Functiones et Approximatio
55.2 (2016), 177–197
doi: 10.7169/facm/2016.55.2.3

AN EXPLICIT RESULT FOR PRIMES BETWEEN CUBES

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Abstract: We prove that there is a prime between n^3 and $(n+1)^3$ for all $n \geqslant \exp(\exp(33.3))$. This is done by first deriving the Riemann–von Mangoldt explicit formula for the Riemann zeta-function with explicit bounds on the error term. We use this along with other recent explicit estimates regarding the zeroes of the Riemann zeta-function to obtain the result. Furthermore, we show that there is a prime between any two consecutive mth powers for $m \geqslant 5 \times 10^9$. Notably, many of the explicit estimates developed in this paper can also find utility elsewhere in the theory of numbers.

Keywords: prime numbers, Riemann zeta-function, Legendre's conjecture.

1. Introduction

Legendre's conjecture asserts that there is at least one prime between any two consecutive squares. Proving this assertion is currently out of reach; even the conditional sledgehammer of the Riemann hypothesis does not suffice in furnishing a proof (see Cramér [3]). Therefore, the purpose of this paper is to consider the weaker problem of primes between cubes, where some progress has already been made.

Consider first the more general problem of showing the existence of at least one prime in the interval $(x, x+x^{\theta})$ for some $\theta \in (0,1)$ and for all sufficiently large x. In 1930, Hoheisel [8] was able to furnish such a result for $\theta = 1 - 1/33000$, that is, there is a prime in the interval

$$(x, x + x^{32999/33000})$$

for all sufficiently large x. His proof made use of asymptotic estimates on the distribution of zeroes of the Riemann zeta-function $\zeta(s)$, namely a zero-free region and a zero-density estimate. Landau's explicit formula for the Riemann zeta-function then allows a connection to be made between the zeroes and the primes. Using Hoheisel's ideas, Ingham [9] was able to prove a more general theorem,

2010 Mathematics Subject Classification: primary: 11Y35; secondary: 11N05

specifically that if one has a bound of the form

$$\zeta(1/2+it) \ll t^c$$

for some $c > 0$, then one can take

$$\theta = \frac{1+4c}{2+4c} + \epsilon$$

for any $\epsilon > 0$. Notably, Hardy and Littlewood were able to give a value of $c = 1/6+\epsilon$, which corresponds to $\theta = 5/8+\epsilon$. From this, setting $x = n^3$ gives

$$(x, x+x^{5/8}) = (n^3, n^3+n^{15/8+\epsilon}) \subset (n^3, (n+1)^3),$$

that is, there is a prime between any two sufficiently large consecutive cubes. Actually, the somewhat larger interval

$$(x, x+3x^{2/3})$$

is sufficient for primes between cubes and as such is the interval we use throughout this paper.

The primary purpose of this paper is to combine explicit results on the Riemann zeta-function to prove the following theorem.

Theorem 1.1. *There is a prime between* n^3 *and* $(n+1)^3$ *for all* $n \geqslant \exp(\exp(33.3))$.

We should note that Cheng [2] has purported to have proved the above theorem for the range $n \geqslant \exp(\exp(15))$. We note, however, that he incorrectly infers that

$$n^3 \geqslant \exp(\exp(45))$$

implies

$$n \geqslant \exp(\exp(15))$$

in establishing his result. There are some other errors also, notably in his proof of Theorem 3 in his paper [2], the first inequality is incorrect and he has used Chebyshev's ψ-function instead of the θ-function.

Clearly, our method does not establish a complete result for the case of cubes, and so we then determine the least integer m such that a complete result can be established.

Theorem 1.2. *Let* $m \geqslant 5 \cdot 10^9$*. Then there is a prime between* n^m *and* $(n+1)^m$ *for all* $n \geqslant 1$.

To prove Theorem 1.1 and Theorem 1.2, we require a version of the Riemann–von Mangoldt explicit formula with explicit bounds on the error term. Such a formula is proven in Section 2, and can be stated as follows. Note that the stipulation that $x > e^{60}$ is suitable as one can elicit results up to this point using the techniques of Ramaré and Saouter [15].

Theorem 1.3. *Let $x > e^{60}$ be half an odd integer and suppose that $50 < T < x$. Then*

$$\psi(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + O^*\left(\frac{2x\log^2 x}{T}\right) \tag{1}$$

where the function on the left is Chebyshev's ψ-function, $\rho = \beta + i\gamma$ denotes a zero of $\zeta(s)$, and the notation $f = g + O^*(h)$ is equivalent to $|f - g| \leqslant h$.

The reader should note that whilst this paper seemingly culminates in the proof of Theorem 1.1, the various explicit results presented herein also find application elsewhere in the theory of numbers. For example, one could use Theorem 1.3 to obtain an explicit bound on the error term in the Prime Number Theorem that is asymptotically better than the result of Mossinghoff and Trudgian [12]. In addition, Kerr [11] has used Lemma 2.5 to obtain lower bounds for the Riemann zeta-function on short intervals of the critical line.

Finally, we should also mention the striking result of Baker, Harman and Pintz [1], that the interval $(x, x + x^{0.525})$ contains a prime for all sufficiently large x. This is tantalisingly close to $\theta = 1/2$, which would furnish a proof of Legendre's conjecture with at most finitely many exceptions.

2. Proof of Theorem 1.3

Our considerations begin with the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p, & n = p^m,\ p \text{ is prime}, m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

and the Chebyshev function $\psi(x) = \sum_{n\leqslant x} \Lambda(n)$. The utility of this function presents itself in the Riemann–von Mangoldt explicit formula

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}) \tag{2}$$

where x is any positive non-integer and the sum is over all nontrivial zeroes ρ of $\zeta(s)$ (see Davenport [4, Ch. 17] for details). One can see that estimates on the zeroes of $\zeta(s)$ can be used to resolve the structure of the prime numbers. However, the explicit formula as seen above relies on estimates over *all* of the nontrivial zeroes of $\zeta(s)$ and so is impractical for certain applications. We often find more use in a *truncated* version of the explicit formula where the sum is over the zeroes $\rho = \beta + i\gamma$ that satisfy $|\gamma| < T$ for some height T. It is the purpose of this section to provide such a formula *viz* Theorem 1.3.

For the most part, we proceed as laid out in Davenport [4]. For $c > 0$, we define the contour integral

$$\delta(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 0 & \text{if } 0 < x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

The importance of this integral becomes apparent when one wishes to study the sum of an arithmetic function up to some value x, particularly when that function is generated by a Dirichlet series. In our case we consider that, for a positive non-integer x and $c > 1$, we have

$$\begin{aligned}\psi(x) = \sum_{n \leqslant x} \Lambda(n) &= \sum_{n=1}^{\infty} \Lambda(n) \delta\left(\frac{x}{n}\right) \\ &= \sum_{n=1}^{\infty} \Lambda(n) \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s}\right] \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right) \frac{x^s}{s} ds.\end{aligned}$$

Notice that keeping $c > 1$ gives absolute convergence to the series in the above equation, and thus justifies the interchange of integration and summation. The Dirichlet series in the above equation is known to be equal to $-\zeta'(s)/\zeta(s)$, and so we have that

$$\sum_{n \leqslant x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} ds.$$

In a more general form this is known as Perron's formula. We may thus estimate the sum of the von Mangoldt function through some knowledge of certain analytic properties of $\zeta'(s)/\zeta(s)$. Our first step is to truncate the path of the integral to a finite segment, namely $(c-iT, c+iT)$. We define for $T > 0$ the truncated integral

$$I(x, T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} ds.$$

The next lemma is a variant of the first lemma in Davenport [4, Ch.17], and will bound the induced error term upon estimating $\delta(x)$ by $I(x, T)$. The proof is omitted here, though one can see Theorem 15 of Estermann [7] for a complete proof.

Lemma 2.1. *For $x > 0$ with $x \neq 1$, $c > 0, T > 0$ we have*

$$\delta(x) = I(x, T) + O^*\left(\frac{x^c}{\pi T |\log x|}\right).$$

From the above lemma, we have that

$$\begin{aligned}\psi(x) &= \sum_{n=1}^{\infty} \Lambda(n) \delta\left(\frac{x}{n}\right) \\ &= \sum_{n=1}^{\infty} \Lambda(n) \left[I(\frac{x}{n}, T) + O^*\left(\frac{1}{\pi T}\left(\frac{x}{n}\right)^c \left|\log \frac{x}{n}\right|^{-1}\right)\right] \\ &= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + \frac{1}{\pi T} O^*\left(\sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c \left|\log \frac{x}{n}\right|^{-1}\right).\end{aligned}$$

Our next lemma bounds the sum in the above formula. We keep $x > e^{60}$ to make the error terms small.

Lemma 2.2. *Let $x > e^{60}$ be half an odd integer and set $c = 1 + 1/\log x$. Then*

$$\sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^c \left|\log \frac{x}{n}\right|^{-1} < 3.1 x \log^2 x. \tag{3}$$

Proof. Some care needs to be taken here. When x and n are quite close, the reciprocal log will become large. Thus, we introduce the parameter $\alpha \in (1, 2)$ and break up the infinite sum:

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{[x/\alpha]} + \sum_{n=[x/\alpha]+1}^{[x]-1} + \sum_{n=[x]}^{[x]+1} + \sum_{n=[x]+2}^{[\alpha x]} + \sum_{n=[\alpha x]+1}^{\infty}$$

On the right side of the above formula, denote the ith sum by S_i. The reader should be convinced by this division; S_3 deals with the most inflated terms, namely when n is either side of x. Then S_2 and S_4 measure the remainder of the region which is *close* to x. We also note that S_1 and S_5 contribute little and can be estimated almost trivially.

Considering the range of n in S_1 and S_5, we have

$$\left|\log \frac{x}{n}\right| > \log \alpha.$$

Inserting this into these sums, pulling out terms which are independent of n, and extending the range of summation to $\mathbb{N}$ we arrive at

$$S_1 + S_5 < \frac{x^c}{\log \alpha} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c} = \frac{x^c}{\log \alpha}\left(-\frac{\zeta'(c)}{\zeta(c)}\right).$$

We then use the main theorem from Delange [5] to obtain

$$S_1 + S_5 < \frac{ex \log x}{\log \alpha}. \tag{4}$$

We now turn our attention to S_3, which is the sum of only two things. It follows, using the fact that $[x] = x - 1/2$ and the trivial bound $\Lambda(n) \leqslant \log n$, that

$$\begin{aligned} S_3 &= \left(\frac{x}{x-\frac{1}{2}}\right)^c \Lambda(x-1/2) \left|\log \frac{x}{x-\frac{1}{2}}\right|^{-1} + \left(\frac{x}{x+\frac{1}{2}}\right)^c \Lambda(x+1/2) \left|\log \frac{x}{x+\frac{1}{2}}\right|^{-1} \\ &< 2\left(\frac{x}{x-\frac{1}{2}}\right)^c \log(x+1/2)\Big(\log x - \log(x-1/2)\Big)^{-1} \end{aligned}$$

We can estimate trivially with

$$\log x - \log(x-1/2) = \int_{x-1/2}^{x} \frac{dt}{t} > \frac{1}{2(x-1/2)}$$

and the bound

$$\left(\frac{x}{x-\frac{1}{2}}\right)^c < 1.1$$

for $x > \exp(60)$ to get that

$$S_3 < 4.4 \log(x + 1/2).$$

This will actually be of little consequence to the final sum (as we will soon see), and so we feel no remorse in collecting here the weaker but tidier bound

$$S_3 < 5x \log x. \tag{5}$$

For S_2, we estimate $x/n < \alpha$ and $\Lambda(n) \leqslant \log n$ to get

$$S_2 < \alpha^c \log x \sum_{n=[x/\alpha]+1}^{[x]-1} \left| \log \frac{x}{n} \right|^{-1}.$$

If we let $n = [x] - v$, then the problem becomes that of summing over $v = 1, 2, \ldots, [x] - [x/\alpha] - 1$. We have

$$\left| \log \frac{x}{n} \right| = \log \frac{x}{n} > \log \frac{[x]}{n} = -\log\left(1 - \frac{v}{[x]}\right) > \frac{v}{[x]}$$

and thus

$$S_2 < \alpha^c x \log x \sum_{v=1}^{[x]-[x/\alpha]-1} \frac{1}{v}.$$

One can estimate this by the known bound $\sum_{n \leqslant x} 1/n \leqslant \log x + \gamma + 1/x$ where $\gamma \approx 0.5772\ldots$ is Euler's constant to get:

$$S_2 < \alpha^c x \log x \left(\log(x - x/\alpha) + \gamma + \frac{1}{x - x/\alpha} \right). \tag{6}$$

The sum S_4 is similar to this; we use $\Lambda(n) \leqslant \log(\alpha x)$ and $x/n < 1$ to get the bound

$$S_4 < \log(\alpha x) \sum_{n=[x]+2}^{[\alpha x]} \frac{1}{\log(n/x)}.$$

As $1 < \alpha < 2$, we have upon setting $n = [x] + 1 + v$ that

$$
\begin{aligned}
S_4 &< \log(\alpha x) \sum_{v=1}^{[x]} \frac{1}{\log\left(\frac{[x]+1+v}{x}\right)} \\
&< \log(\alpha x) \sum_{v=1}^{[x]} \frac{1}{\log\left(\frac{[x]+1+v}{[x]+1}\right)} \\
&< \log(\alpha x) \sum_{v=1}^{[x]} \frac{1}{\log\left(1 + \frac{v}{[x]+1}\right)}.
\end{aligned}
$$

Using the estimate $\log(1+x) > 2x/3$ for $0 < x < 1$ we have that

$$
\begin{aligned}
S_4 &< \frac{3}{2} \log(\alpha x)([x]+1) \sum_{v \leqslant x} \frac{1}{v} \\
&< \frac{3}{2} \log(\alpha x)([x]+1) \left(\log x + \gamma + \frac{1}{x} \right)
\end{aligned}
\tag{7}
$$

Finally, one may combine (4), (5), (6), (7) to get an inequality of the form

$$
\sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^{c} \left| \log \frac{x}{n} \right|^{-1} < f(\alpha, x). \tag{8}
$$

The result follows now from choosing $\alpha = 1.2$ and letting $x > e^{60}$.

■

The immediate result of Lemma 2.2 is that

$$
\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds + O^*\left(\frac{3.1 x \log^2 x}{\pi T} \right)
$$

for $x > e^{60}$, $c = 1 + 1/\log x$ and $T > 0$. We now look to shifting the line of integration so that we might involve the residues of the integrand. In doing so, we incur errors which only slightly increase the above error term. That is, the bulk of the error has already been obtained, and so we can be excused for not pursuing the best possible bounds in the remainder of this section. Let $U \geqslant 2$ be an even number and define the line segments

$$
\begin{aligned}
C_1 &= [c - iT, c + iT] & \qquad C_2 &= [c + iT, -U + iT] \\
C_3 &= [-U + iT, -U - iT] & \qquad C_4 &= [-U - iT, c - iT]
\end{aligned}
$$

and their union C along with the corresponding integrals

$$
I_i = \frac{1}{2\pi i} \int_{C_i} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.
$$

One can note that I_4 is the conjugate of I_3, and we will later use this fact to bound both at once. We also denote by I the integral around the rectangle C. Note that we need to account for the fact that while T is stipulated not to be the ordinate of a zero of $\zeta(s)$, it might be undesirably close to such. We show in Lemma 2.7 that there is always some good choice of T nearby, and so some work will be required later to shift our horizontal paths. Also note that any work we do in bounding I_2 will also hold for I_4 and so it follows that

$$|\psi(x) - I| < 2|I_2| + |I_3| + 3.1\frac{x\log^2 x}{\pi T}. \tag{9}$$

One can use Cauchy's theorem (see Davenport [4] for full details) to show that

$$I = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0<2m<U} \frac{x^{-2m}}{2m}$$

where $\rho = \beta + i\gamma$ denotes a zero of $\zeta(s)$. Noting that the rightmost summation is a partial sum of the series for $\log(1 - x^{-2})/2$, we can write that

$$\psi(x) = x - \sum_{|\gamma|<T} \frac{x^\rho}{\rho} + E(x, T, U)$$

where

$$|E(x,T,U)| < \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log(1 - x^{-2}) + 2|I_2| + |I_3| + 2.8\frac{x\log^2 x}{\pi T}. \tag{10}$$

It remains to bound $|I_2|$ and $|I_3|$ by deriving and making use of explicit estimates for $|\zeta'(s)/\zeta(s)|$ in appropriate regions.

We first establish a bound on the lengths of the rectangle C that intersect with the half-plane $\sigma \leqslant -1$. Our contour, or rather U, is chosen so that we might avoid the poles of $\tan \pi s/2$ which occur at the odd integers.

Lemma 2.3. *We have that*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| < 9 + \log|s|$$

on the intersection of C with $\sigma \leqslant -1$.

Proof. Consider the logarithmic derivative of the functional equation:

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\log 2\pi - \frac{1}{2}\pi\tan\frac{\pi s}{2} + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}.$$

Let $\sigma \geqslant 2$ (so that $1 - \sigma \leqslant -1$) and notice that $|\frac{1}{2}\pi\tan\frac{\pi s}{2}| < 2$ so long as s is distanced by at least 1 from odd integers on the real axis (this justifies our choice of U). We can then use

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} - \int_0^\infty \frac{[u] - u + 1/2}{(u+s)^2} du \tag{11}$$

to bound $|\Gamma'(s)/\Gamma(s)|$ trivially. The result then follows by observing that

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leqslant -\frac{\zeta'(2)}{\zeta(2)} < \frac{3}{5} \tag{12}$$

and putting it all together. ■

We now look to the harder task of establishing a bound over the region that includes the critical strip, as is essential for the estimation of I_2.

Lemma 2.4. *Let $s = \sigma + it$ where $\sigma > -1$ and $t > 50$. Then*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O^*(2\log t), \tag{13}$$

Proof. We start with the equation (see 12.8 of Davenport [4])

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log\pi + \frac{\Gamma'(\frac{s}{2}+1)}{2\Gamma(\frac{s}{2}+1)} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) \tag{14}$$

where $B = \gamma/2 - 1 + \frac{1}{2}\log 4\pi$. Successively, we set $s_0 = 2 + it$ and $s = \sigma + it$ and then find the difference between the two expressions. The terms involving the Γ-function are dealt with using (11), whereas the rest are estimated either trivially or with (12) to arrive at the result. ■

We can estimate the sum in Lemma 2.4 by breaking it into two smaller sums S_1 and S_2, where S_1 ranges over the zeroes $\rho = \beta + i\gamma$ with $|\gamma - t| \geqslant 1$ and S_2 is over the remaining zeroes.

Lemma 2.5. *Let $s = \sigma + it$, where $\sigma > -1$ and $t > 50$. Then*

$$S_1 = \sum_{|t-\gamma|\geqslant 1} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) = O^*(16\log t).$$

Proof. We can estimate the summand as follows (see Davenport [4, Ch. 15]):

$$\left| \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right| < \frac{3}{(t-\gamma)^2}. \tag{15}$$

We then have that

$$S_1 < \sum_{|t-\gamma|\geqslant 1} \frac{3}{(t-\gamma)^2} \leqslant \sum_\rho \frac{6}{1+(t-\gamma)^2}.$$

By letting $\sigma = 2$, taking real parts in (14) and estimating as in the proof of Lemma 2.4 we have

$$\sum_{\rho} \Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) < \frac{2}{3}\log t$$

for $t > 50$. We then use the two simple facts

$$\Re\left(\frac{1}{s-\rho}\right) = \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2} > \frac{1}{4+(t-\gamma)^2}$$

and

$$\Re\left(\frac{1}{\rho}\right) = \frac{\beta}{|\rho|^2} > 0$$

to get

$$\sum_{\rho} \frac{1}{4+(t-\gamma)^2} < \frac{2}{3}\log t.$$

Putting it all together we have

$$\begin{aligned} S_1 &< \sum_{\rho} \frac{6}{1+(t-\gamma)^2} \\ &< 24\sum_{\rho} \frac{1}{4+(t-\gamma)^2} \\ &< 16\log t. \end{aligned}$$

■

We now wish to estimate the remaining sum

$$S_2 = \sum_{|\gamma - t|<1} \left(\frac{1}{s-\rho} - \frac{1}{2+it-\rho}\right).$$

To do this, we first note that as $|2+it-\rho| > 1$, the contribution of the second term to the sum can be estimated trivially by

$$N(t+1) - N(t-1)$$

where $N(T)$ denotes the number of zeroes of $\zeta(s)$ in the critical strip up to height T. Now we prove the following result.

Lemma 2.6. *We have that*

$$N(t+1) - N(t-1) < \log t \tag{16}$$

for all $t > 50$.

Proof. We can use Corollary 1 of Trudgian [17] with $T_0 = 50$ to verify that the bound holds as long as $t > 250000$. To prove it for the remaining range, we can use Odlyzko's tables [13] of the zeroes of the Riemann zeta-function. A short algorithm written in Python reads in zeroes from the table and checks that the bound (16) holds in the remaining range. Specifically, the algorithm runs a check on the values of t from 50 to 250000 in increments of 0.01. To verify that the lemma is true for all values of t, we check the sharper inequality

$$N(t+1.01) - N(t-1) < \log t$$

at these discrete values and from this it follows that the result is true for all $t > 50$. ■

It follows from the above lemma that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|<1} \frac{1}{s-\rho} + O^*(19\log t).$$

Thus, finally, we are concerned with bounding the magnitude of

$$S_2' = \sum_{|\gamma-t|<1} \frac{1}{s-\rho}.$$

Of course, the problem here is that s might be close to a zero ρ, and this will give us trouble when we attempt to bound the line integrals. We search instead for a better value of t, say $t_0 \in (t-1, t+1)$, which will give a better bound. We will use this in the next section to shift our horizontal line of integration to a better height.

Lemma 2.7. *Let $t > 50$. There exists $t_0 \in (t-1, t+1)$ that does not depend on σ and such that*

$$\left| \sum_{|\gamma-t|<1} \frac{1}{(\sigma + it_0) - \rho} \right| < \log^2 t + \log t. \tag{17}$$

Proof. By (16), there are at most $\log t$ terms in S_2'. The imaginary ordinates of these zeroes partition the region of the strip into no more than $\log t + 1$ zero-free sections. Trivially, there will always be such a section of height

$$\frac{2}{\log t + 1}$$

and choosing the midpoint, say t_0, of this region will guarantee a distance of

$$\frac{1}{\log t + 1}$$

from any zero. As such, we have, letting $s = \sigma + it_0$, that

$$\sum_{|\gamma-t|<1} \frac{1}{|s-\rho|} \leqslant \sum_{|\gamma-t|<1} \frac{1}{|\gamma - t|} \leqslant \sum_{|\gamma-t|<1} (\log t + 1) \leqslant \log^2 t + \log t.$$ ■

Finally, we can put the previous three lemmas together to get the following.

Lemma 2.8. *Let $\sigma > -1, t > 50$. Then there exists $t_0 \in (t-1, t+1)$ such that for every $\sigma > -1$ we have*

$$\left| \frac{\zeta'(\sigma + it_0)}{\zeta(\sigma + it_0)} \right| < \log^2 t + 20 \log t.$$

That is, if our contour is somewhat close to a zero, we can shift it slightly to a line where we have good bounds.

2.1. Integral Estimates

We now bound the error term $E(x, T, U)$ in (10), by estimating each integral trivially. Using Lemma 2.3, we have

$$\begin{aligned} |I_3| &= \frac{1}{2\pi} \left| \int_{-U-iT}^{-U+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \right| \\ &< \int_{-T}^{T} \frac{9 + \log \sqrt{U^2 + T^2}}{2\pi x^U T} dt \\ &= \frac{18 + 2\log \sqrt{U^2 + T^2}}{2\pi x^U}. \end{aligned}$$

We save this, for soon we will combine our estimates and bound them in unison upon an appropriate choice for U. Consider now the problem of estimating I_2, and the issue that T might be close to the ordinate of a zero. From Lemma 2.7, there exists some $T_0 \in (T-1, T+1)$ that we should integrate over instead. We thus aim to shift the line of integration from C_2 to

$$C_2' = [-U + iT_0, c + iT_0].$$

It follows from Cauchy's theorem that

$$|I_2| < \sum_{T-1<\rho<T+1} \left| \frac{x^\rho}{\rho} \right| + |I_5| + |I_6| + |I_7| + |I_8|$$

where

$$I_5 = \frac{1}{2\pi i} \int_{-U+iT}^{-U+iT_0} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \qquad I_6 = \frac{1}{2\pi i} \int_{-U+iT_0}^{-1+iT_0} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds$$

$$I_7 = \frac{1}{2\pi i} \int_{-1+iT_0}^{c+iT_0} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \qquad I_8 = \frac{1}{2\pi i} \int_{c+iT}^{c+iT_0} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

From (16), we can estimate the sum by

$$\sum_{T-1<\Im\rho<T+1} \left|\frac{x^\rho}{\rho}\right| < \sum_{T-1<\Im\rho<T+1} \frac{x}{T-1} < \frac{2x\log T}{T-1}.$$

We can bound I_5 in the same way as I_3 to obtain

$$|I_5| < \frac{18 + 2\log\sqrt{U^2+(T+1)^2}}{2\pi x^U T}.$$

Bounding I_6 is done using Lemma 2.3:

$$|I_6| < \frac{9 + \log\sqrt{U^2+(T+1)^2}}{2\pi x(T-1)}$$

We also use Lemma 2.7 to get

$$|I_7| < \frac{e}{2\pi(T-1)}(\log^2(T+1) + \log(T+1)).$$

To get an upper bound for I_8, we notice that $\Re s = 1 + 1/\log x$ and so following the line of working which led to (4) gives

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| < \log x.$$

This is by the working involved in (4). Following through we get the bound

$$|I_8| < \frac{ex\log x}{\pi(T-1)}.$$

Now, throwing all of our estimates for the terms in (10) together, implanting the information that $T \leqslant x$, $x > e^{60}$ and letting U be equal to the even integer closest to x we obtain Theorem 1.3. It now remains to apply this result to the problem of primes between powers.

3. Proof of Theorems 1.1 and 1.2

3.1. An indicator function for intervals

We define the Chebyshev θ-function as

$$\theta(x) = \sum_{p\leqslant x} \log p$$

and consider that $\theta(x+h) - \theta(x)$ is positive if and only if there is at least one prime in the interval $(x, x+h]$. Clearly, one removes the contribution of prime powers to $\psi(x)$ to get $\theta(x)$. We let $h > 0$ and substitute $x+h$ and then x into Theorem 1.3. Taking the difference then gives us that

$$\psi(x+h) - \psi(x) > h - \left| \sum_{|\gamma|<T} \frac{(x+h)^\rho - x^\rho}{\rho} \right| - \frac{4(x+h)\log^2(x+h)}{T}. \tag{18}$$

Whilst the above will tell us information about prime powers, we are actually interested in primes. We thus require the following lemma; a combination of Proposition 3.1 and 3.2 of Dusart [6].

Lemma 3.1. *Let* $x \geqslant 121$. *Then*

$$0.9999x^{1/2} < \psi(x) - \theta(x) < 1.00007x^{1/2} + 1.78x^{1/3}.$$

It follows from (9) and the above lemma that

$$\begin{aligned}\theta(x+h) - \theta(x) > h - \left| \sum_{|\gamma|<T} \frac{(x+h)^\rho - x^\rho}{\rho} \right| - \frac{4(x+h)\log^2(x+h)}{T} \\ - 1.00007(x+h)^{1/2} - 1.78(x+h)^{1/3} + 0.9999x^{1/2}.\end{aligned} \tag{19}$$

Given that we are interested in the case where $h = 3x^{2/3}$, it remains to choose $T = T(x)$ and find x_0 such that the above is positive for all $x > x_0$.

3.2. Estimating the sum over the zeroes

In consideration of (19), we let

$$S = \left| \sum_{|\gamma|<T} \frac{(x+h)^\rho - x^\rho}{\rho} \right|. \tag{20}$$

We then have that

$$S = \left| \sum_{|\gamma|<T} \int_x^{x+h} t^{\rho-1} \right| \leqslant \sum_{|\gamma|<T} \int_x^{x+h} t^{\beta-1} \leqslant h \sum_{|\gamma|<T} x^{\beta-1}. \tag{21}$$

From the identity

$$\begin{aligned}\sum_{|\gamma|<T} (x^{\beta-1} - x^{-1}) &= \sum_{|\gamma|<T} \int_0^\beta x^{\sigma-1} \log x \, d\sigma \\ &= \int_0^1 \sum_{\beta>\sigma, |\gamma|<T} x^{\sigma-1} \log x \, d\sigma,\end{aligned}$$

it follows that

$$\sum_{|\gamma|<T} x^{\beta-1} = 2x^{-1}N(T) + 2\int_0^1 N(\sigma, T) x^{\sigma-1} \log x \, d\sigma, \tag{22}$$

where $N(\sigma, T)$ denotes the number of zeroes ρ of the Riemann zeta-function with $0 < \gamma < T$ and $\beta > \sigma$.

We can estimate the above sum, and thus S, with the assistance of some explicit bounds. Firstly note, that by Corollary 1 of Trudgian [16] we have that

$$N(T) < \frac{T\log T}{2\pi}$$

for all $T > 15$, say. Explicit estimates for $N(\sigma, T)$ are rare, though have come to light recently through the likes of Kadiri [10] and Ramaré [14], who have produced zero-density estimates of rather different shape to each other. Ramaré's estimate, which is an explicit and asymptotically better version of Ingham's [9] original density estimate, is required for the problem of primes between cubes. We give the result here, which is a corollary of Theorem 1.1 of [14].

Lemma 3.2. *Let $T \geqslant 2000$ and $\sigma \geqslant 0.52$. Then*

$$N(\sigma, T) \leqslant 9.7(3T)^{8(1-\sigma)/3}\log^{5-2\sigma} T + 103\log^2 T.$$

The following zero-free region, given by Ford, will also be required.

Lemma 3.3. *Let $T \geqslant 3$. Then there are no zeroes of $\zeta(s)$ in the region given by $\sigma \geqslant 1-\nu(T)$ where*

$$\nu(T) = \frac{1}{57.54\log^{2/3} T(\log\log T)^{1/3}}.$$

It is useful to carry out the bulk of the calculations with A in place of the constant 9.7 in Lemma 3.2 and c in place of the 57.54 in Lemma 3.3. Doing so allows us later on to see the importance of improvements of these constants, and thus gives direction to future efforts on this problem.

We split the integral in (22) into two parts, one over the interval $0 \leqslant \sigma \leqslant 5/8$, where $N(\sigma, T)$ may as well be bounded by $N(T)$, and another over $5/8 \leqslant \sigma \leqslant 1-\nu(T)$. By inserting the relevant estimates, we get

$$\begin{aligned}\sum_{|\gamma|<T} x^{\beta-1} &< 2x^{-1}N(T) + 2x^{-1}N(T)\log x\int_0^{5/8} x^\sigma dx \\ &\quad + 2Ax^{-1}(3T)^{8/3}\log x\log^5 T\int_{5/8}^{1-\nu(T)}\left(\frac{x}{(3T)^{8/3}\log^2 T}\right)^\sigma d\sigma \\ &\quad + 103x^{-1}\log x\log^2 T\int_{5/8}^{1-\nu(T)} x^\sigma d\sigma. \qquad (23)\end{aligned}$$

The working out is routine, yet tedious. We give the details to the extent that the reader can follow the process. We introduce the parameter $k \in (\frac{2}{3}, 1)$, which will play a part in the relationship between T and x. The reasons for the range of values of k will become clear soon. Now let $T = T(x)$ be the solution to the equation

$$\frac{x}{(3T)^{8/3}\log^2 T} = \exp(\log^k x).$$

Upon performing the integration in (23), we directly substitute in the above relationship, along with the bound for $N(T)$ and the fact that $\log T < (3/8)\log x$, to get

$$\sum_{|\gamma|<T} x^{\beta-1} < \frac{e^{-\frac{3}{8}\log^k x}\log^{1/4} x}{3^{3/4}8^{1/4}\pi} + \frac{27A}{256}\log^{4-k} x(e^{-\nu(T)\log^k x} - e^{-(3/8)\log^k x}) + \frac{927A}{32}\log^2 x(e^{-\nu(T)\log x} - x^{-3/8}). \tag{24}$$

There is some cancellation in the above. First, we need to estimate one of the exponential terms involving $\nu(T)$. We have that

$$e^{-v(T)\log x} = \exp\Bigg(-\frac{\log x}{c\log^{2/3} T(\log\log T)^{1/3}}\Bigg) < \exp\Bigg(-\frac{4}{3^{2/3}c}\Big(\frac{\log x}{\log\log x}\Big)^{1/3}\Bigg).$$

Now, upon expansion of (24) and using the above we can notice that

$$-\frac{27A}{256}(\log x)^{4-k}e^{-(3/8)\log^k x} + \frac{927A}{32}\log^2 x(e^{-\nu(T)\log x} - x^{-3/8}) < 0.$$

This is clear if one looks at the dominant terms. It follows that

$$\sum_{|\gamma|<T} x^{\beta-1} < \frac{e^{-\frac{3}{8}\log^k x}\log^{1/4} x}{3^{3/4}8^{1/4}\pi} + \frac{27A}{256}(\log x)^{4-k}e^{-\nu(T)\log^k x}. \tag{25}$$

The remaining exponential term involving $\nu(T)$ is dealt with as before; this is then coupled with (20) and (21) to get

$$S < \frac{he^{-\frac{3}{8}\log^k x}\log^{1/4} x}{3^{3/4}8^{1/4}\pi} + \frac{27Ah}{256}(\log x)^{4-k}\exp\Bigg(-\frac{4}{3^{2/3}c}\frac{\log^{k-2/3} x}{(\log\log x)^{1/3}}\Bigg).$$

3.3. Estimates for inequalities

It is now clear that we may write (19) as

$$\theta(x+h) - \theta(x) > h - f(x,h,k,A,c) - g(x,h,k) - E(x,h,k)$$

where

$$f(x,h,k,A,c) = \frac{27Ah}{256}(\log x)^{4-k}\exp\Bigg(-\frac{4}{3^{2/3}c}\frac{\log^{k-2/3} x}{(\log\log x)^{1/3}}\Bigg),$$

$$g(x,h,k) = 12\Big(\frac{3}{8}\Big)^{3/4}\frac{(x+h)\log^{11/4}(x+h)}{x^{3/8}}\exp(\frac{3}{8}\log^k x),$$

$$E(x,h,k) = -\frac{h(\log x)^{1/4}\exp(-\frac{3}{8}\log^k x)}{6^{3/4}\pi} - 1.00007(x+h)^{1/2} - 1.78(x+h)^{1/3} + 0.9999x^{1/2}.$$

First, we look to bound the error. Noting that $x > e^{60}$, we set $h = 3x^{2/3}$ and use the fact that $k = 2/3$ will give us the worst possible error to get

$$\frac{E(x, 3x^{2/3}, 2/3)}{3x^{2/3}} < 10^{-3}.$$

Thus, one can show that positivity holds if the following two inequalities are simultaneously satisfied:

1. $f(x, h, k, A, c) < \frac{1}{2}(1 - 10^{-3})h$,
2. $g(x, h, k) < \frac{1}{2}(1 - 10^{-3})h$.

This splitting simplifies our working greatly whilst perturbing the solution negligibly. To be convinced of this, one could consider the right hand side of each of the above inequalities as being equal to h, in some better-than-possible scenario. It turns out that the improvements would hardly be noticeable. We will, however, mention at the end of this paper some direction for future attempts at improving the work on primes between cubes.

Now, in the first inequality, we take the logarithm of both sides and set $x = e^y$ to get

$$\log\left(\frac{27A}{256}\right) + (4 - k)\log y - \frac{4}{3^{2/3}c}\frac{y^{k-2/3}}{\log^{1/3} y} < \log(\frac{1}{2}(1 - 10^{-3})) \tag{26}$$

This is easy to solve for y given knowledge of A, k and c. There are some notes to make here first. We can see that A, the constant in front of Ramaré's zero-density estimate has little contribution, for being in the argument of the logarithm. However, c plays a much larger part from where it is positioned. We can also see the reason for $k > 2/3$, in that it guarantees a solution.

We deal with the second inequality in the same way, but first we notice that

$$\frac{g(x, h, k)}{h} < \frac{2\log^{11/4} x}{x^{1/24}} \exp(\frac{3}{8}\log^k x).$$

This is obtained using the main result of Ramaré and Saouter [15] to bound

$$x + h < \frac{x}{1 - \Delta^{-1}}$$

where $\Delta = 28314000$ as given in their paper. Thus, using the same approach as before we get

$$\frac{11}{4}\log y + \frac{3}{8}y^k - \frac{1}{24}y < \log(\frac{1}{4}(1 - 10^{-3})). \tag{27}$$

We notice here our reason for having $k < 1$. One can also see the reason for leaving k free to vary in $(2/3, 1)$. There should be an optimal value of k, where the solution range of the above two inequalities are equal and their intersection is minimised.

Here, we set $A = 9.7$, $c = 57.54$ and use the `Manipulate` function of MATHEMATICA to "hunt" for a good value of k. It turns out that upon choosing $k = 0.9359$, we have that both inequalities are satisfied (we check this using Python) for $y > 8 \times 10^{14}$, or $x^{1/3} > \exp(\exp(33.217))$, which proves our main result.

3.4. Notes for future improvements

Using the explicit methods of this paper, better estimates for zero-densities, zero-free regions and the error term of Landau's explicit formula could effectively be implemented to furnish a new estimate. The following preemptive discussion might be useful for one looking to do such a thing.

Let's consider first improving the zero-density estimate given by Ramaré. Say, for the sake of discussion, one could obtain a value of $A = 10^{-4}$. Then we would obtain our result instead with $n \geqslant \exp(\exp(32.7))$, an improvement which would probably not be worth the efforts required to obtain such a value of A.

Ramaré has communicated that one could use the Brun-Titchmarsh theorem to remove a power from the logarithm in the error term of (1.3). This does not seem to improve the overall result; a shortcoming, perhaps, of the numerical methods used by the author.

There are other parameters where one might wish to direct future efforts. In Ramaré's zero density estimate, one might consider the power $5 - 2\sigma$ of the logarithm to be $L - 2\sigma$. The main difference in our working would be $(L - 1 - k)$ in place of $(4 - k)$ in the reduced form of our second inequality. The following table summarises the improvements which would follow; a prime between n^3 and $(n+1)^3$ for all $n \geqslant n_0$.

L	$\log\log n_0$
5	33.217
4	31.8
3	29.8
2	22.19

Turning now to the error term of Theorem 1.3 one could also consider a smaller constant in place of 2. This constant, however, would appear in the logarithm of the right hand side of (27), and thus make little difference.

Wolke has derived the explicit formula with an error term which is

$$O\left(\frac{x \log x}{T \log(x/T)}\right) = O\left(\frac{x}{T}\right)$$

for the choice of $T(x)$ used in this paper. One may propose all sorts of "good" explicit constants for the above error term and try them via the methods of this paper, but there will be no major improvements.

Changes in the constant c are more effective, though seemingly much more difficult to obtain. A value of $c = 40$ would yield only $n \geqslant \exp(\exp(31.88))$, and $c = 20$ would give $n \geqslant \exp(\exp(29.6))$. The removal of the $(\log\log T)^{1/3}$ would give a similar result.

Thus one expects a major result, or perhaps many minor ones, to make significant progress on this problem.

3.5. Higher powers

In lieu of a complete result on the problem of primes between cubes, we consider instead primes between mth powers, where m is some positive integer. Appropriately, we choose $h = mx^{1-1/m}$, and we are able to prove the following result.

Theorem 3.4. *Let $m \geqslant 4.971 \times 10^9$. Then there is a prime between n^m and $(n+1)^m$ for all $n \geqslant 1$.*

The result seems absurd on a first glance as the value of m is quite large. We shall leave it to others to attempt to bring the value down.

We now prove the above theorem as follows; for our choice of h, it follows that inequality (27) becomes

$$\frac{11}{4}\log y - \left(\frac{3}{8} - \frac{1}{m}\right) y + \frac{3}{8} y^k < \log\left(\frac{m}{12}(1 - 10^{-3})\right) \tag{28}$$

whereas inequality (26) remains the same. As before, we can, for some given m, choose k and find n_0 such that there is a prime between n^m and $(n+1)^m$ for all $n \geqslant n_0$ by solving both inequalities. Some results are given in the following table.

m	k	$\log\log n_0$
4	0.9635	29.240
5	0.9741	27.820
6	0.9796	27.230
7	0.983	26.427
1000	0.9998	19.807

One can see that this method has its limitations, even in the case of higher powers. Nonetheless, we have that there is a prime in $(n^{1000}, (n+1)^{1000})$ for all $n \geqslant \exp(\exp(19.807))$. It follows that, for $m \geqslant 1000$, there is a prime between n^m and $(n+1)^m$ for all

$$n \geqslant \exp\left(\frac{1000\exp(19.807)}{m}\right). \tag{29}$$

We could choose $m = 1000\exp(19.807) \approx 4 \times 10^{11}$ to get primes between n^m and $(n+1)^m$ for all $n \geqslant e$. Bertrand's postulate improves this to all $n \geqslant 1$.

However, we can use Corollary 2 of Trudgian [17] to improve on this value of m. This states that for all $x \geqslant 2898239$ there exists a prime in the interval

$$\left[x, x\left(1 + \frac{1}{111\log^2 x}\right)\right].$$

If we set $x = n^m$, we might ask when the above interval falls into $[n^m, n^m + mn^{m-1}]$. One can rearrange the inequality

$$n^m\left(1 + \frac{1}{111\log^2(n^m)}\right) < n^m + mn^{m-1}$$

to get

$$\frac{n}{\log^2 n} < 111m^3. \tag{30}$$

We wish to choose the lowest value of m for which the solution sets of (29) and (30) first coincide. It is not to hard to see that this equates to solving simultaneously the equations

$$n = \exp\left(\frac{1000\exp(19.807)}{m}\right)$$

and

$$\frac{n}{\log^2 n} = 111m^3.$$

We do this by substituting the first equation directly into the second to get

$$\exp\left(\frac{1000\exp(19.807)}{m}\right) = 111(1000\exp(19.807))^2 m$$

which can easily be solved with MATHEMATICA to prove Theorem 3.4.

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Received: 6 October 2015; **revised:** 11 May 2016