathrm{gyr}\,[(yH)\oplus_\sigma(xH), xH] &= (\mathrm{gyr}\,[xH,(yH)\oplus_\sigma(xH)])^{-1} \\ &= (\mathrm{gyr}\,[xH,yH])^{-1} = \mathrm{gyr}\,[yH,xH].\end{aligned}$$

Thus, the assumptions (i)-(iv) imply that $(G/H, \oplus_\sigma)$ is a gyrocommutative gyrogroup, Q.E.D.

Definition 30. *If $(V, \oplus, \otimes)$ is a left gyrovector space and $(V, \oplus)$ is a gyrocommutative gyrogroup then $(V, \oplus, \otimes)$ is called a gyrovector space.*

The theory of gyrogroups and gyrovector spaces is developed in Ungar's book [16].

Corollary 31. *Let G be a connected Lie group, $H \subset G$ be a closed connected subgroup and $\sigma : G/H \to G$ be a non-positively curved analytic section of $\pi : G \to G/H$. Assume that $S := \sigma(G/H)$ is subject to the following properties:*

a) $\left[\mathrm{Lie}\,(H), T^{\mathbb{R}}_{1_G} S\right] \subseteq T^{\mathbb{R}}_{1_G} S$

b) $\left[T^{\mathbb{R}}_{1_G}S, T^{\mathbb{R}}_{1_G}S\right] \subseteq \mathrm{Lie}\,(H)$

c) *there is an anti-involution* $\tau : G \to G$, *whose fixed point set* $Fix(\tau) = S$.

Then the operations (10) *and* (9) *turn* G/H *into a Lie gyrovector space.*

Proof: By Corollary 26, a non-positively curved section $\sigma : G/H \to G$, subject to a) determines a left Lie gyrovector space $(G/H, \oplus_\sigma, \otimes_\sigma)$.

It suffices to verify that the infinitesimal assumptions a)-c) imply the sufficient conditions iii) and iv) from Lemma 29 for $(G/H, \oplus_\sigma)$ to be a gyrocommutative gyrogroup.

First, we infer iii) from a) and b). More precisely, for $\forall u, v \in T^{\mathbb{R}}_{1_G}S$ there exists a unique $w \in T^{\mathbb{R}}_{1_G}S$, such that $\sigma(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H) = \mathrm{Exp}\,(w)$, i.e., $\mathrm{Exp}\,(-w)\mathrm{Exp}\,(u)\mathrm{Exp}\,(v) \in H$. Recall that the Campbell-Hausdorff map

$$\mathcal{F} : \mathrm{Lie}\,(G) \times \mathrm{Lie}\,(G) \longrightarrow \mathrm{Lie}\,(G)$$

defined by $\mathrm{Exp}\,(x)\mathrm{Exp}\,(y) = \mathrm{Exp}\,(\mathcal{F}(x, y))$ for $x, y \in \mathrm{Lie}\,(G)$, is given by the series

$$\mathcal{F}(x, y) = \sum_{m,k_i,l_i} \frac{(-1)^{m+k_1+l_1+\ldots+k_m+l_m}\mathrm{ad}\,_y^{l_m}\mathrm{ad}\,_x^{k_m}\ldots\mathrm{ad}\,_y^{l_1}\mathrm{ad}\,_x^{k_1-1}(x)}{m(k_1+l_1+\ldots+k_m+l_m)l_m!k_m!\ldots l_1!k_1!} \qquad (26)$$

where the summation is over all the natural numbers m and all the non-negative integers k_i, l_i with $k_i + l_i > 0$. Each of the terms $\mathrm{ad}\,_y^{l_m}\mathrm{ad}\,_x^{k_m}\ldots\mathrm{ad}\,_y^{l_1}\mathrm{ad}\,_x^{k_1-1}(x)$ is considered to be of total degree $k_1+l_1+\ldots+k_m+l_m$ with respect to x and y. Let us denote by $[\mathcal{F}(-w, \mathcal{F}(u, v))]_0$ the sum of the terms of $\mathcal{F}(-w, \mathcal{F}(u, v))$, which are of even total degree with respect to u, v, w. Similarly, put $[\mathcal{F}(-w, \mathcal{F}(u, v))]_1$ for the sum of the terms of odd total degree. The conditions a), b) imply that $[\mathcal{F}(-w, \mathcal{F}(u, v))]_0 \in \mathrm{Lie}\,(H)$ and $[\mathcal{F}(-w, \mathcal{F}(u, v))]_1 \in T^{\mathbb{R}}_{1_G}S$ for arbitrary u, v, $w \in T^{\mathbb{R}}_{1_G}S$. Therefore, $\mathcal{F}(-w, \mathcal{F}(u, v)) = [\mathcal{F}(-w, \mathcal{F}(u, v))]_0 + [\mathcal{F}(-w, \mathcal{F}(u, v))]_1$ belongs to $\mathrm{Lie}\,(H)$ if and only if $[\mathcal{F}(-w, \mathcal{F}(u, v))]_1 = 0$. If so, then a simultaneous change of the signs of u, v and w yields

$$\begin{aligned}[\mathcal{F}(w, \mathcal{F}(-u, -v))]_0 &= [\mathcal{F}(-w, \mathcal{F}(u, v))]_0, [\mathcal{F}(w, \mathcal{F}(-u, -v))]_1 \\ &= -[\mathcal{F}(-w, \mathcal{F}(u, v))]_1 = 0.\end{aligned}$$

Thus, $\mathcal{F}(-w, \mathcal{F}(u, v)) \in \mathrm{Lie}\,(H)$ forces $\mathcal{F}(w, \mathcal{F}(-u, -v)) \in \mathrm{Lie}\,(H)$. Equivalently, $\mathrm{Exp}\,(-w)\mathrm{Exp}\,(u)\mathrm{Exp}\,(v) \in H$ suffices for $\mathrm{Exp}\,(w)\mathrm{Exp}\,(-u)\mathrm{Exp}\,(-v) \in H$, provided (a) and (b). As a result,

$$\begin{aligned}&[\sigma(\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)H)]^{-1} = [\mathrm{Exp}\,(w)]^{-1} = \mathrm{Exp}\,(-w) \\ &\Longrightarrow \sigma(\mathrm{Exp}\,(-u)\mathrm{Exp}\,(-v)H) = \mathrm{Exp}\,(-w)\end{aligned}$$

for $\forall u, v \in T^{\mathbb{R}}_{1_G} S$.

Next, note that (c) suffices for (iv). More precisely, an anti-involution $\tau : G \to G$ is a bijection with $\tau^2 = \mathrm{Id}_G$ and $\tau(ab) = \tau(b)\tau(a)$ for $\forall a, b \in G$. If $Fix(\tau) = S$ then arbitrary $x, y \in S$ satisfy the twisted group property $\tau(xyx) = \tau(x)\tau(y)\tau(x)$ $= xyx \in S$, Q.E.D.

Let $(G/H, \oplus_\sigma, \otimes_\sigma)$ be a quasi left Lie gyrovector space, subject to the automorphic inverse property $\ominus_\sigma(\ominus_\sigma x \oplus_\sigma y) = x \ominus_\sigma y$ for $\forall x, y \in G/H$ and g be a left G-invariant metric on G/H. Then combining (i) and (iii) from Corollary 15, one concludes that

$$d(x, y) = ||x \ominus_\sigma y||$$

for the distance function d, associated with g and the norm (13).

Let G be a noncompact semisimple Lie group and $K \subset G$ be a maximal compact subgroup. By means of the faithful (i.e., injective) adjoint representation

$$\mathrm{ad} : \mathrm{Lie}\,(G) \longrightarrow \mathrm{End}(\mathrm{Lie}\,(G))$$
$$\mathrm{ad}_x(y) := [x, y] \quad \text{for } \forall x, y \in \mathrm{Lie}\,(G)$$

one introduces a non-degenerate bilinear form

$$B : \mathrm{Lie}\,(G) \times \mathrm{Lie}\,(G) \longrightarrow \mathrm{Lie}\,(G)$$
$$B(x, y) := \mathrm{Tr}\ (\mathrm{ad}_x \mathrm{ad}_y)$$

and considers the orthogonal complement

$$\mathfrak{p} = \{x \in \mathrm{Lie}\,(G) \ ; \ \mathrm{Tr}\ \left(\mathrm{ad}_x \mathrm{ad}_{\mathrm{Lie}\,(K)}\right) = 0\}.$$

of $\mathrm{Lie}\,(K)$. There is a direct sum Cartan decomposition

$$\mathrm{Lie}\,(G) = \mathfrak{p} + \mathrm{Lie}\,(K)$$

associated with a Cartan involution

$$\theta : \mathrm{Lie}\,(G) = \mathfrak{p} + \mathrm{Lie}\,(K) \longrightarrow \mathfrak{p} + \mathrm{Lie}\,(K) = \mathrm{Lie}\,(G)$$
$$\theta(u + a) := -u + a \quad \text{for } \forall u \in \mathfrak{p},\ \forall a \in \mathrm{Lie}\,(K).$$

By Lemma VI.1.2. [6], the bilinear form

$$B_\theta : \mathrm{Lie}\,(G) \times \mathrm{Lie}\,(G) \longrightarrow \mathbb{R}$$
$$B_\theta(x, y) := -B(x, \theta(y)) = -\mathrm{Tr}\ \left(\mathrm{ad}_x \mathrm{ad}_{\theta(y)}\right) \quad \text{for } \forall x, y \in \mathrm{Lie}\,(G)$$

is symmetric and positive definite. The left G-invariant metric on G, whose restriction on $T^{\mathbb{R}}_{1_G}G = \mathrm{Lie}\,(G)$ coincides with B_θ, is called Killing form of G. The homogeneous space G/K is a Riemmanian symmetric space of noncompact type, associated with G. The Riemmanian structure on G/K is given by the left G-invariant metric g with $\pi^* g_{\breve{o}} = B_\theta$.

In [8] Krammer and Urbantke have constructed a gyrocommutative gyrogroup structure on any Riemannian symmetric space G/K of noncompact type. This result is extended by the following

Corollary 32. *Let G be a noncompact semisimple Lie group with exponential map* $\mathrm{Exp}\,:\mathrm{Lie}\,(G)\to G$ *and Cartan decomposition* $\mathrm{Lie}\,(G) = \mathfrak{p} + \mathrm{Lie}\,(K)$. *Then the Riemannian symmetric space G/K of noncompact type admits a nonpositively curved analytic section*

$$\sigma : G/K \longrightarrow G, \quad \sigma(\mathrm{Exp}\,(u)K) := \mathrm{Exp}\,(u) \textit{ for } \forall u \in \mathfrak{p} = T^{\mathbb{R}}_{1_G}\sigma\,(G/K)\,,$$

whose associated operations (10) *and* (9) *determine a Lie gyrovector space* $(G/K, \oplus_\sigma, \otimes_\sigma)\,.$

We name these gyrovector spaces after Cartan, because their associated sections σ arise from the Cartan decompositions.

Proof of Corollary 32: According to Theorem VI.1.1 iii) [6], the composition

$$\pi\mathrm{Exp}\,:\mathfrak{p} \longrightarrow G/K$$

of the exponential map $\mathrm{Exp}\,:\mathfrak{p}\to S := \mathrm{Exp}\,(\mathfrak{p}) \subset G$ and the canonical projection

$$\pi : S \to (SK)/K = G/K$$

is a global analytic diffeomorphism. Therefore

$$G/K = \{\pi\mathrm{Exp}\,(u) = \mathrm{Exp}\,(u)K \quad ; \quad u \in \mathfrak{p}\}.$$

The restrictions $\mathrm{Exp}\,|_{\mathfrak{p}}, \pi|_S$ are analytic diffeomorphisms. Thus,

$$\sigma := \mathrm{Exp}\,\left(\pi\mathrm{Exp}\,\right)^{-1}$$

is a global analytic diffeomorphism of G/K onto $\sigma\,(G/K) = \mathrm{Exp}\,(\mathfrak{p}) = S$ with $\pi\sigma = (\pi\mathrm{Exp}\,)\,(\pi\mathrm{Exp}\,)^{-1} = \mathrm{Id}\,_{G/K}$ and $\sigma(K) = \mathrm{Exp}\,(0) = 1_G$. In other words, $\sigma : G/K \to G$ is an analytic section of $\pi : G \to G/K$. Moreover, the exponential map of G restricts to a global diffeomorphism

$$\mathrm{Exp}\,:\mathfrak{p} = T^{\mathbb{R}}_{1_G}S \longrightarrow S = \sigma\,(G/K)$$

so that σ is non-positively curved (cf. Proposition 12 and Definition 13).

The proof will be completed by checking the assumptions a), b), c) from Corollary 31. The inclusions

$$[\mathrm{Lie}\,(K), \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathrm{Lie}\,(K)$$

are well known properties of the Cartan decomposition $\mathrm{Lie}\,(G) = \mathfrak{p} + \mathrm{Lie}\,(K)$ (cf. IV.5 [6]). Further, the Cartan involution $\theta(u + a) = -u + a$ for $\forall u \in \mathfrak{p}$, $\forall a \in \mathrm{Lie}\,(K)$ is a Lie algebra homomorphism of $\mathrm{Lie}\,(G)$ and gives rise to a group homomorphism

$$\Theta : G = \sigma\,(G/K)\,K = \mathrm{Exp}\,(\mathfrak{p})K \longrightarrow G$$
$$\Theta(\mathrm{Exp}\,(u)k) := k\mathrm{Exp}\,(-u) \quad \text{for } \forall u \in \mathfrak{p}, \forall k \in K$$

called Cartan involution of G. Let us consider the diffeomorphism

$$\tau : G \longrightarrow G$$
$$\tau(X) := [\Theta(X)]^{-1} \quad \text{for } \forall X \in G.$$

By $\tau^2(X) = \left\{\Theta\left[(\Theta(X))^{-1}\right]\right\}^{-1} = \Theta\left\{\left[(\Theta(X))^{-1}\right]^{-1}\right\} = \Theta^2(X) = X$ there follows $\tau^2 = \mathrm{Id}\,_G$. For arbitrary $X, Y \in G$ one checks straightforward by that

$$\tau(XY) = [\Theta(XY)]^{-1} = [\Theta(X)\Theta(Y)]^{-1} = [\Theta(Y)]^{-1}\,[\Theta(X)]^{-1} = \tau(Y)\tau(X)$$

and concludes that τ is an anti-involution. For $X = \mathrm{Exp}\,(u)k \in G$ with $u \in \mathfrak{p}$, $k \in K$ note that $\tau(X) = [\Theta(X)]^{-1} = X$ if and only if

$$k\mathrm{Exp}\,(-u) = \Theta(X) = X^{-1} = k^{-1}\mathrm{Exp}\,(-u)$$

which is equivalent to $k = k^{-1}$. Thus, $S = \{\mathrm{Exp}\,(u)\,;\, u \in \mathfrak{p}\}$ consists of fixed points for τ and $Fix(\tau) \subseteq \{\mathrm{Exp}\,(u)k\,;\, u \in \mathfrak{p}, k \in K, k^2 = 1_G\} = SK^{(2)}$, where $K^{(2)} := \{k \in K\,;\, k^2 = 1_G\}$ is the normal subgroup of K, constituted by its elements of order 2. Towards the proof of the discreteness of $K^{(2)}$ in K, let us fix a faithful finite dimensional linear representation of K and $\mathrm{Lie}\,(K)$. Then note that $a \in \mathrm{Lie}\,(K)$ with $[\mathrm{Exp}\,(a)]^2 = \mathrm{Exp}\,(2a) = 1_K$ requires the matrix of a to be semisimple and with eigenvalues from $\pi\mathrm{i}\mathbb{Z}$. That suffices for

$$K^{(2)} = \mathrm{Exp}\,\{a \in \mathrm{Lie}\,(K) \;\; ; \;\; \mathrm{Exp}\,(a) \in K^{(2)}\}$$

to be discrete in K. Similar considerations justify that the elements of order 2 from G form a discrete normal subgroup $G^{(2)} \subset G$. Note that the Lie groups

$G/G^{(2)}$ and $K/K^{(2)}$ have Lie algebras $\mathrm{Lie}\left(G/G^{(2)}\right) = \mathrm{Lie}\,(G)$, respectively, $\mathrm{Lie}\left(K/K^{(2)}\right) = \mathrm{Lie}\,(K)$. Taking into account that $G^{(2)} \cap K = K^{(2)}$, one observes also that

$$\left(G/G^{(2)}\right) / \left(K/K^{(2)}\right) \simeq G/K.$$

Thus, without loss of generality, K can be assumed to have no elements of order 2. That implies $Fix(\tau) = S$, Q.E.D.

The Cartan gyrovector space structure on a Hermitian symmetric space of noncompact type is generalized in [5] by Friedman and Ungar to a gyrogroup structure on a bounded symmetric domain in an arbitrary complex Banach space.

We conclude the characterization of the Cartan gyrovector spaces by showing that they are the only members of a certain class of Lie gyrovector spaces.

Corollary 33. *Let $(G/H, \oplus_\sigma, \otimes_\sigma)$ be a Lie gyrovector space, associated with a non-positively curved real analytic section $\sigma : G/H \to G$ of $\pi : G \to G/H$. Suppose that*

$$\left[\mathrm{Lie}\,(H), T^{\mathbb{R}}_{1_G}\sigma\,(G/H)\right] \subseteq T^{\mathbb{R}}_{1_G}\sigma\,(G/H)$$

and

$$\mathcal{F}(u, \mathcal{F}(v,u)) \in T^{\mathbb{R}}_{1_G}\sigma\,(G/H) \quad \textit{for } \forall u, v \in T^{\mathbb{R}}_{1_G}\sigma\,(G/H)$$

where $\mathcal{F} : \mathrm{Lie}\,(G) \times \mathrm{Lie}\,(G) \to \mathrm{Lie}\,(G)$ stands for the Campbell-Hausdorff series (26).

Then $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a Cartan gyrovector space. In particular, there is a noncompact semisimple Lie group G_o and a maximal compact subgroup $K_o \subset G_o$, such that G/H is isomorphic as an analytic manifold to the Riemannian symmetric space G_o/K_o of noncompact type.

Proof: By definition, one has to fix an analytic Riemannian metric g on G/H and to prove that $\forall p \in G/H$ is an isolated fixed point of an involutive isometry $\psi_p : M \to M$ for g.

First of all, the sections $\sigma : G/H \to G$ are in a bijective correspondence with the decompositions $G = \sigma\,(G/H)\,H$ into products of disjoint subgroups $H \subset G$ and subsets $\sigma\,(G/H) \subset G$. Namely, a section $\sigma : G/H \to G$ of $\pi : G \to G/H$ gives rise to correctly defined maps

$$s : G \longrightarrow S := \sigma\,(G/H)$$
$$s(x) := \sigma(xH)$$

and

$$h : G \longrightarrow H$$
$$h(x) := [s(x)]^{-1} x$$

such that $x = s(x)h(x)$ for $\forall x \in G$. If $\sigma(xH) = a \in S \cap H$ then $xH = \pi\sigma(xH) = \pi(a) = H$ implies that $\sigma(xH) = \sigma(H) = 1_G$, so that $S \cap H = \{1_G\}$. Thus, the decomposition $x = s(x)h(x)$ is unique for all $x \in G$. Conversely, any decomposition $G = SH$ into a product of a subgroup $H \subset G$ and a subset $S \subset G$ with $S \cap H = \{1_G\}$ determines maps $s : G \to S$ and $h : G \to H$, such that $x = s(x)h(x)$ for $\forall x \in G$. In particular, for arbitrary $a \in H$ there holds

$$s(xa)h(xa) = xa = s(x)h(x)a$$

so that $s(xa) = s(x)$ is constant on any coset xH. That allows to define a map

$$\sigma : G/H \longrightarrow G$$
$$\sigma(xH) := s(x).$$

After checking that $\pi\sigma(xH) = s(x)H = s(x)h(x)H = xH$ for $\forall xH \in G/H$ and $\sigma(H) = s(1_G) = 1_G$, one concludes that σ is a section of $\pi : G \to G/H$.

To any section $\sigma : G/H \to G$ of $\pi : G \to G/H$ we associate a groupoid $(G/H, \oplus_\sigma)$, setting

$$(xH) \oplus_\sigma (yH) = \sigma(xH)yH \quad \text{for } \forall x, y \in G.$$

On the other hand, an arbitrary decomposition $G = SH$, $S \cap H = \{1_G\}$ determines a groupoid $(S, \oplus^s)$, where

$$x \oplus^s y = s(xy) \quad \text{for } \forall x, y \in S$$

and $s : G \to S$ is the decomposition map. Whenever $\sigma : G/H \to G$ is associated with $G = SH$, the groupoids $(G/h, \oplus_\sigma)$ and $(S, \oplus^s)$ are isomorphic, as far as

$$(x \oplus^s y)\, H = s(xy)H = \sigma(xyH)H = xyH = (xH) \oplus_\sigma (yH) \quad \text{for } \forall x, y \in S.$$

For an arbitrary decomposition $G = SH$ with $S \cap H = \{1_G\}$ the subspace $T^{\mathbb{R}}_{1_G} S \subset \mathrm{Lie}\,(G)$ is transversal to $\mathrm{Lie}\,(H)$, $T^{\mathbb{R}}_{1_G} S \cap \mathrm{Lie}\,(H) = \{0\}$. Let

$$g_{\breve{o}} : \mathrm{Lie}\,(G) \times \mathrm{Lie}\,(G) \longrightarrow \mathbb{R}$$

be a positive definite symmetric bilinear form, with respect to which $T_{1_G}^{\mathbb{R}} S$ and $\mathrm{Lie}\,(H)$ are orthogonal. Then the real analytic family $g = \{g_{xH}\}_{xH \in G/H}$ of positive definite symmetric bilinear forms

$$g_{xH} : T_{xH}^{\mathbb{R}}(G/H) \times T_{xH}^{\mathbb{R}}(G/H) \longrightarrow \mathbb{R}$$
$$g_{xH}(u,v) := g_{\check{o}}\left(\left(\mathrm{d}x^{-1}\right)_x u, \left(\mathrm{d}x^{-1}\right)_x v\right)$$

is a left G-invariant metric on G/H.

We are looking for an involutive g-isometry $\psi_{\check{o}} : G/H \to G/H$ with an isolated fixed point $\check{o} \equiv H$. To this end, let us recall that the section $\sigma : G/H \to G$ is non-positively curved exactly when the exponential map $\mathrm{Exp} : \mathrm{Lie}\,(G) \to G$ restricts to a global diffeomorphism $\mathrm{Exp} : T_{1_G}^{\mathbb{R}} S \to S$ onto its image $S := \sigma\,(G/H)$. Therefore

$$S = \mathrm{Exp}\,\left(T_{1_G}^{\mathbb{R}} S\right) = \mathrm{Exp}\,\left(-T_{1_G}^{\mathbb{R}} S\right) = S^{-1}$$

is closed under inversion. According to Corollary 26, S is normalized by H, i.e., $hSh^{-1} \subseteq S$ for $\forall h \in H$, provided $T_{1_G}^{\mathbb{R}} S$ is $\mathrm{ad}_{\,\mathrm{Lie}\,(H)}$-invariant. There holds also the twisted group property

$$\mathrm{Exp}\,(u)\mathrm{Exp}\,(v)\mathrm{Exp}\,(u) = \mathrm{Exp}\,(u)\mathrm{Exp}\,(\mathcal{F}(v,u)) = \mathrm{Exp}\,(\mathcal{F}(u,\mathcal{F}(v,u)) \in S$$

for $\forall u, v \in T_{1_G}^{\mathbb{R}} S$. By assumption, $(G/H, \oplus_\sigma, \otimes_\sigma)$ is a Lie gyrovector space, so that $(S, \oplus^s)$ is a gyrocommutative gyrogroup. A result of Foguel and Ungar from [3] establishes that whenever S is a twisted group, closed under inversion and normalized by H, the groupoid $(S, \oplus^s)$ is a gyrocommutative gyrogroup if and only if there is an involutive group automorphism $\psi : G \to G$ with $\psi(x) = x^{-1}$ for $\forall x \in S$ and $\psi(y) = y$ for $\forall y \in H$. Since $\psi(xy)H = x^{-1}yH = x^{-1}H = \psi(x)H$ for $\forall x \in S$, $\forall y \in H$, there is a correctly defined map

$$\psi_{\check{o}} : G/H \longrightarrow G/H$$
$$\psi_{\check{o}}\,(xH) := \psi(x)H \quad \text{for } \forall x \in G.$$

Clearly, $\psi^2 = \mathrm{Id}_{\,G}$ implies $\psi_{\check{o}}^2 = \mathrm{Id}_{\,G/H}$. The origin $\check{o} \equiv H$ is the only fixed point of $\psi_{\check{o}}$, because $\psi_{\check{o}}(xH) = xH$ for $x \in S$ requires $x \in H \cap S = \{1_G\}$, whereas $x = 1_G$. Due to the G-invariance of g, it suffices to show that the differential

$$(\mathrm{d}\psi_{\check{o}})_{\check{o}} : T_{\check{o}}^{\mathbb{R}}(G/H) \longrightarrow T_{\check{o}}^{\mathbb{R}}(G/H)$$

is orthogonal with respect to $g_{\check{o}}$, in order to conclude that $\psi_{\check{o}}$ is a g-isometry. Indeed, the diffeomorphism $\pi : S \to G/H$ induces a linear isomorphism $(\mathrm{d}\pi)_{1_G}$:

$T^{\mathbb{R}}_{1_G} S \to T^{\mathbb{R}}_{\check{o}}(G/H)$ and

$$\begin{aligned} & g_{\check{o}}\left((\mathrm{d}\psi_{\check{o}})_{\check{o}} (\mathrm{d}\pi)_{1_G} u, (\mathrm{d}\psi_{\check{o}})_{\check{o}} (\mathrm{d}\pi)_{1_G} v \right) \\ = & g_{\check{o}}\left(-(\mathrm{d}\pi)_{1_G} u, -(\mathrm{d}\pi)_{1_G} v \right) = g_{\check{o}}\left((\mathrm{d}\pi)_{1_G} u, (\mathrm{d}\pi)_{1_G} v \right) \text{ for } \forall u, v \in T^{\mathbb{R}}_{1_G} S. \end{aligned}$$

For an arbitrary point $p = xH \in G/H$, $x \in S$ note that

$$\psi_p := x\psi_{\check{o}} x^{-1} : G/H \longrightarrow G/H$$

is an involutive g-isometry, as far as $\psi_{\check{o}}$ has the same property and the left multiplications by x, x^{-1} are isometries. Further, $\psi_p(yH) = yH$ for $y \in S$ if and only if $\psi_{\check{o}}\left(x^{-1}yH\right) = x^{-1}yH$. Consequently, $p = xH$ is the only fixed point of ψ_p, Q.E.D.

6. Example: The Cartan Gyrovector Space Structure on the Upper Half-plane

The Möbius gyrovector space structure on the unit disc $SU(1,1)/S(U_1 \times U_1)$ has been extensively studied by A. Ungar in [16], [17] and others. Here we illustrate the considerations from the previous sections on the example of the upper half-plane

$$\mathcal{H} = SL(2,\mathbb{R})/SO(2).$$

The Lie algebra

$$sl(2,\mathbb{R}) = \left\{ m = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad ; \quad a, b, c \in \mathbb{R} \right\}.$$

Its compact real form

$$su(2) := \{ m \in sl(2,\mathbb{R}) \quad ; \quad m + {}^t\overline{m} = 0 \} = \left\{ \begin{pmatrix} \mathrm{i}r & \zeta \\ -\overline{\zeta} & -\mathrm{i}r \end{pmatrix} ; \ r \in \mathbb{R}, \ \zeta \in \mathbb{C} \right\}$$

so that the maximal compact subalgebra

$$so(2) = sl(2,\mathbb{R}) \cap su(2) = \left\{ \xi_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \quad \xi_0 \in \mathbb{R} \right\}.$$

The infinitesimal Cartan decomposition $sl(2,\mathbb{R}) = \mathfrak{p}_o + so(2)$ holds for

$$\mathfrak{p}_o := sl(2,\mathbb{R}) \cap (\sqrt{-1} su(2)) = \left\{ \xi_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \xi_1, \xi_2 \in \mathbb{R} \right\}.$$

Lemma 34. *The exponential map*

$$\text{Exp}\ : sl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ;\quad a,b,c \in \mathbb{R} \right\}$$
$$\longrightarrow SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} ;\quad A,B,C,D \in \mathbb{R},\ \ AD - BC = 1 \right\}$$

restricts to a diffeomorphism

$$\text{Exp}\ : \mathfrak{p}_o = \left\{ m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix} ;\quad \xi_1, \xi_2 \in \mathbb{R} \right\}$$
$$\longrightarrow \text{Exp}\,(\mathfrak{p}_o) = \left\{ M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_2 & \dfrac{1 + x_2^2}{x_1} \end{pmatrix} ;\quad x_1, x_2 \in \mathbb{R}, x_1 > 0 \right\}$$

where

$$\text{Exp}\ \ m \begin{pmatrix} \xi_1 \\ \\ \xi_2 \end{pmatrix} = M \begin{pmatrix} \cosh(\rho) + \dfrac{\xi_1}{\rho} \sinh(\rho) \\ \\ \dfrac{\xi_2}{\rho} \sinh(\rho) \end{pmatrix} \tag{27}$$

for $(\xi_1, \xi_2) \neq (0,0)$, $\rho := \sqrt{\xi_1^2 + \xi_2^2} \in \mathbb{R}$, $\rho > 0$, $\cosh(\rho) := \dfrac{e^\rho + e^{-\rho}}{2}$, $\sinh(\rho) := \dfrac{e^\rho - e^{-\rho}}{2}$,

$$\text{Exp}\ \ m \begin{pmatrix} 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{28}$$

and its inverse

$$\begin{aligned} \text{Exp}^{-1} M \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} &= m \begin{pmatrix} \dfrac{(x_1^2 - x_2^2 - 1)}{2x_1} \dfrac{\rho}{\sinh(\rho)} \\ x_2 \dfrac{\rho}{\sinh(\rho)} \end{pmatrix} \\ &= \frac{\rho}{\sinh(\rho)} m \begin{pmatrix} \dfrac{x_1^2 - x_2^2 - 1}{2x_1} \\ x_2 \end{pmatrix} \end{aligned} \tag{29}$$

for

$$\rho := \ln \left[\frac{(1 + x_1^2 + x_2^2) + \sqrt{(1 + x_1^2 + x_2^2)^2 - 4x_1^2}}{2x_1} \right],\ (x_1, x_2) \neq (1,0) \tag{30}$$

$$\text{Exp}^{\,-1} M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = m \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

Proof: The formulae (28) and (31) are straightforward. From now on, let us assume that $(\xi_1, \xi_2) \neq (0,0)$. The characteristic polynomial

$$\det \begin{pmatrix} \xi_1 - \lambda & \xi_2 \\ \xi_2 & -\xi_1 - \lambda \end{pmatrix} = \lambda^2 - \xi_1^2 - \xi_2^2 = 0$$

has roots $\pm\rho$ for $\rho := \sqrt{\xi_1^2 + \xi_2^2}$, $\rho > 0$. The columns of the orthogonal matrix T, satisfying

$$m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = T \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} {}^tT$$

are solutions of the homogeneous linear systems

$$\begin{pmatrix} \xi_1 \mp \rho & \xi_2 \\ \xi_2 & -\xi_1 \mp \rho \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For instance,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ -\xi_1 \pm \rho \end{pmatrix}$$

work out. Their lengths are, respectively,

$$\nu_{1,2} = \sqrt{\xi_2^2 + (-\xi_1 \pm \rho)^2} = \sqrt{2\rho(\rho \mp \xi_1)}.$$

Thus, one can choose

$$T = \begin{pmatrix} \dfrac{\xi_2}{\sqrt{2\rho(\rho - \xi_1)}} & \dfrac{\xi_2}{\sqrt{2\rho(\rho + \xi_1)}} \\ \dfrac{\rho - \xi_1}{\sqrt{2\rho(\rho - \xi_1)}} & \dfrac{-(\rho + \xi_1)}{\sqrt{2\rho(\rho + \xi_1)}} \end{pmatrix}.$$

Consequently,

$$\operatorname{Exp} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^k = T \left(\sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix}^k \right) {}^tT$$

$$= T \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} {}^tT = \begin{pmatrix} \dfrac{\xi_2}{\sqrt{2\rho(\rho - \xi_1)}} & \dfrac{\xi_2}{\sqrt{2\rho(\rho + \xi_1)}} \\ \dfrac{\rho - \xi_1}{\sqrt{2\rho(\rho - \xi_1)}} & \dfrac{-(\rho + \xi_1)}{\sqrt{2\rho(\rho + \xi_1)}} \end{pmatrix} \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} {}^tT$$

$$= \begin{pmatrix} \frac{\xi_2 e^{\rho}}{\sqrt{2\rho(\rho-\xi_1)}} & \frac{\xi_2 e^{-\rho}}{\sqrt{2\rho(\rho+\xi_1)}} \\ \frac{(\rho-\xi_1)e^{\rho}}{\sqrt{2\rho(\rho-\xi_1)}} & \frac{-(\rho+\xi_1)e^{-\rho}}{\sqrt{2\rho(\rho+\xi_1)}} \end{pmatrix} \begin{pmatrix} \frac{\xi_2}{\sqrt{2\rho(\rho-\xi_1)}} & \frac{\rho-\xi_1}{\sqrt{2\rho(\rho-\xi_1)}} \\ \frac{\xi_2}{\sqrt{2\rho(\rho+\xi_1)}} & \frac{-(\rho+\xi_1)}{\sqrt{2\rho(\rho+\xi_1)}} \end{pmatrix}$$

$$= M \begin{pmatrix} \cosh(\rho) + \frac{\xi_1}{\rho}\sinh(\rho) \\ \frac{\xi_2}{\rho}\sinh(\rho) \end{pmatrix}.$$

Here one can use the symmetry of the diagonal matrix $\begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix}$, in order to infer the symmetry of $\operatorname{Exp} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$. Further, $\det \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix} = 1$ and $\det(T)\det({}^tT) = \det(T{}^tT) = \det(I_2) = 1$ reveal that $\operatorname{Exp} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in SL(2,\mathbb{R})$.

Conversely, for any $M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1, x_2 \in \mathbb{R}$, $x_1 > 0$, there exist uniquely determined $\xi_1, \xi_2 \in \mathbb{R}$ with $\operatorname{Exp} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. More precisely, if $\rho := \sqrt{\xi_1^2 + \xi_2^2} \neq 0$,

$$\cosh(\rho) + \frac{\xi_1}{\rho}\sinh(\rho) = x_1, \quad \frac{\xi_2}{\rho}\sinh(\rho) = x_2,$$

then

$$x_1 \left(\cosh(\rho) - \frac{\xi_1}{\rho}\sinh(\rho)\right) = \cosh^2(\rho) - \frac{\xi_1^2}{\rho^2}\sinh^2(\rho)$$
$$= 1 + \left(1 - \frac{\xi_1^2}{\rho^2}\right)\sinh^2(\rho) = 1 + \frac{\xi_2^2}{\rho^2}\sinh^2(\rho) = 1 + x_2^2.$$

Consequently,

$$\frac{1 + x_2^2}{x_1} + x_1 = 2\cosh(\rho),$$

whereas (30). In particular, $\rho = 0$ if and only if $\frac{1 + x_1^2 + x_2^2}{2x_1} = \cosh(0) = 1$, $(x_1 - 1)^2 + x_2^2 = 0$, i.e., $(x_1, x_2) = (1, 0)$. That allows to determine

$$\xi_2 = x_2 \frac{\rho}{\sinh(\rho)}, \quad \xi_1 = (x_1 - \cosh(\rho))\frac{\rho}{\sinh(\rho)} = \frac{(x_1^2 - x_2^2 - 1)}{2x_1}\frac{\rho}{\sinh(\rho)}$$

for $(x_1, x_2) \neq (1, 0)$, Q.E.D.

Proposition 35. *The operations of the Cartan gyrovector space*

$$(SL(2,\mathbb{R})/SO(2), \oplus, \otimes)$$

on the upper half-plane

$$SL(2,\mathbb{R})/SO(2) = \left\{ M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} SO(2) \;\; ; \;\; x_1, x_2 \in \mathbb{R}, x_1 > 0 \right\}$$

are given explicitly by

$$M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} SO(2) \oplus M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} SO(2) = M \begin{pmatrix} z_1(x,y) \\ z_2(x,y) \end{pmatrix} SO(2) \quad \textit{for}$$

$$t_1(x,y) = (x_1y_1 + x_2y_2)^2 + \left[x_1y_2 + (1+y_2^2)\frac{x_2}{y_1} \right]^2 \tag{32}$$

$$\begin{aligned} t_2(x,y) &= (x_1y_1 + x_2y_2) \left[x_2y_1 + (1+x_2^2)\frac{y_2}{x_1} \right] \\ &\quad + \left[x_1y_2 + (1+y_2^2)\frac{x_2}{y_1} \right] \left[x_2y_2 + \frac{(1+x_2^2)(1+y_2^2)}{x_1y_1} \right] \end{aligned} \tag{33}$$

$$z_1(x,y) = \frac{\sqrt{t_1}\left[t_1(t_1^2 + t_2^2 - 1) + 1 - t_1^2 + t_2^2 \right]}{\left[(1-t_1)^2 + t_2^2 \right] \sqrt{(1+t_1)^2 + t_2^2}} \tag{34}$$

$$z_2(x,y) = \frac{\sqrt{t_1}\, t_2}{\sqrt{(1+t_1)^2 + t_2^2}} \tag{35}$$

and by

$$t \otimes M \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} SO(2) = M \begin{pmatrix} b_1(t,a) \\ b_2(t,a) \end{pmatrix} SO(2) \quad \textit{for}$$

$$a_1 > 0,\; (a_1, a_2) \neq (0,0),\; r = \ln \left[\frac{(1 + a_1^2 + a_2^2) + \sqrt{(1 + a_1^2 + a_2^2)^2 - 4a_1^2}}{2a_1} \right]$$

$$t \neq 0, \quad s = t \frac{r}{\sinh(r)} \frac{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2a_2^2}}{2a_1}$$

$$b_1(t,a) = \cosh(s) + \frac{a_1^2 - a_2^2 - 1}{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2a_2^2}} \sinh(s) \tag{36}$$

$$b_2(t,a) = \frac{2a_1a_2}{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2a_2^2}} \sinh(s) \tag{37}$$

$$t \otimes M \begin{pmatrix} 1 \\ 0 \end{pmatrix} SO(2) = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} SO(2) \tag{38}$$

$$0 \otimes M \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} SO(2) = M \begin{pmatrix} 1 \\ 0 \end{pmatrix} SO(2) \tag{39}$$

Proof: It follows from Lemma 34 that the coset representatives $M\begin{pmatrix}x_1\\x_2\end{pmatrix} \in \operatorname{Exp}(\mathfrak{p}_o)$. According to Corollary 32

$$M\begin{pmatrix}x_1\\x_2\end{pmatrix} SO(2) \oplus M\begin{pmatrix}y_1\\y_2\end{pmatrix} SO(2) = M\begin{pmatrix}x_1\\x_2\end{pmatrix} M\begin{pmatrix}y_1\\y_2\end{pmatrix} SO(2).$$

We look for a positive definite symmetric matrix $M\begin{pmatrix}z_1\\z_2\end{pmatrix} \in SL(2,\mathbb{R})$, such that

$$M\begin{pmatrix}x_1\\x_2\end{pmatrix} M\begin{pmatrix}y_1\\y_2\end{pmatrix} = M\begin{pmatrix}z_1\\z_2\end{pmatrix} U$$

for some $U \in SO(2)$. To this end, let $P(x,y) := M\begin{pmatrix}x_1\\x_2\end{pmatrix} M\begin{pmatrix}y_1\\y_2\end{pmatrix}$ and

$$N(x,y) := \sqrt{P(x,y)^t P(x,y)}$$

be the positive definite square root of the positive definite symmetric matrix $P(x,y)^t P(x,y)$. More precisely, if

$$P(x,y)^t P(x,y) = S(x,y)\Delta(x,y)^t S(x,y)$$

for an orthogonal matrix $S(x,y)$ and a positive definite diagonal matrix $\Delta(x,y)$, then $N(x,y)$ is defined as

$$N(x,y) := S(x,y)\sqrt{\Delta(x,y)}\left({}^t S(x,y)\right).$$

In the case under consideration, $\det P(x,y) = 1$, whereas $\det \Delta(x,y) = 1$ and

$$\Delta(x,y) = \begin{pmatrix}\delta(x,y) & 0\\ 0 & [\delta(x,y)]^{-1}\end{pmatrix}, \ \sqrt{\Delta(x,y)} = \begin{pmatrix}\sqrt{\delta(x,y)} & 0\\ 0 & \left(\sqrt{\delta(x,y)}\right)^{-1}\end{pmatrix}$$

for real positive $\delta(x,y)$, $\sqrt{\delta(x,y)}$. In particular, $\det\sqrt{\Delta(x,y)} = 1$ specifies that $N(x,y) \in SL(2,\mathbb{R})$. Moreover, $N(x,y)$ is symmetric and positive definite, so that $N(x,y) \in \operatorname{Exp}(\mathfrak{p}_o)$. Due to ${}^t N = N$, one observes that

$${}^t(N^{-1}P)(N^{-1}P) = {}^t P(N^2)^{-1}P = {}^t P(P^t P)^{-1}P = I_2.$$

Consequently, $U := N^{-1}P \in O(2)$ and, moreover, $U \in SO(2)$, due to $\det(N) = \det(P) = 1$. Thus, $M(z) = N(x,y)$.

For arbitrary $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^+\times\mathbb{R}$ we obtain explicitly the corresponding $(z_1,z_2)\in\mathbb{R}^+\times\mathbb{R}$ with $M(z)=N(x,y)$. It is immediate that

$$P(x,y):=\begin{pmatrix} x_1 & x_2 \\ x_2 & \dfrac{1+x_2^2}{x_1}\end{pmatrix}\begin{pmatrix} y_1 & y_2 \\ y_2 & \dfrac{1+y_2^2}{y_1}\end{pmatrix}$$
$$=\begin{pmatrix} x_1y_1+x_2y_2 & x_1y_2+\dfrac{x_2}{y_1}(1+y_2^2) \\ x_2y_1+(1+x_2^2)\dfrac{y_2}{x_1} & x_2y_2+\dfrac{(1+x_2^2)(1+y_2^2)}{x_1y_1}\end{pmatrix}$$

whereas

$$P(x,y)^tP(x,y)=\begin{pmatrix} x_1y_1+x_2y_2 & x_1y_2+\dfrac{x_2}{y_1}(1+y_2^2) \\ x_2y_1+(1+x_2^2)\dfrac{y_2}{x_1} & x_2y_2+\dfrac{(1+x_2^2)(1+y_2^2)}{x_1y_1}\end{pmatrix}$$
$$\times\begin{pmatrix} x_1y_1+x_2y_2 & x_2y_1+(1+x_2^2)\dfrac{y_2}{x_1} \\ x_1y_2+\dfrac{x_2}{y_1}(1+y_2^2) & x_2y_2+\dfrac{(1+x_2^2)(1+y_2^2)}{x_1y_1}\end{pmatrix}=M\begin{pmatrix} t_1(x,y) \\ t_2(x,y)\end{pmatrix}$$

for $t_1(x,y)$, $t_2(x,y)$ with (32), (33). Note that $\det\ M\begin{pmatrix} t_1(x,y) \\ t_2(x,y)\end{pmatrix}=1$ guarantees $t_1(x,y)>0$. Then the characteristic polynomial of $M\begin{pmatrix} t_1 \\ t_2\end{pmatrix}$ is

$$\det\begin{pmatrix} t_1-\lambda & t_2 \\ t_2 & \dfrac{1+t_2^2}{t_1}-\lambda\end{pmatrix}=\lambda^2-\frac{(1+t_1^2+t_2^2)}{t_1}\lambda+1=0$$

with roots

$$\lambda_{1,2}(t)=\frac{(1+t_1^2+t_2^2)\pm\sqrt{(1+t_1^2+t_2^2)^2-4t_1^2}}{2t_1}.$$

The homogeneous linear systems

$$\begin{pmatrix} t_1-\lambda_{1,2}(t) & t_2 \\ t_2 & \dfrac{1+t_2^2}{t_1}-\lambda_{1,2}(t)\end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$$

have solutions

$$\begin{pmatrix} t_2 \\ \lambda_{1,2}(t) - t_1 \end{pmatrix}$$

of length

$$\nu_{1,2}(t) = \sqrt{t_2^2 + (\lambda_{1,2}(t) - t_1)^2} = \sqrt{\frac{(1 - t_1^2 + t_2^2)}{t_1}\lambda_{1,2}(t) + (t_1^2 + t_2^2 - 1)}.$$

Therefore, the matrix

$$S(t_1, t_2) = \begin{pmatrix} \dfrac{t_2}{\nu_1(t)} & \dfrac{t_2}{\nu_2(t)} \\ \dfrac{\lambda_1(t) - t_1}{\nu_1(t)} & \dfrac{\lambda_2(t) - t_1}{\nu_2(t)} \end{pmatrix}$$

is orthogonal and

$$M\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = S(t_1, t_2)\begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix}{}^tS(t_1, t_2).$$

By construction,

$$M\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} := S(t_1, t_2)\begin{pmatrix} \sqrt{\lambda_1(t)} & 0 \\ 0 & \sqrt{\lambda_2(t)} \end{pmatrix}{}^tS(t_1, t_2)$$

$$M\begin{pmatrix} z_1(t) \\ \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \dfrac{t_2}{\nu_1(t)} & \dfrac{t_2}{\nu_2(t)} \\ \dfrac{\lambda_1(t) - t_1}{\nu_1(t)} & \dfrac{\lambda_2(t) - t_1}{\nu_2(t)} \end{pmatrix}\begin{pmatrix} \sqrt{\lambda_1(t)} & 0 \\ \\ 0 & \sqrt{\lambda_2(t)} \end{pmatrix}{}^tS(t_1, t_2)$$

$$M\begin{pmatrix} z_1(t) \\ \\ z_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{t_2\sqrt{\lambda_1(t)}}{\nu_1(t)} & \dfrac{t_2\sqrt{\lambda_2(t)}}{\nu_2(t)} \\ \dfrac{(\lambda_1(t) - t_1)\sqrt{\lambda_1(t)}}{\nu_1(t)} & \dfrac{(\lambda_2(t) - t_1)\sqrt{\lambda_2(t)}}{\nu_2(t)} \end{pmatrix}\begin{pmatrix} \dfrac{t_2}{\nu_1(t)} & \dfrac{\lambda_1(t) - t_1}{\nu_1(t)} \\ \dfrac{t_2}{\nu_2(t)} & \dfrac{\lambda_2(t) - t_1}{\nu_2(t)} \end{pmatrix}.$$

Consequently,

$$z_1(t) = \frac{t_2^2\sqrt{\lambda_1(t)}}{\nu_1^2(t)} + \frac{t_2^2\sqrt{\lambda_2(t)}}{\nu_2^2(t)}$$

$$z_2(t) = \frac{t_2[\lambda_1(t) - t_1]\sqrt{\lambda_1(t)}}{\nu_1^2(t)} + \frac{t_2[\lambda_2(t) - t_1]\sqrt{\lambda_2(t)}}{\nu_2^2(t)}.$$

Taking into account $\lambda_1(t)\lambda_2(t) = 1$ and $\lambda_1(t) + \lambda_2(t) = \dfrac{1 + t_1^2 + t_2^2}{t_1}$, one calculates

$$\begin{aligned}\nu_1^2\nu_2^2 =&[(\lambda_1(t) - t_1)^2 + t_2^2][(\lambda_2(t) - t_1)^2 + t_2^2] = (t_1 - \lambda_1(t))^2(t_1 - \lambda_2(t))^2 \\ &+ t_2^2\left[\lambda_1^2(t) + \lambda_2^2(t) - 2t_1(\lambda_1(t) + \lambda_2(t)) + 2t_1^2\right] + t_2^4 \\ =&t_2^4 + t_2^2\left[\frac{(1 + t_1^2 + t_2^2)^2}{t_1^2} - 2 - 2(1 + t_1^2 + t_2^2) + 2t_1^2\right] + t_2^4 \\ =&\frac{t_2^2}{t_1^2}[(1 + t_1)^2 + t_2^2][(1 - t_1)^2 + t_2^2].\end{aligned}$$

Further,

$$\begin{aligned}z_1(t) &= \frac{t_2^2}{\nu_1^2(t)\nu_2^2(t)}\left(\nu_1^2(t)\sqrt{\lambda_2(t)} + \nu_2^2(t)\sqrt{\lambda_1(t)}\right) \\ &= \frac{t_2^2}{\nu_1^2\nu_2^2}\left[\frac{1 - t_1^2 + t_2^2}{t_1} + t_1^2 + t_2^2 - 1\right]\left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right) \\ &= \frac{\sqrt{t_1}\left[t_1(t_1^2 + t_2^2 - 1) + 1 - t_1^2 + t_2^2\right]}{[(1 - t_1)^2 + t_2^2]\sqrt{(1 + t_1)^2 + t_2^2}}\end{aligned}$$

shows (34), due to $\sqrt{\lambda_1(t)}\sqrt{\lambda_2(t)} = 1$, $t_1 \geq 0$ and

$$\begin{aligned}\sqrt{\lambda_1(t)} + \sqrt{\lambda_2(t)} =&\sqrt{(\sqrt{\lambda_1(t)} + \sqrt{\lambda_2(t)})^2} \\ =&\sqrt{\lambda_1(t) + \lambda_2(t) + 2} = \frac{\sqrt{(1 + t_1)^2 + t_2^2}}{\sqrt{t_1}}.\end{aligned}$$

Next, (35) follows by

$$\begin{aligned}z_2(t) =&\frac{t_2}{\nu_1^2\nu_2^2}\left\{\left[\frac{1 - t_1^2 + t_2^2}{t_1}\lambda_2(t) + (t_1^2 + t_2^2 - 1)\right](\lambda_1(t) - t_1)\sqrt{\lambda_1(t)}\right. \\ &\left.+ \left[\frac{1 - t_1^2 + t_2^2}{t_1}\lambda_1(t) + (t_1^2 + t_2^2 - 1)\right](\lambda_2(t) - t_1)\sqrt{\lambda_2(t)}\right\} \\ =&\frac{t_2}{\nu_1^2\nu_2^2}\left\{\left[\frac{1-t_1^2+t_2^2}{t_1} - t_1(t_1^2+t_2^2-1) - (1-t_1^2+t_2^2)\right]\left(\sqrt{\lambda_1(t)}+\sqrt{\lambda_2(t)}\right)\right. \\ &\left.+ (t_1^2 + t_2^2 - 1)\left[(\lambda_1(t))^{\frac{3}{2}} + (\lambda_2(t))^{\frac{3}{2}}\right]\right\} = \frac{\sqrt{t_1}t_2}{\sqrt{(1 + t_1)^2 + t_2^2}}\end{aligned}$$

after expressing

$$[\lambda_1(t)]^{\frac{3}{2}} + [\lambda_2(t)]^{\frac{3}{2}} = \left(\sqrt{\lambda_1(t)} + \sqrt{\lambda_2(t)}\right)\left(\lambda_1(t) - \sqrt{\lambda_1(t)}\sqrt{\lambda_2(t)} + \lambda_2(t)\right)$$
$$= \frac{\sqrt{(1+t_1)^2 + t_2^2}}{\sqrt{t_1}}\left[\frac{1+t_1^2+t_2^2}{t_1} - 1\right].$$

According to Corollary 32, the scalar multiplication of the Cartan gyrovector space $SL(2,\mathbb{R})/SO(2)$ is given by

$$t \otimes M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} SO(2) = \mathrm{Exp}\left(t\mathrm{Exp}^{-1} M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) SO(2)$$

for $\forall t \in \mathbb{R}$, $\forall M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathrm{Exp}\,(\mathfrak{p}_o)$. Then by Lemma 34 there holds

$$\mathrm{Exp}\left(t\mathrm{Exp}^{-1} M\begin{pmatrix} a_1 \\ \\ a_2 \end{pmatrix}\right) = \mathrm{Exp}\, m\begin{pmatrix} t\dfrac{r}{\sinh(r)}\dfrac{a_1^2 - a_2^2 - 1}{2a_1} \\ t\dfrac{r}{\sinh(r)} a_2 \end{pmatrix}$$
$$= M\begin{pmatrix} b_1(t,a) \\ \\ b_2(t,a) \end{pmatrix}$$

for

$$a_1 > 0, \quad (a_1, a_2) \neq (1,0), r = \ln\left[\frac{(1+a_1^2+a_2^2) + \sqrt{(1+a_1^2+a_2^2)^2 - 4a_1^2}}{2a_1}\right] > 0$$
$$t \neq 0, \quad \rho = |t|\frac{r}{\sinh(r)}\frac{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2 a_2^2}}{2a_1} > 0$$
$$b_1(t,a) = \cosh(\rho) + \mathrm{sign}\,(t)\frac{a_1^2 - a_2^2 - 1}{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2 a_2^2}}\sinh(\rho)$$
$$b_2(t,a) = 2\mathrm{sign}\,(t)\frac{a_1 a_2}{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2 a_2^2}}\sinh(\rho).$$

Here $\mathrm{sign}\,(t) = 1$ for $t > 0$ or $\mathrm{sign}\,(t) = -1$ for $t < 0$. Let us introduce

$$s := \mathrm{sign}\,(t)\rho = t\frac{r}{\sinh(r)}\frac{\sqrt{(a_1^2 - a_2^2 - 1)^2 + 4a_1^2 a_2^2}}{2a_1}$$

and observe that $\mathrm{sign}\,(s) = \mathrm{sign}\,(t)$. Making use of the identities $\cosh\,(\mathrm{sign}\,(s)s) = \cosh(s)$, $\sinh\,(\mathrm{sign}\,(s)s) = \mathrm{sign}\,(s)\sinh(s)$, one obtains (36) and (37). The equalities (38) and (39) follow from (9), (31), (28), Q.E.D.

Let us construct explicitly the norm

$$\left\| \operatorname{Exp} m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} SO(2) \right\| := \frac{\sqrt{B_\theta \left(m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \right)}}{2\sqrt{2}}$$

for $\forall m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix} \in \mathfrak{p}_o = \mathfrak{p}\left(SL(2,\mathbb{R})/SO(2)\right)$, associated with the restriction of the Killing form

$$B_\theta \left(m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right) = -\operatorname{Tr} \left(\operatorname{ad}_{m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}} \operatorname{ad}_{\theta \left(m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right)} \right)$$

$$B_\theta \left(m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right) = \operatorname{Tr} \left(\operatorname{ad}_{m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}} \operatorname{ad}_{m \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}} \right)$$

on $\mathfrak{p}_o$. To this end, let us introduce the 2×2-matrix units E_{ij}, $1 \leq i, j \leq 2$, with single nonzero entry 1 at the intersection of the i-th row with the j-th column. Then $\mathfrak{p}_o$ is the real span of

$$\varepsilon_1 := E_{11} - E_{22} \quad \text{and} \quad \varepsilon_2 := E_{12} + E_{21}.$$

More precisely,

$$m \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_1 \varepsilon_1 + \xi_2 \varepsilon_2.$$

The isotropy subalgebra $so(2) \subset sl(2,\mathbb{R})$ is the real line, generated by

$$\varepsilon_o := E_{12} - E_{21}.$$

For an arbitrary $\zeta \in sl(2,\mathbb{R})$ let us identify

$$\operatorname{ad}_\zeta : sl(2,\mathbb{R}) \longrightarrow sl(2,\mathbb{R})$$

with its 3×3-matrix with respect to the basis ε_o, ε_1, ε_2. It is straightforward that

$$\operatorname{ad}_{\varepsilon_1}(\varepsilon_o) = [\varepsilon_1, \varepsilon_o] = 2\varepsilon_2, \quad \operatorname{ad}_{\varepsilon_1}(\varepsilon_1) = 0, \quad \operatorname{ad}_{\varepsilon_1}(\varepsilon_2) = 2\varepsilon_o$$

whereas

$$\operatorname{ad}_{\varepsilon_1} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Similarly,

$$\mathrm{ad}_{\varepsilon_2}(\varepsilon_o) = -2\varepsilon_1, \quad \mathrm{ad}_{\varepsilon_2}(\varepsilon_1) = -2\varepsilon_o, \quad \mathrm{ad}_{\varepsilon_2}(\varepsilon_2) = 0$$

reveal that

$$\mathrm{ad}_{\varepsilon_2} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently,

$$B_\theta(\varepsilon_1, \varepsilon_1) = \mathrm{Tr}\ (\mathrm{ad}_{\varepsilon_1})^2 = \mathrm{Tr}\ \left[\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}^2\right] = 8$$

$$B_\theta(\varepsilon_1, \varepsilon_2) = \mathrm{Tr}\ (\mathrm{ad}_{\varepsilon_1}\mathrm{ad}_{\varepsilon_2}) = \mathrm{Tr}\ \left[\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = 0$$

$$B_\theta(\varepsilon_2, \varepsilon_2) = \mathrm{Tr}\ (\mathrm{ad}_{\varepsilon_2})^2 = \mathrm{Tr}\ \left[\begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2\right] = 8.$$

Thus, we obtain the following

Corollary 36. *Let* $(SL(2,\mathbb{R})/SO(2), \oplus, \otimes)$ *be the Cartan gyrovector space on the upper half-plane, described in Proposition* 35*,*

$$\left\|\mathrm{Exp}\ \ m\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} SO(2)\right\| := \sqrt{\xi_1^2 + \xi_2^2}$$

$$d(x,y) = ||x \ominus y|| \quad \textit{for } \forall x, y \in SL(2,\mathbb{R})/SO(2).$$

Then d *is the distance function of the left* $SL(2,\mathbb{R})$*-invariant Killing metric* g *on the upper half-plane* $SL(2,\mathbb{R})/SO(2)$*. The left* $\oplus$*-translations and the gyrations are isometries for* g*,* d *and the geodesics for the Killing metric on* $SL(2,\mathbb{R})/SO(2)$ *are the gyro-lines* $\gamma(t) = x \oplus (t \otimes y)$*,* $t \in \mathbb{R}$ *for fixed* $x, y \in SL(2,\mathbb{R})/SO(2)$*.*

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