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COUNTING LATTICE POINTS IN CERTAIN RATIONAL POLYTOPES AND GENERALIZED DEDEKIND SUMS

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Abstract: Let $\mathcal{P} \subset \mathbf{R}^n$ be a rational convex polytope with vertices at the origin and on each positive coordinate axes. On the basis of the study on counting lattice points in $t\mathcal{P}$ with positive integer t, which is deeply connected with reciprocity laws for generalized Dedekind sums, we study the number of lattice points in the shifted polytope of $t\mathcal{P}$ by a fixed rational point. Certain generalized multiple Dedekind sums appear naturally in the main result.

Keywords: rational polytopes, lattice points, Ehrhart quasipolynomial, Dedekind sums.

1. Introduction

Let $\mathcal{P} \subset \mathbf{R}^n$ be a rational convex polytope and for $t \in \mathbf{N}$, put

$$L_{\mathcal{P}}(t) = \sharp\left(t\mathcal{P} \cap \mathbf{Z}^n\right),$$

the number of lattice points in $t\mathcal{P}$. It is known that $L_{\mathcal{P}}(t)$ is expressed as

$$L_{\mathcal{P}}(t) = c_n(t)t^n + \cdots + c_1(t)t + c_0(t)$$

with periodic functions $c_0(t), \cdots, c_n(t)$ and is called the Ehrhart quasipolynomial of $\mathcal{P}$([15]). Further the problem of finding an explicit expression of $L_{\mathcal{P}}(t)$ is deeply connected with reciprocity laws for certain generalized Dedekind sums. Historically, the first example appeared in [16], where Mordell studied the number of lattice points in the interior of the tetrahedron

$$\mathcal{P} = \left\{(x,y,z) \in \mathbf{R}^3_{\geqslant 0} \;\middle|\; \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leqslant 1\right\}$$

for $a, b, c \in \mathbf{N}$ and obtained a formula connected with a three-term relation of the classical Dedekind sums.

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Generalizations for higher dimensional case are studied in [2], [3] and [4] etc., in which Dedekind-Rademacher sums or Fourier-Dedekind sums appear naturally. Here, along the content of this paper, put

$$\mathcal{P}(\boldsymbol{a}) = \left\{(x_1, \cdots, x_n) \in \mathbf{R}^n_{\geqslant 0} \,\middle|\, a_1x_1 + \cdots a_nx_n \leqslant 1\right\}$$

and

$$L(t : \boldsymbol{a}) = L_{\mathcal{P}(\boldsymbol{a})}(t)$$

for $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbf{N}^n$. Let us recall the result for this case. It is obvious that $L(t : \boldsymbol{a})$ is equal to the Taylor coefficient of z^t of the function

$$\begin{aligned} F(z : \boldsymbol{a}) &\overset{\text{def}}{=} \left(\prod_{i=1}^{n}(1 + z^{a_i} + z^{2a_i} + \cdots)\right)(1 + z + z^2 + \cdots) \\ &= \left(\prod_{i=1}^{n}\frac{1}{1 - z^{a_i}}\right)\frac{1}{1 - z}. \end{aligned}$$

Making use of this, Beck, Dias and Robins studied in [3] an explicit expression of $L(t : \boldsymbol{a})$ under the condition of $\gcd\{a_i, a_j\} = 1$ for all $i \neq j$. In order to state the result precisely, let us define the Fourier-Dedekind sum by

$$\sigma_l(c_1, \cdots, c_n : c) = \frac{1}{c}\sum_{\substack{\zeta^c = 1\\ \zeta \neq 1}}\frac{\zeta^l}{(\zeta^{c_1} - 1)\cdots(\zeta^{c_n} - 1)} \tag{1.1}$$

for $c, c_1, \cdots, c_n \in \mathbf{N}$ and $l \in \mathbf{Z}$, and put

$$R_{-t}(\boldsymbol{a}) = -\operatorname{Res}\left(z^{-t-1}F(z : \boldsymbol{a}) : z = 1\right).$$

Then, it is shown in [3] that

$$L(t : \boldsymbol{a}) = R_{-t}(\boldsymbol{a}) + (-1)^n\sum_{i=1}^{n}\sigma_{-t}(a_1, \cdots, \widehat{a_i}, \cdots, a_n, 1 : a_i). \tag{1.2}$$

Note that if we put

$$p(t : \boldsymbol{a}) = \sharp\{(m_1, \cdots, m_n) \in \mathbf{Z}^n_{\geqslant 0} \,|\, a_1m_1 + \cdots a_nm_n = t\},$$

then

$$L(t : \boldsymbol{a}) = p(t : (\boldsymbol{a}, 1)),$$

where $(\boldsymbol{a}, 1) = (a_1, \cdots, a_n, 1) \in \mathbf{N}^{n+1}$. In [4], Beck, Gessel and Komatsu studied a formula for the polynomial part of $p(t : \boldsymbol{a})$. From Theorem and Proposition of [4] and Remark 1 of [3], we see that $R_{-t}(\boldsymbol{a})$ equals the polynomial part of $p(t : (\boldsymbol{a}, 1))$ and is expressed as

$$R_{-t}(\boldsymbol{a}) = \frac{1}{a_1\cdots a_n}\sum_{m=0}^{n}\frac{(-1)^m}{(n-m)!}\sum_{\substack{p_1, \cdots, p_n, q \in \mathbf{Z}_{\geqslant 0}\\ p_1 + \cdots + p_n + q = m}} a_1^{p_1}\cdots a_n^{p_n}\frac{B_{p_1}\cdots B_{p_n}B_q}{p_1!\cdots p_n!q!}t^{n-m}, \tag{1.3}$$

where B_p is the pth Bernoulli number.

As for the value at $t = 0$, it is known that if $\mathcal{P}$ is an integral polytope, $L_{\mathcal{P}}(t)$ is a polynomial of t of which the constant term equals the Euler characteristic $\chi(\mathcal{P})$ of $\mathcal{P}$. It is also known that $\chi(\mathcal{P}) = 1$ if $\mathcal{P}$ is convex. In our case, since $a_1 \cdots a_n \cdot \mathcal{P}(\boldsymbol{a})$ is integral and convex, we have $L(0, \boldsymbol{a}) = 1$. We note that this can also be interpreted as $L(0, \boldsymbol{a}) = \sharp\,(0 \cdot \mathcal{P}(\boldsymbol{a}) \cap \mathbf{Z}^n)$. In addition the formula (1.2) also holds for $t = 0$.

Now the classical Dedekind sum $s(a, b)$ is defined by

$$s(a,b) = \sum_{\lambda \bmod b} \left(\left(\frac{\lambda}{b}\right)\right)\left(\left(\frac{a\lambda}{b}\right)\right), \tag{1.4}$$

where $a \in \mathbf{Z}$, $b \in \mathbf{N}$ and

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z}. \end{cases}$$

If $a, b \in \mathbf{N}$ with $\gcd\{a, b\} = 1$, we have a well-known reciprocity law such as

$$s(a,b) + s(b,a) = \frac{1}{12}\left(\frac{b}{a} + \frac{a}{b} + \frac{1}{ab}\right) - \frac{1}{4}. \tag{1.5}$$

([14], [18]). In the special case of $n = 2$ and $t = 0$, we have

$$\sigma_0(a, 1 : b) = -s(a,b) + \frac{1}{4} - \frac{1}{4ab}$$

and the formula (1.2), together with (1.3), naturally reduces to (1.5).

In this paper, as a generalization of $L(t : \boldsymbol{a})$, we study the formula for the number of the lattice points in the shifted polytope of $t\mathcal{P}(\boldsymbol{a})$ by a fixed rational point, namely the formula expressing $\sharp\,((-\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})) \cap \mathbf{Z}^n)$ for $\boldsymbol{\alpha} \in \mathbf{Q}^n$. The special case of $n = 2$, in which $\mathcal{P}(\boldsymbol{a})$ is a rectangled triangle in $\mathbf{R}^2$, is studied in [5]. In our main result, we enlarge the range of t as $t \in \mathbf{Q}_{\geqslant 0}$ and multiple versions of the Dedekind-Rademacher sums will appear naturally. Let us give a description of each section.

In Section 2, we first recall the definition and basic properties of Bernoulli functions and give a definition of generalized Dedekind sums which appear in our main result.

In Section 3, as important tools for the study of lattice points in rational polytopes, we describe the integer-point transforms of rational polytopes or cones in $\mathbf{R}^n$ and well-known Brion's Theorem. Then we state the main result as a natural application of Brion's Theorem to the polytope $-\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})$. As a Corollary of the main result, we also show a generalized reciprocity law for multiple Dedekind-Rademacher sums.

In order to prove the main result, we prepare two equations as Lemmas in Section 4 and complete the proof in Section 5.

2. Notations and definitions

Let $B_p(X)$ be the pth Bernoulli polynomial defined by

$$\frac{te^{tX}}{e^t-1}=\sum_{p=0}^{\infty}B_p(X)\frac{t^p}{p!}.$$

and let $B_p=B_p(0)$, the pth Bernoulli number. For any $x\in\mathbf{Q}$, write $x=[x]+\{x\}$ with $[x]\in\mathbf{Z}$ and $0\leqslant\{x\}<1$ and define $\tilde{B}_p(x)=B_p(\{x\})$, which is periodic of period 1 and satisfies a distribution relation such as

$$\sum_{\lambda \bmod k}\tilde{B}_p\left(x+\frac{\lambda}{k}\right)=k^{1-p}\tilde{B}_p(kx) \tag{2.1}$$

for any $k\in\mathbf{N}$ and $x\in\mathbf{Q}$. Let $P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}$, $q\in\mathbf{Z}_{\geqslant 0}$, $\boldsymbol{a}=(a_1,\cdots,a_n)\in\mathbf{Z}^n, b\in\mathbf{Z}$, $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)\in\mathbf{Q}^n$ and $\beta\in\mathbf{Q}$, and we define the following multiple Dedekind sum:

$$\begin{aligned}&\mathcal{S}_{(P,q)}\begin{pmatrix}\boldsymbol{a} & b\\ \boldsymbol{\alpha} & \beta\end{pmatrix}\\ &\quad=\sum_{\lambda_1,\cdots,\lambda_n \bmod b}\left(\prod_{j=1}^{n}\tilde{B}_{p_j}\left(\frac{\lambda_j+\alpha_j}{b}\right)\right)\tilde{B}_q\left(\frac{\sum_{j=1}^n a_j(\lambda_j+\alpha_j)}{b}+\beta\right).\end{aligned} \tag{2.2}$$

In the special case of $n=1$, the sum is reduced to the classical Dedekind sum (1.4) as

$$\mathcal{S}_{(1,1)}\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}=\frac{1}{4}+s(a,b).$$

In addition, we also have

$$\mathcal{S}_{(p,q)}\begin{pmatrix}a & b\\ \alpha & \beta\end{pmatrix}=\sum_{\lambda \bmod b}\tilde{B}_p\left(\frac{\lambda+\alpha}{b}\right)\tilde{B}_q\left(\frac{a(\lambda+\alpha)}{b}+\beta\right),$$

which essentially includes the sums defined by Apostol as (1.3) in [1], by Rademacher as (1.3) in [17] and by Carlitz as (1.2) in [8], (1.7) in [10] and (1.12) in [12]. We also note that in [10] and [13], Carlitz had already studied the sum (2.2) in the case of $P=(1,\cdots,1)$, $\boldsymbol{\alpha}=(0,\cdots,0)$ and $\beta=0$ with rather modified forms.

In the case of $P=(1,\cdots,1)$, $q=1$, $\boldsymbol{\alpha}=\mathbf{0}=(0,\cdots,0)$ and $\beta=t/b$, the sum (2.2) is reduced to the Fourier-Dedekind sum (1.1) in such a way that

$$\mathcal{S}_{(1,\cdots,1,1)}\begin{pmatrix}\boldsymbol{a} & b\\ \mathbf{0} & t/b\end{pmatrix}=\sigma_{-t}(-\boldsymbol{a},1:b)+\frac{B_1^n}{b}. \tag{2.3}$$

In the case of $(P,q)=(p_1,\cdots,p_n,q)\in\mathbf{Z}^{n+1}_{\geqslant 0}-\mathbf{N}^{n+1}$ and $\gcd(b,a_j)=1$ for $1\leqslant j\leqslant n$, we can derive by (2.1) that

$$\mathcal{S}_{(P,q)}\begin{pmatrix}\boldsymbol{a} & b\\ \boldsymbol{\alpha} & \beta\end{pmatrix}=b^{n-(p_1+\cdots+p_n+q)}\left(\prod_{j=1}^{n}\tilde{B}_{p_j}(\alpha_j)\right)\tilde{B}_q(\boldsymbol{a}\cdot\boldsymbol{\alpha}+b\beta), \tag{2.4}$$

where $\boldsymbol{a}\cdot\boldsymbol{\alpha}=a_1\alpha_1+\cdots+a_n\alpha_n$, the inner product of $\boldsymbol{a}$ and $\boldsymbol{\alpha}$.

3. Integer-point transforms

Let $S \subset \mathbf{R}^n$ be a rational cone or polytope, The integer-point transform of S is defined by

$$\sigma(u_1, \cdots, u_n : S) = \sum_{(m_1,\cdots,m_n)\in S\cap\mathbf{Z}^n} u_1^{m_1} \cdots u_n^{m_n}. \tag{3.1}$$

If S is a polytope, the right-hand side of (3.1) is a finite sum. If S is a cone, the right-hand side of (3.1) is a Laurent series of $u_1^{\varepsilon_1}, \cdots, u_n^{\varepsilon_n}$, where $\varepsilon_j = 1$ or -1 for $1 \leqslant j \leqslant n$ and can also be expressed as a rational function of $u_1, \cdots, u_n$ (cf. Chapter 3.2 of [6]).

Let $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbf{N}^n$, $b \in \mathbf{N}$, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbf{Q}^n$ and $\beta \in \mathbf{Q}$. In what follows, we consider the range of t as $t \in \mathbf{Q}_{\geqslant 0}$.

Proposition 3.1. *Let $t \in \mathbf{Q}_{\geqslant 0}$ and let $K(t)$ denote the cone in $\mathbf{R}^{n+1}$ defined by*

$$K(t) = \Big\{(x_1, \cdots, x_n, y) \in \mathbf{R}^{n+1} \,\Big|\, \sum_{j=1}^n a_j(x_j + \alpha_j) + b(y + \beta) \leqslant t, \\ x_j + \alpha_j \geqslant 0 \ (1 \leqslant j \leqslant n)\Big\}.$$

Then we have

$$\begin{aligned}
&\sigma(u_1, \cdots, u_n, v : K(t)) \\
&\quad = \sum_{0\leqslant\lambda_1,\cdots,\lambda_n\leqslant b-1} \left(\prod_{j=1}^n \frac{u_j^{\lambda_j - [\alpha_j]}}{1 - u_j^b v^{-a_j}}\right) \frac{v^{[-\frac{1}{b}\sum_{j=1}^n a_j(\lambda_j + \{\alpha_j\}) - \beta + \frac{t}{b}]}}{1 - v^{-1}} && (3.2) \\
&\quad = u_1^{-\alpha_1} \cdots u_n^{-\alpha_n} v^{-\beta + \frac{t}{b}} \sum_{0\leqslant\lambda_1,\cdots,\lambda_n\leqslant b-1} \left(\prod_{j=1}^n \frac{(u_j^b v^{-a_j})^{\frac{\lambda_j + \{\alpha_j\}}{b}}}{1 - u_j^b v^{-a_j}}\right) \\
&\qquad \times \frac{v^{-\{-\frac{1}{b}\sum_{j=1}^n a_j(\lambda_j + \{\alpha_j\}) - \beta + \frac{t}{b}\}}}{1 - v^{-1}}. && (3.3)
\end{aligned}$$

Proof. If $(m_1, \cdots, m_n, m) \in K(t) \cap \mathbf{Z}^{n+1}$, then we have $m_j + \alpha_j \geqslant 0$ for each $1 \leqslant j \leqslant n$ and

$$m \leqslant -\frac{1}{b} \sum_{j=1}^n a_j(m_j + \alpha_j) - \beta + \frac{t}{b}.$$

This implies $m_j + [\alpha_j] \in \mathbf{Z}_{\geqslant 0}$ and we can express

$$m_j = -[\alpha_j] + \lambda_j + b l_j \qquad \text{with } \ 0 \leqslant \lambda_j \leqslant b - 1 \ \text{ and } \ l_j \in \mathbf{Z}_{\geqslant 0}$$

and

$$\begin{aligned} m &= \left[-\frac{1}{b}\sum_{j=1}^{n} a_j(m_j+\alpha_j) - \beta + \frac{t}{b} \right] - l \qquad \text{with} \;\; l \in \mathbf{Z}_{\geqslant 0} \\ &= \left[-\frac{1}{b}\sum_{j=1}^{n} a_j(\lambda_j+\{\alpha_j\}) - \beta + \frac{t}{b} \right] - \sum_{j=1}^{n} a_j l_j - l. \end{aligned}$$

Hence

$$\begin{aligned} &\sigma(u_1,\cdots,u_n,v:K(t)) \\ &\quad= \sum_{l_1,\cdots,l_n,l\geqslant 0}\ \sum_{0\leqslant\lambda_1,\cdots,\lambda_n\leqslant b-1} \left(\prod_{j=1}^{n} u_j^{-[\alpha_j]+\lambda_j} \right) v^{[-\frac{1}{b}\sum_{j=1}^{n} a_j(\lambda_j+\{\alpha_j\})-\beta+\frac{t}{b}]} \\ &\qquad \times \left(\prod_{j=1}^{n} (u_j^b v^{-a_j})^{l_j} \right) \cdot v^{-l} \\ &\quad= \sum_{0\leqslant\lambda_1,\cdots,\lambda_n\leqslant b-1} \left(\prod_{j=1}^{n} \frac{u_j^{\lambda_j-[\alpha_j]}}{1-u_j^b v^{-a_j}} \right) \frac{v^{[-\frac{1}{b}\sum_{j=1}^{n} a_j(\lambda_j+\{\alpha_j\})-\beta+\frac{t}{b}]}}{1-v^{-1}}. \end{aligned}$$

Thus, we obtain (3.2) and equation (3.3) is directly derived from (3.2) by making use of $[x] = x - \{x\}$ for any $x \in \mathbf{Q}$. ■

Now suppose that $\boldsymbol{a} = (a_1,\cdots,a_n) \in \mathbf{N}^n$ with $\gcd\{a_i,a_j\} = 1$ for all $i \neq j$ and as in the introduction, put

$$\mathcal{P}(\boldsymbol{a}) = \{(x_1,\cdots,x_n) \in \mathbf{R}^n_{\geqslant 0} \mid a_1x_1+\cdots+a_nx_n \leqslant 1\}.$$

Let $\mathrm{A}_1,\mathrm{A}_2,\cdots,\mathrm{A}_n$ denote the points $\left(\frac{1}{a_1},0,\cdots,0\right), \left(0,\frac{1}{a_2},0,\cdots,0\right),\cdots,$ $\left(0,\cdots,0,\frac{1}{a_n}\right)$, respectively. Then for $t>0$, the vertices of $t\mathcal{P}(\boldsymbol{a})$ are the origin and $t\mathrm{A}_1,\cdots,t\mathrm{A}_n$. For each $1 \leqslant i \leqslant n$, let $K_i(t)$ denote the tangent cone of $t\mathrm{A}_i$. Then

$$K_i(t) = \{(t-\mu_i)\overrightarrow{\mathrm{OA}_i} + \sum_{j\neq i} \mu_j \overrightarrow{\mathrm{A}_i\mathrm{A}_j} \mid \mu_1,\cdots,\mu_n \geqslant 0\} \tag{3.4}$$

$$= \{(x_1,\cdots,x_n) \in \mathbf{R}^n \mid a_1x_1+\cdots+a_nx_n \leqslant t, \; x_j \geqslant 0 \text{ for } 1\leqslant j\leqslant n \text{ with } j\neq i\}. \tag{3.5}$$

In addition , we put

$$K_0(t) = \mathbf{R}^n_{\geqslant 0}, \tag{3.6}$$

which is the tangent cone of the origin for $t\mathcal{P}(\boldsymbol{a})$. Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in \mathbf{Q}^n$ and let us consider the shifted polytope

$$\begin{aligned} -\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a}) = \{(x_1, \cdots, x_n) \in \mathbf{R}^n \mid a_1(x_1+\alpha_1) + \cdots + a_n(x_n+\alpha_n) \leqslant t, \\ x_j + \alpha_i \geqslant 0 \, (1 \leqslant j \leqslant n)\} \end{aligned}$$

and put

$$L(t : \boldsymbol{a}, \boldsymbol{\alpha}) = \sharp \left((-\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})) \cap \mathbf{Z}^n \right).$$

Then the vertices of $-\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})$ are the points $-\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha} + t\mathrm{A}_i$ for $1 \leqslant i \leqslant n$ and their tangent cones are $-\boldsymbol{\alpha} + K_0(t) = -\boldsymbol{\alpha} + \mathbf{R}^n_{\geqslant 0}$ and

$$\begin{aligned} -\boldsymbol{\alpha} + K_i(t) = \{(x_1, \cdots, x_n) \in \mathbf{R}^n \mid a_1(x_1+\alpha_1) + \cdots + a_n(x_n+\alpha_n) \leqslant t, \\ x_j + \alpha_j \geqslant 0 \ \text{ for } \ 1 \leqslant j \leqslant n \ \text{ with } \ j \neq i\}, \end{aligned}$$

respectively. Applying (3.3), we see that

$$\begin{aligned} &\sigma(u_1, \cdots, u_n, : -\boldsymbol{\alpha} + K_i(t)) \\ &\quad = u_1^{-\alpha_1} \cdots u_n^{-\alpha_n} u_i^{\frac{t}{a_i}} \sum_{0 \leqslant \lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \lambda_n \leqslant a_i - 1} \left(\prod_{j \neq i} \frac{(u_j^{a_i} u_i^{-a_j})^{\frac{\lambda_j + \{\alpha_j\}}{a_i}}}{1 - u_j^{a_i} u_i^{-a_j}} \right) \\ &\qquad \times \frac{u_i^{-\{-\frac{1}{a_i} \sum_{j \neq i} a_j(\lambda_j + \{\alpha_j\}) - \alpha_i + \frac{t}{a_i}\}}}{1 - u_i^{-1}} \end{aligned} \tag{3.7}$$

for $1 \leqslant i \leqslant n$. For $i = 0$, we have

$$\begin{aligned} \sigma(u_1, \cdots, u_n, : -\boldsymbol{\alpha} + K_0(t)) &= \sum_{(m_1, \cdots, m_n) \in (-\boldsymbol{\alpha} + \mathbf{R}^n_{\geqslant 0}) \cap \mathbf{Z}^n} u_1^{m_1} \cdots u_n^{m_n} \\ &= \prod_{i=1}^n \sum_{m_i \geqslant -[\alpha_i]} u_i^{m_i} = \prod_{i=1}^n \frac{u_i^{-[\alpha_i]}}{1 - u_i} \\ &= u_1^{-\alpha_1} \cdots u_n^{-\alpha_n} \prod_{i=1}^n \frac{u_i^{\{\alpha_i\}}}{1 - u_i}. \end{aligned} \tag{3.8}$$

Now we have the following theorem due to Brion ([7] or Theorem 9.7 of [6]).

Theorem 3.2 (Brion). *Suppose $\mathcal{P} \subset \mathbf{R}^n$ is a rational convex polytope. For each vertix v of $\mathcal{P}$, let K_v denote the tangent cone of v. Then we have*

$$\sigma(u_1, \cdots, u_n : \mathcal{P}) = \sum_{v : \text{a vertix of } \mathcal{P}} \sigma(u_1, \cdots, u_n : K_v).$$

Applying Brion's Theorem to $\mathcal{P} = -\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})$, we deduce that

$$\sigma(u_1, \cdots, u_n : -\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})) = \sum_{i=0}^{n} \sigma(u_1, \cdots, u_n : -\boldsymbol{\alpha} + K_i(t)). \tag{3.9}$$

For each $P = (p_1, \cdots, p_n) \in \mathbf{Z}^n_{\geqslant 0}$, we put $P_i = (p_1, \cdots, \widehat{p_i}, \cdots, p_n)$ for $1 \leqslant i \leqslant n$. Similarly we put $\boldsymbol{a}_i = (a_1, \cdots, \widehat{a_i}, \cdots, a_n)$ and $\boldsymbol{\alpha}_i = (\alpha_1, \cdots, \widehat{\alpha_i}, \cdots, \alpha_n)$. Then taking $u_i = e^{a_i x_i}$ for $1 \leqslant i \leqslant n$ and combining equations (3.7), (3.8), (3.9) and the definition (2.2), we obtain

$$\begin{aligned} &e^{a_1\alpha_1x_1+\cdots+a_n\alpha_nx_n} \sum_{(m_1,\cdots,m_n)\in(-\boldsymbol{\alpha}+t\mathcal{P}(\boldsymbol{a}))\cap\mathbf{Z}^n} e^{a_1m_1x_1+\cdots+a_nm_nx_n} \\ &= (-1)^n \sum_{i=1}^{n} e^{tx_i} \sum_{P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}} \mathcal{S}_{(P_i,p_i)} \begin{pmatrix} -\boldsymbol{a}_i & a_i \\ \boldsymbol{\alpha}_i & -\alpha_i + \dfrac{t}{a_i} \end{pmatrix} \\ &\quad \times \left(\prod_{j\neq i} \frac{(a_ia_j(x_j - x_i))^{p_j-1}}{p_j!} \right) \frac{(-a_ix_i)^{p_i-1}}{p_i!} \\ &\quad + (-1)^n \sum_{P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}} \prod_{i=1}^{n} \frac{\tilde{B}_{p_i}(\alpha_i)}{p_i!}(a_ix_i)^{p_i-1} \end{aligned} \tag{3.10}$$

Here we give a supplementary explanation for the case of $t = 0$. We define $K_i(0)$ by (3.4) or equivalently by (3.5) if $i \geqslant 1$ and by (3.6) if $i = 0$. Then (3.7) and (3.8) are also valid for $t = 0$. Further we have $-\boldsymbol{\alpha} + 0 \cdot \mathcal{P}(\boldsymbol{a}) = \{-\boldsymbol{\alpha}\}$ and we define $L(0 : \boldsymbol{a}, \boldsymbol{\alpha}) = \sharp(\{-\boldsymbol{\alpha}\} \cap \mathbf{Z}^n)$, which is 1 or 0 according as $\boldsymbol{\alpha} \in \mathbf{Z}^n$ or $\boldsymbol{\alpha} \notin \mathbf{Z}^n$. In the same way we can define $\sigma(u_1, \cdots, u_n : \{-\boldsymbol{\alpha}\}) = u^{-\alpha_1} \cdots u^{-\alpha_n}$ or 0 according as $\boldsymbol{\alpha} \in \mathbf{Z}^n$ or $\boldsymbol{\alpha} \notin \mathbf{Z}^n$. Since $\mathbf{Z}^n$ is discrete in $\mathbf{R}^n$, $L(t_0 + \varepsilon : \boldsymbol{a}, \boldsymbol{\alpha})$ and $\sigma(u_1, \cdots, u_n : -\boldsymbol{\alpha} + (t_0 + \varepsilon)\mathcal{P}(\boldsymbol{a}))$ remain invariant for any fixed $t_0 \in \mathbf{Q}_{\geqslant 0}$ and sufficiently small $\varepsilon \geqslant 0$. By considering the case of $t_0 = 0$, (3.10) also holds for $t = 0$.

Now for $t \in \mathbf{Q}_{\geqslant 0}$, we have

$$L(t : \boldsymbol{a}, \boldsymbol{\alpha}) = \sigma(1, \cdots, 1 : -\boldsymbol{\alpha} + t\mathcal{P}(\boldsymbol{a})),$$

which also equals the left-hand side of (3.10) at $(x_1, \cdots, x_n) = (0, \cdots, 0)$. In the rest of this paper, we shall study the right-hand side of (3.10) and deduce the following main result.

Theorem 3.3. *For any $t \in \mathbf{Q}_{\geqslant 0}$, we have*

$$L(t : \boldsymbol{a}, \boldsymbol{\alpha}) = P(t : \boldsymbol{a}, \boldsymbol{\alpha}) + (-1)^n \sum_{i=1}^{n} Q_i(t : \boldsymbol{a}, \boldsymbol{\alpha}), \tag{3.11}$$

where

$$P(t:\boldsymbol{a},\boldsymbol{\alpha})$$
$$=\frac{1}{a_1\cdots a_n}\sum_{m=0}^{n}\sum_{\substack{p_1,\cdots,p_n,p\in\mathbf{Z}_{\geqslant 0}\\ p_1+\cdots+p_n+p=m}}(-1)^m\left(\prod_{i=1}^{n}\frac{a_i^{p_i}\tilde{B}_{p_i}(\alpha_i)}{p_i!}\right)\frac{\tilde{B}_p(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)}{p!}\cdot\frac{t^{n-m}}{(n-m)!}$$

symbolically

$$=\frac{1}{a_1\cdots a_n}\Big(t-\Big(a_1\tilde{B}(\alpha_1)+\cdots+a_n\tilde{B}(\alpha_n)+\tilde{B}(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)\Big)\Big)^n\frac{1}{n!} \tag{3.12}$$

and

$$Q_i(t:\boldsymbol{a},\boldsymbol{\alpha})=\mathcal{S}_{(1,\cdots,1)}\begin{pmatrix}-\boldsymbol{a}_i & a_i\\ \boldsymbol{\alpha}_i & -\alpha_i+\dfrac{t}{a_i}\end{pmatrix}-\frac{1}{a_i}\left(\prod_{j\neq i}\tilde{B}_1(\alpha_j)\right)\tilde{B}_1(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t).$$

Taking $t=0$ and making use of the symbolical expression as in (3.12), we can easily dedeuce a generalized reciprocity law for multiple Dedekind-Rademacher sums, which we show as the following.

Corollary 3.4. *We have*

$$\begin{aligned}a_1\cdots a_n\sum_{i=1}^{n}\mathcal{S}_{(1,\cdots,1)}\begin{pmatrix}-\boldsymbol{a}_i & a_i\\ \boldsymbol{\alpha}_i & -\alpha_i\end{pmatrix}&\\ =-\frac{1}{n!}\Big(a_1\tilde{B}(\alpha_1)+\cdots+a_n\tilde{B}(\alpha_n)+\tilde{B}(-\boldsymbol{a}\cdot\boldsymbol{\alpha})\Big)^n&\\ +\sum_{i=1}^{n}\left(\prod_{j\neq i}a_j\tilde{B}_1(\alpha_j)\right)\tilde{B}_1(-\boldsymbol{a}\cdot\boldsymbol{\alpha})+\varepsilon,&\end{aligned} \tag{3.13}$$

where $\varepsilon=(-1)^n a_1\cdots a_n$ *or* 0 *according as* $\boldsymbol{\alpha}\in\mathbf{Z}^n$ *or* $\boldsymbol{\alpha}\notin\mathbf{Z}^n$.

4. Preliminary results

Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\triangle(\boldsymbol{x})=\triangle(x_1,\cdots,x_n)=\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)$, the difference product of $x_1,\cdots,x_n$. Then as is well known for the Vandermonde determinant, we have

$$\begin{vmatrix}x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1\\ \vdots & \vdots & & \vdots & \vdots\\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1\end{vmatrix}=\triangle(\boldsymbol{x}).$$

For the proof of Theorem 3.3, we shall need the following two lemmas.

Lemma 4.1. *Let $n \geqslant 2$ and $N \in \mathbf{Z}_{\geqslant 0}$. Then we have*

$$\begin{vmatrix} x_1^N & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^N & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \triangle(\boldsymbol{x}) \sum_{\substack{p_1,\cdots,p_n \in \mathbf{Z}_{\geqslant 0} \\ p_1+\cdots+p_n = N-n+1}} x_1^{p_1} \cdots x_n^{p_n}. \tag{4.1}$$

Lemma 4.2. *Let $n, N \in \mathbf{Z}_{\geqslant 0}$ and $l \in \mathbf{Z}$. Then we have*

$$\sum_{j=0}^{N} (-1)^j \binom{N}{j} \binom{l+j}{n} = \begin{cases} (-1)^N \dbinom{l}{n-N} & \textit{if} \quad N \leqslant n \\ 0 & \textit{otherwise.} \end{cases}$$

Concerning Lemma 4.1, let us recall the Schur polynomial, which is defined by

$$s(\boldsymbol{x} : (\lambda_j)) = \frac{\det(x_i^{\lambda_j+n-j})}{\triangle(\boldsymbol{x})}$$

for $(\lambda_j) = (\lambda_1, \cdots, \lambda_n) \in \mathbf{Z}_{\geqslant 0}$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and expressed by making use of the corresponding Young diagrams for (λ_j). If $N \geqslant n-1$, (4.1) is a direct consequence of the special case of $(\lambda_j) = (N-n+1, 0, \cdots, 0)$, in which the Schur polynomial becomes the complete symmetric polynomial of degree $N-n+1$ in n variables $x_1, \cdots, x_n$. Direct proof for this case is also possible by making use of induction on N. Note that in the case of $0 \leqslant N < n-1$, (4.1) is also valid since both sides become 0.

As for Lemma 4.2, consider the following equation

$$T^l (1+T)^N = \sum_{j=0}^{N} \binom{N}{j} T^{l+j}.$$

Differentiating both sides $n$times, we obtain

$$\sum_{j=0}^{N} \binom{n}{j} \frac{d^{n-j} T^l}{dT^{n-j}} \cdot \frac{d^j}{dT^j} (1+T)^N = \sum_{j=0}^{N} \binom{N}{j} \frac{d^n T^{l+j}}{dT^n},$$

namely

$$\sum_{j=0}^{N} \binom{n}{j} \binom{l}{n-j} (n-j)! T^{l-n+j} \cdot \binom{N}{j} j! (1+T)^{N-j} = \sum_{j=0}^{N} \binom{N}{j} \binom{l+j}{n} n! T^{l+j-n}.$$

By taking $T = -1$, the result follows immediately.

5. Proof of Theorem 3.3

In order to study the right-hand side of (3.10), we put $Z_0 = \mathbf{Z}^n_{\geqslant 0}$ and $Z_1 = \mathbf{N}^n$, and introduce the following functions $G_k(\boldsymbol{x}) = G_k(x_1, \cdots, x_n)$ and $H_k(\boldsymbol{x}) = H_k(x_1, \cdots, x_n)$ for $k = 0, 1$:

$$G_k(\boldsymbol{x}) = \sum_{i=1}^{n} e^{tx_i} \sum_{P=(p_1,\cdots,p_n)\in Z_k} \mathcal{S}_{(P_i,p_i)} \begin{pmatrix} -\boldsymbol{a}_i & a_i \\ \boldsymbol{\alpha}_i & -\alpha_i + \dfrac{t}{a_i} \end{pmatrix} \times \left(\prod_{j\neq i} \frac{(a_i a_j (x_j - x_i))^{p_j - 1}}{p_j!} \right) \frac{(-a_i x_i)^{p_i - 1}}{p_i!}$$

$$H_k(\boldsymbol{x}) = \frac{1}{a_1 \cdots a_n} \sum_{i=1}^{n} e^{tx_i} \sum_{P=(p_1,\cdots,p_n)\in Z_k} \left(\prod_{j\neq i} a_j^{p_j} \tilde{B}_{p_j}(\alpha_j) \right) \tilde{B}_{p_i}(-\boldsymbol{a}\cdot\boldsymbol{\alpha} + t) \times \left(\prod_{j\neq i} \frac{(x_j - x_i)^{p_j - 1}}{p_j!} \right) \frac{(-x_i)^{p_i - 1}}{p_i!}.$$

Then by (2.4), we see that

$$G_0(\boldsymbol{x}) - G_1(\boldsymbol{x}) = H_0(\boldsymbol{x}) - H_1(\boldsymbol{x}). \tag{5.1}$$

Taking $\boldsymbol{x} = (x, \cdots, x)$, we have

$$G_1(x, \cdots, x) = e^{tx} \sum_{i=1}^{n} \sum_{p=1}^{\infty} \mathcal{S}_{(1,\cdots,1,p)} \begin{pmatrix} -\boldsymbol{a}_i & a_i \\ \boldsymbol{\alpha}_i & -\alpha_i + \dfrac{t}{a_i} \end{pmatrix} \frac{(-a_i x)^{p-1}}{p!}$$

and

$$H_1(x, \cdots, x) = e^{tx} \sum_{i=1}^{n} \frac{1}{a_i} \sum_{p=1}^{\infty} \left(\prod_{j\neq i} \tilde{B}_1(\alpha_j) \right) \tilde{B}_p(-\boldsymbol{a}\cdot\boldsymbol{\alpha} + t) \frac{(-x)^{p-1}}{p!}.$$

Especially for $\boldsymbol{x} = (0, \cdots, 0)$, we have

$$G_1(0, \cdots, 0) = \sum_{i=1}^{n} \mathcal{S}_{(1,\cdots,1,1)} \begin{pmatrix} -\boldsymbol{a}_i & a_i \\ \boldsymbol{\alpha}_i & -\alpha_i + \dfrac{t}{a_i} \end{pmatrix} \tag{5.2}$$

and

$$H_1(0, \cdots, 0) = \sum_{i=1}^{n} \frac{1}{a_i} \left(\prod_{j\neq i} \tilde{B}_1(\alpha_j) \right) \tilde{B}_1(-\boldsymbol{a}\cdot\boldsymbol{\alpha} + t). \tag{5.3}$$

As for $H_0(\boldsymbol{x})$, we first note that

$$x_i^{p_i - 1} \left(\prod_{j\neq i} (x_j - X)^{p_j} \right) (x_i - X)^p \Bigg|_{X = x_i} = \begin{cases} x_i^{p_i - 1} \prod_{j\neq i} (x_j - x_i)^{p_j} & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p \geqslant 1. \end{cases}$$

for $1 \leqslant i \leqslant n$. Changing the roles of p_i and p, we can express

$$H_0(\boldsymbol{x}) = \frac{1}{a_1 \cdots a_n} \sum_{i=1}^{n} e^{tx_i} \sum_{P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}} \sum_{p=0}^{\infty} \left(\prod_{j=1}^{n} \frac{a_j^{p_j} \tilde{B}_{p_j}(\alpha_j)}{p_j!} \right) \frac{\tilde{B}_p(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)}{p!} \times \frac{(-x_i)^{p-1} \prod_{j=1}^{n} (x_j - X)^{p_j} \Big|_{X=x_i}}{\prod_{j\neq i} (x_j - x_i)}.$$

For each $P = (p_1, \cdots, p_n) \in \mathbf{Z}^n_{\geqslant 0}$ and $p \in \mathbf{Z}_{\geqslant 0}$, we put

$$\mathcal{B}(P,p) = \left(\prod_{j=1}^{n} \frac{a_j^{p_j} \tilde{B}_{p_j}(\alpha_j)}{p_j!} \right) \frac{\tilde{B}_p(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)}{p!}$$

and

$$s(P) = p_1 + \cdots + p_n,$$

and express

$$\prod_{j=1}^{n} (x_i - X)^{p_j} = \sum_{k=0}^{s(P)} c_k(\boldsymbol{x} : P) X^k$$

with $c_k(\boldsymbol{x} : P) \in \mathbf{Z}[x_1, \cdots, x_n]$. Then

$$\begin{aligned}
&\triangle(\boldsymbol{x}) H_0(\boldsymbol{x}) \\
&= \frac{1}{a_1 \cdots a_n} \sum_{p=0}^{\infty} \sum_{P\in\mathbf{Z}^n_{\geqslant 0}} \mathcal{B}(P,p) \sum_{i=1}^{n} e^{tx_i} \triangle(x_1, \cdots, \widehat{x_i}, \cdots, x_n)(-1)^{p+n-i-1} x_i^{p-1} \\
&\quad \times \sum_{k=0}^{s(P)} c_k(\boldsymbol{x} : P) x_i^k \\
&= \frac{(-1)^n}{a_1 \cdots a_n} \sum_{p=0}^{\infty} \sum_{P\in\mathbf{Z}^n_{\geqslant 0}} (-1)^p \mathcal{B}(P,p) \sum_{k=0}^{s(P)} c_k(\boldsymbol{x} : P) \begin{vmatrix} e^{tx_1} x_1^{p+k-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ e^{tx_n} x_n^{p+k-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} \\
&= \frac{(-1)^n}{a_1 \cdots a_n} \sum_{p=0}^{\infty} \sum_{P\in\mathbf{Z}^n_{\geqslant 0}} (-1)^p \mathcal{B}(P,p) \sum_{k=0}^{s(P)} c_k(\boldsymbol{x} : P) \\
&\quad \times \sum_{m=0}^{\infty} \frac{t^m}{m!} \begin{vmatrix} x_1^{m+p+k-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{m+p+k-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}.
\end{aligned}$$

Applying Lemma 4.1, we have

$$\begin{vmatrix} x_1^{m+p+k-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{m+p+k-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \triangle(\boldsymbol{x}) \sum_{\substack{Q=(q_1,\cdots,q_n)\in\mathbf{Z}^n_{\geqslant 0} \\ s(Q)=m+p+k-n}} x_1^{q_1}\cdots x_n^{q_n}$$

except for the case of $m = p = k = 0$. If $m = p = k = 0$, the determinant above becomes

$$\begin{vmatrix} x_1^{-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix} = \frac{(-1)^{n-1}}{x_1\cdots x_n}\triangle(\boldsymbol{x}).$$

Hence we deduce that

$$\begin{aligned} &H_0(\boldsymbol{x}) \\ &= \frac{(-1)^n}{a_1\cdots a_n}\sum_{p=0}^{\infty}\sum_{P\in\mathbf{Z}^n_{\geqslant 0}}(-1)^p\mathcal{B}(P,p)\sum_{k=0}^{s(P)}c_k(\boldsymbol{x}:P)\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\substack{Q=(q_1,\cdots,q_n)\in\mathbf{Z}^n_{\geqslant 0} \\ s(Q)=m+p+k-n}} x_1^{q_1}\cdots x_n^{q_n} \\ &\quad - \frac{1}{a_1\cdots a_n}\sum_{P\in\mathbf{Z}^n_{\geqslant 0}}\frac{\mathcal{B}(P,0)c_0(\boldsymbol{x}:P)}{x_1\cdots x_n}. \end{aligned}$$

Now taking $\boldsymbol{x} = (x,\cdots,x)$, we have

$$\prod_{j=1}^{n}(x_j - X)^{p_j} = (x-X)^{s(P)} = \sum_{k=0}^{s(P)}\binom{s(P)}{k}x^{s(P)-k}(-1)^k X^k,$$

which implies

$$c_k(x,\cdots,x:P) = \binom{s(P)}{k}(-1)^k x^{s(P)-k}.$$

Hence

$$\begin{aligned} H_0(x,\cdots,x) = \frac{(-1)^n}{a_1\cdots a_n}&\sum_{p=0}^{\infty}\sum_{P\in\mathbf{Z}^n_{\geqslant 0}}\mathcal{B}(P,p)\sum_{k=0}^{s(P)}\binom{s(P)}{k}(-1)^{p+k} \\ &\times\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{\substack{Q\in\mathbf{Z}^n_{\geqslant 0} \\ s(Q)=m+p+k-n}} x^{s(P)+s(Q)-k} \\ &- \frac{1}{a_1\cdots a_n}\sum_{P\in\mathbf{Z}^n_{\geqslant 0}}\mathcal{B}(P,0)x^{s(P)-n}. \end{aligned}$$

Note that for any $l \in \mathbf{Z}_{\geqslant 0}$, the number of $Q \in \mathbf{Z}_{\geqslant 0}^n$ satisfying $s(Q) = l$ is what is called the number of repeated combination and equals $\binom{l+n-1}{n-1}$. It follows that

$$
\begin{aligned}
H_0(x, \cdots, x) = \frac{(-1)^n}{a_1 \cdots a_n} & \sum_{p=0}^{\infty} \sum_{P \in \mathbf{Z}_{\geqslant 0}^n} \mathcal{B}(P, p) \sum_{k=0}^{s(P)} \binom{s(P)}{k} (-1)^{p+k} \\
& \times \sum_{m=0}^{\infty} \frac{t^m}{m!} \binom{m+p+k-1}{n-1} x^{m+p+s(P)-n} \\
& - \frac{(-1)^n}{a_1 \cdots a_n} \sum_{P \in \mathbf{Z}_{\geqslant 0}^n} \mathcal{B}(P, 0) \binom{-1}{n-1} x^{s(P)-n} \\
& - \frac{1}{a_1 \cdots a_n} \sum_{P \in \mathbf{Z}_{\geqslant 0}^n} \mathcal{B}(P, 0) x^{s(P)-n}.
\end{aligned}
$$

Note that the last two summations in the right-hand side of this equation are canceled since $\binom{-1}{n-1} = (-1)^{n-1}$. Then applying Lemma 4.2, we see that

$$
\begin{aligned}
& H_0(x, \cdots, x) \\
& = \frac{(-1)^n}{a_1 \cdots a_n} \sum_{p=0}^{\infty} \sum_{\substack{P \in \mathbf{Z}_{\geqslant 0}^n \\ s(P) \leqslant n-1}} \mathcal{B}(P, p) \sum_{m=0}^{\infty} \frac{t^m}{m!} (-1)^{p+s(P)} \binom{m+p-1}{n-1-s(P)} x^{m+p+s(P)-n}.
\end{aligned} \tag{5.4}
$$

Now we see from (3.10) that

$$
\begin{aligned}
& L(t : \boldsymbol{a}, \boldsymbol{\alpha}) = \text{constant term of} \\
& \qquad (-1)^n \left(G_0(x, \cdots, x) + \sum_{P=(p_1, \cdots, p_n) \in \mathbf{Z}_{\geqslant 0}^n} \left(\prod_{i=1}^n \frac{a_i^{p_i - 1} \tilde{B}_{p_i}(\alpha_i)}{p_i!} \right) x^{s(P)-n} \right).
\end{aligned}
$$

From (5.1), (5.2) and (5.3), we also see that the constant term of $G_0(x, \cdots, x) - H_0(x, \cdots, x)$ equals

$$
\begin{aligned}
& G_1(0, \cdots, 0) - H_1(0, \cdots, 0) \\
& \quad = \sum_{i=1}^n \left(\mathcal{S}_{(1, \cdots, 1, 1)} \begin{pmatrix} -\boldsymbol{a}_i & a_i \\ \boldsymbol{\alpha}_i & -\alpha_i + \dfrac{t}{a_i} \end{pmatrix} - \frac{1}{a_i} \Big(\prod_{j \neq i} \tilde{B}_1(\alpha_j)\Big) \tilde{B}_1(-\boldsymbol{a} \cdot \boldsymbol{\alpha} + t) \right).
\end{aligned}
$$

It follows from (5.4) that

$$
\begin{aligned}
&L(t:\boldsymbol{a},\boldsymbol{\alpha})\\
&=(-1)^n\sum_{i=1}^n\left(\mathcal{S}_{(1,\cdots,1)}\begin{pmatrix}-\boldsymbol{a}_i & a_i\\ \boldsymbol{\alpha}_i & -\alpha_i+\dfrac{t}{a_i}\end{pmatrix}-\frac{1}{a_i}\big(\prod_{j\neq i}\tilde{B}_1(\alpha_j)\big)\tilde{B}_1(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)\right)\\
&\quad+\frac{1}{a_1\cdots a_n}\sum_{\substack{P\in\mathbf{Z}^n_{\geqslant 0}\\ s(P)\leqslant n-1}}\ \sum_{\substack{p,m\geqslant 0\\ s(P)+m+p=n}}(-1)^{p+s(P)}\mathcal{B}(P,p)\frac{t^m}{m!}\\
&\quad+\frac{(-1)^n}{a_1\cdots a_n}\sum_{\substack{P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}\\ s(P)=n}}\prod_{i=1}^n\frac{a_i^{p_i}\tilde{B}_{p_i}(\alpha_i)}{p_i!}\\
&=\frac{1}{a_1\cdots a_n}\sum_{\substack{P=(p_1,\cdots,p_n)\in\mathbf{Z}^n_{\geqslant 0}\\ s(P)\leqslant n}}\ \sum_{\substack{p,m\geqslant 0\\ s(P)+m+p=n}}(-1)^{s(P)+p}\mathcal{B}(P,p)\frac{t^m}{m!}\\
&\quad+(-1)^n\sum_{i=1}^n\left(\mathcal{S}_{(1,\cdots,1)}\begin{pmatrix}-\boldsymbol{a}_i & a_i\\ \boldsymbol{\alpha}_i & -\alpha_i+\dfrac{t}{a_i}\end{pmatrix}-\frac{1}{a_i}\big(\prod_{j\neq i}\tilde{B}_1(\alpha_j)\big)\tilde{B}_1(-\boldsymbol{a}\cdot\boldsymbol{\alpha}+t)\right),
\end{aligned}
$$

which is easily transformed into the right-hand side of (3.11). This completes the proof of Theorem 3.3. ■

As for relations to preceding results mainly by Beck, Carlitz and Rademacher, we note the following.

Remark 5.1. In the case of $\boldsymbol{\alpha}=(0,\cdots,0)$ and $t\in\mathbf{Z}_{\geqslant 0}$, $P(t:\boldsymbol{a},\boldsymbol{\alpha})$ reduces to the right-hand side of (1.3) and $Q_i(t:\boldsymbol{a},\boldsymbol{\alpha})$ to $\sigma_{-t}(a_1,\cdots,\widehat{a_i},\cdots,a_n:a_i)$ by virtue of (2.3). Hence (3.11) reduces to the formula (1.2).

Remark 5.2. In the case of $n=2$ and $t=0$, some calculations show that (3.13) reduces to the reciprocity law for Dedekind-Rademacher sums (Theorem 2 of [17] or the formula in the case $p=1$ for (4.4) of [11]). In addition, multiplying both sides of (3.10) by $(x_1-x_2)x_1x_2$ and examining the coefficient of $x_1^rx_2^s$ carefully for each $r,s\in\mathbf{Z}_{\geqslant 0}$, we can also derive the formula (2.15) of [12], which also reduces to (3.2) of [8] and (4.1) of [9] if $\boldsymbol{\alpha}\in\mathbf{Z}^2$.

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