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POLYNOMIALS WITH POLAR DERIVATIVES

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Abstract: In this paper, we present an integral inequality for the polar derivative of polynomials. Our theorem includes as special cases several interesting generalizations of some Zygmund type inequalities for polynomials.

Keywords: polar derivative, Zygmund inequality.

1. Introduction

Let P_n be the class of polynomials of degree n and we define $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$, the polar derivative of the polynomial $P(z)$ with respect to the point α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Various results have been obtained for polar derivatives of polynomials. Aziz and Rather [1] proved that, if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \geqslant 1$ and $r \geqslant 1$,

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leqslant n(|\alpha|+1) C_r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{1.1}$$

where

$$C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{-1}{r}}. \tag{1.2}$$

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2. Main results

In this paper, we generalize the above result for $r > 0$. The above inequality will be a consequence from the more fundamental inequality presented by the following theorem:

Theorem. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex numbers α, β with $|\alpha| \geqslant 1, |\beta| \leqslant 1$ and $r > 0$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \left(\frac{|\alpha| - 1}{2} \right) P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leqslant n \Big\{ (|\alpha| + 1) + |\beta|(|\alpha| - 1) \Big\} C_r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \quad (2.1)$$

where C_r is defined by (1.2).

Remark. If we put $\beta = 0$, in the above result, we get (1.1) extended to the case $r \in (0, \infty)$. Dividing the two sides of (2.1) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result.

Corollary. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number β with $|\beta| \leqslant 1$ and $r > 0$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} P'(e^{i\theta}) + \frac{n\beta}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leqslant n\big(|\beta| + 1\big) C_r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \quad (2.2)$$

where C_r is defined by (1.2).

The above inequality generalizes some inequalities obtained by De-Bruijn [3] as well as Rahman and Schmeisser [5].

3. Lemmas

For the proof of the Theorem, we need the following lemmas:

Lemma 1. *If $Q \in P_n$ be a polynomial such that $Q(z) \neq 0$ for $|z| > 1$ and $P \in P_n$. If $|P(z)| \leqslant |Q(z)|$ for $|z| = 1$, then for all complex numbers α, β with $|\alpha| \geqslant 1, |\beta| \leqslant 1$,*

$$\left| zD_\alpha P(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) P(z) \right| \leqslant \left| zD_\alpha Q(z) + n\beta \left(\frac{|\alpha| - 1}{2} \right) Q(z) \right|, \quad \text{for } |z| \geqslant 1.$$

The above lemma is due to Liman, Mohapatra and Shah [4].

Lemma 2. *If $P \in P_n$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for every $r > 0$ and γ real,*

$$\int_0^{2\pi} \int_0^{2\pi} \left| P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta}) \right|^r d\theta d\gamma \leqslant 2\pi n^r \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta.$$

The above lemma is due to Aziz and Rather [2].

4. Proof of the Theorem

Since $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, the polynomial $Q(z) = z^n\overline{P(1/\bar{z})} \in P_n$ and $Q(z) \neq 0$ in $|z| > 1$. By Lemma 1, we have for complex numbers α, β with $|\alpha| \geqslant 1, |\beta| \leqslant 1$,

$$\left|zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \leqslant \left|zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|, \quad \text{for } |z| = 1. \tag{4.1}$$

Now for every real γ and $t \geqslant 1$, it can be easily verified that $|t + e^{i\gamma}| \geqslant |1 + e^{i\gamma}|$. Observe that for any $r > 0$ and $a,\ b \in \mathbb{C}$ such that $|b| \geqslant |a|$ we have

$$\int_0^{2\pi} \left|a + e^{i\gamma}b\right|^r d\gamma \geqslant |a|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma. \tag{4.2}$$

Indeed, if $a = 0$ the above inequality (4.2) is obvious. In the case of $a \neq 0$ we get

$$\begin{aligned}\int_0^{2\pi} \left|1 + e^{i\gamma}\frac{b}{a}\right|^r d\gamma &= \int_0^{2\pi} \left|1 + e^{i\gamma}\left|\frac{b}{a}\right|\right|^r d\gamma = \int_0^{2\pi} \left|\left|\frac{b}{a}\right| + e^{i\gamma}\right|^r d\gamma \\ &\geqslant \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma.\end{aligned}$$

Now, we can take

$$a = e^{i\theta}D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta}),\ b = e^{i\theta}D_\alpha Q(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta}),$$

because $|b| \geqslant |a|$ from (4.1), we get from (4.2) that

$$\begin{aligned}\int_0^{2\pi} &\left|\left\{e^{i\theta}D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right\}\right. \\ &\qquad \left. + e^{i\gamma}\left\{e^{i\theta}D_\alpha Q(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right\}\right|^r d\gamma \\ &\geqslant \left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma.\end{aligned} \tag{4.3}$$

Integrating both sides of (4.3) with respect to θ from 0 to 2π, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} &\left|\left\{e^{i\theta}D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right\}\right. \\ &\qquad \left. + e^{i\gamma}\left\{e^{i\theta}D_\alpha Q(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right\}\right|^r d\theta d\gamma \\ &\geqslant \int_0^{2\pi} \left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|^r d\theta \int_0^{2\pi} |1 + e^{i\gamma}|^r d\gamma.\end{aligned} \tag{4.4}$$

As $Q(z) = z^n\overline{P(1/\bar{z})}$, we have $P(z) = z^n\overline{Q(1/\bar{z})}$. It can be easily verified that for $0 \leqslant \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})},$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}.$$

Hence

$$\begin{aligned} nP(e^{i\theta}) + e^{i\gamma}nQ(e^{i\theta}) &= e^{i\theta}P'(e^{i\theta}) + e^{i(n-1)\theta}\overline{Q'(e^{i\theta})} \\ &\quad + e^{i\gamma}\Big(e^{i\theta}Q'(e^{i\theta}) + e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\Big) \\ &= e^{i\theta}\Big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big) \\ &\quad + e^{i(n-1)\theta}\Big(\overline{Q'(e^{i\theta})} + e^{i\gamma}\overline{P'(e^{i\theta})}\Big), \end{aligned}$$

which gives

$$\begin{aligned} n\Big|P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\Big| &\leqslant \Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big| + \Big|\overline{Q'(e^{i\theta})} + e^{i\gamma}\overline{P'(e^{i\theta})}\Big|, \\ &= 2\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big|. \end{aligned} \tag{4.5}$$

Also, we have

$$\begin{aligned} &\Big|D_\alpha P(e^{i\theta}) + e^{i\gamma}D_\alpha Q(e^{i\theta})\Big| \\ &\quad = \Big|nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + e^{i\gamma}\Big(nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\Big)\Big| \\ &\quad = \Big|\Big(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\Big) + e^{i\gamma}\Big(nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\Big) \\ &\qquad + \alpha\Big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big)\Big| \\ &\quad = \Big|\Big(\overline{Q'(e^{i\theta})} + e^{i\gamma}\overline{P'(e^{i\theta})}\Big)e^{i(n-1)\theta} + \alpha\Big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big)\Big| \\ &\quad \leqslant \overline{\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big|} + |\alpha|\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big| \\ &\quad = (|\alpha| + 1)\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big|. \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we have for $0 \leqslant \theta < 2\pi$, $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, and γ real,

$$\begin{aligned}
&\left|\left\{e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right\}\right. \\
&\qquad\left. + e^{i\gamma}\left\{e^{i\theta} D_\alpha Q(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right\}\right| \\
&\qquad \leqslant \left|D_\alpha P(e^{i\theta}) + e^{i\gamma} D_\alpha Q(e^{i\theta})\right| + n|\beta|\left(\frac{|\alpha|-1}{2}\right)\left|P(e^{i\theta}) + e^{i\gamma} Q(e^{i\theta})\right| \\
&\qquad \leqslant (|\alpha|+1)\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right| + |\beta|\left(\frac{|\alpha|-1}{2}\right)2\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right| \\
&\qquad = \left\{(|\alpha|+1) + |\beta|(|\alpha|-1)\right\}\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|.
\end{aligned}$$

This gives with the help of Lemma 2 for each $r > 0$ and γ real,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left|\left\{e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right\}\right. \\
&\qquad\left. + e^{i\gamma}\left\{e^{i\theta} D_\alpha Q(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right\}\right|^r d\theta d\gamma \\
&\qquad \leqslant \left\{(|\alpha|+1) + |\beta|(|\alpha|-1)\right\}^r \int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|^r d\theta d\gamma. \\
&\qquad \leqslant 2\pi n^r\left\{(|\alpha|+1) + |\beta|(|\alpha|-1)\right\}^r \int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta.
\end{aligned}$$

From (4.4) we have for every α, β with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, and $r > 0$,

$$\begin{aligned}
&\int_0^{2\pi}\left|e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|^r d\theta \int_0^{2\pi} |1 + e^{i\gamma}|^r d\gamma \\
&\qquad\qquad \leqslant 2\pi n^r\left\{(|\alpha|+1) + |\beta|(|\alpha|-1)\right\}^r \int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta,
\end{aligned}$$

which is equivalent to (2.1).

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