NONVANISHING AND CENTRAL CRITICAL VALUES OF TWISTED *L*-FUNCTIONS OF CUSP FORMS ON AVERAGE

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Abstract: Let f be a cusp form of integral weight $k \geqslant 4$ for $\Gamma_0(N)$ with nebentypus ψ. Generalising work of Kohnen we construct a kernel function for the *L*-function $L(f, \chi, s)$ of f twisted by a primitive Dirichlet character χ and use it to show that the average $\sum_{f \in S_k(N,\psi)} \frac{L(f,\chi,s)}{\langle f,f \rangle} \overline{a_f(1)}$ over an orthogonal basis of $S_k(N, \psi)$ does not vanish on certain rectangles inside the critical strip if the weight k or the level N is big enough. As another application of the kernel function we prove an averaged version of Waldspurger's Theorem.

Keywords: twisted *L*-functions of cusp forms, nonvanishing, Poincaré series, Waldspurger's theorem.

1. Introduction and results

Let f be a cusp form of even weight $k \geqslant 4$ for $\mathrm{SL}_2(\mathbb{Z})$ and let $L^*(f, s)$ denote the completed Hecke *L*-function of f. It has an analytic continuation to $\mathbb{C}$ and satisfies the functional equation $L^*(f, k-s) = (-1)^{k/2} L^*(f, s)$. The generalised Riemann hypothesis for $L^*(f, s)$ states that its only zeros inside the critical strip $(k-1)/2 < \Re(s) < (k+1)/2$ lie on the line $\Re(s) = k/2$. Kohnen [Koh97] constructed a kernel function $R_k(\tau, s)$ for the completed *L*-function $L^*(f, s)$ and used it to show that in a certain sense the generalised Riemann hypothesis for $L^*(f, s)$ holds on average for large weights. More precisely, he proved that for every $t_0 \in \mathbb{R}$ and $\varepsilon > 0$ there exists a constant $C(t_0, \varepsilon) > 0$ such that for every even integer $k > C(t_0, \varepsilon)$ the function

$$\sum_{f \in S_k(1)} \frac{L^*(f, s)}{\langle f, f \rangle}$$

(where the sum runs over an orthogonal basis of normalised Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$) does not vanish at points $s = \sigma + it_0$ with $(k-1)/2 < \sigma < k/2 - \varepsilon$ or $k/2 + \varepsilon < \sigma < (k+1)/2$. The result was generalised by Raghuram [Rag05] to arbitrary level N and primitive nebentypus ψ, using essentially the same arguments.

Following the ideas of Kohnen [Koh97], we consider the kernel function

$$R_{k,N,\psi}(\tau, s, \chi) = \sum_{m=1}^{\infty} \overline{\chi(m)} m^{s-1} P_{k,N,\psi,m}(\tau)$$

(with $P_{k,N,\psi,m} \in S_k(N, \psi)$ being the usual m-th Poincaré series) for the twisted L-function

$$L(f, \chi, s) = \sum_{m=1}^{\infty} \chi(m) a_f(m) m^{-s}$$

of cusp forms $f \in S_k(N, \psi)$ of weight $k \geqslant 4$, level N and nebentypus ψ, where χ is a primitive Dirichlet character whose conductor is coprime to the level N and $a_f(m)$ denotes the m-th Fourier coefficient of f. We compute the Fourier expansion of $R_{k,N,\psi}(\tau, s, \chi)$ and use it to show an analogue of Kohnen's Theorem for $L(f, \chi, s)$. As Raghuram, we generalise Kohnen's result in the weight and the level aspect:

Theorem 1.1. *Let χ be a primitive Dirichlet character mod h.*

(1) *For every positive integer N with $(N, h) = 1$, every positive integer m with $(m, h) = 1$, every $T > 0$ and $\varepsilon > 0$ there exists a constant $C(T, \varepsilon, N, m, h) > 0$ such that for every integer $k > C(T, \varepsilon, N, m, h)$ and every Dirichlet character ψ mod N with $\psi(-1) = (-1)^k$ the function*

$$\sum_{f \in S_k(N,\psi)} \frac{L(f, \chi, s)}{\langle f, f \rangle} \overline{a_f(m)}$$

does not vanish at points $s = \sigma + it$ with $-T < t < T$ and $(k-1)/2 < \sigma < k/2 - \varepsilon$ or $k/2 + \varepsilon < \sigma < (k+1)/2$.

(2) *For every integer $k \geqslant 4$, every positive integer m with $(m, h) = 1$, every $T > 0$ and $\varepsilon > 0$ there exists a constant $C(T, \varepsilon, k, m, h) > 0$ such that for every integer $N > C(T, \varepsilon, k, m, h)$ with $(N, h) = 1$ and every Dirichlet character ψ mod N with $\psi(-1) = (-1)^k$ the function*

$$\sum_{f \in S_k(N,\psi)} \frac{L(f, \chi, s)}{\langle f, f \rangle} \overline{a_f(m)}$$

does not vanish at points $s = \sigma + it$ with $-T < t < T$ and $(k-1)/2 < \sigma < k/2 - \varepsilon$ or $k/2 + \varepsilon < \sigma < (k+1)/2$.

Here the sums run over an orthogonal basis of $S_k(N, \psi)$, not necessarily consisting of normalised Hecke eigenforms.

Note that $L(f, \chi, s)$ and its completion $L^*(f, \chi, s)$ differ by a factor which is independent of f and does not vanish in the critical strip, so we could as well replace $L(f, \chi, s)$ by $L^*(f, \chi, s)$ in the theorem. Further, the proof of the theorem shows that the constants $C(T, \varepsilon, k, m, h)$ and $C(T, \varepsilon, N, m, h)$ are of polynomial growth as functions of m and h, and are decaying to 0 as functions of the fixed parameter k and N, respectively.

As a second application of the kernel function $R_{k,N,\psi}(\tau, s, \chi)$ we prove an averaged version of Waldspurger's Theorem relating the central critical value of the twisted L-function of a cusp form of even weight $2k$ to the square of a Fourier coefficient of a Jacobi cusp form of weight $k+1$. Let us explain our result in some more detail:

It was shown in [SZ88] that the space $J_{k+1,N}^{\mathrm{cusp,new}}$ of cuspidal Jacobi newforms of weight $k+1$ and index N is Hecke-equivariantly isomorphic to the space $S_{2k}^{\mathrm{new},-}(N)$ of elliptic newforms of weight $2k$ and level N whose L-function has a minus sign in its functional equation. This isomorphism is sometimes also referred to as the Shimura correspondence. Let $\phi \in J_{k+1,N}^{\mathrm{cusp,new}}$ be a cuspidal Jacobi newform of weight $k+1$ and index N and let $f \in S_{2k}^{\mathrm{new},-}(N)$ be the normalised newform of weight $2k$ and level N associated to ϕ under the Shimura correspondence. If $D < 0$ with $(D, N) = 1$ is a fundamental discriminant which is a square mod $4N$, say $D \equiv r^2(4N)$ for some $r \in \mathbb{Z}$, the formula of Waldspurger [Wal81] in the explicit form given by Gross, Kohnen and Zagier [GKZ87] states that

$$\frac{|c_\phi(D, r)|^2}{\langle \phi, \phi \rangle} = \frac{(k-1)!}{2^{2k-1}\pi^k N^{k-1}} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}, \tag{1.1}$$

where $L(f, D, s)$ denotes the L-function of f twisted by the Kronecker symbol $\left(\frac{D}{\cdot}\right)$ and $c_\phi(D, r)$ is the (D, r)-th Fourier coefficient of ϕ. We prove an averaged version of Waldspurger's Theorem, where by 'averaged' we mean the formula obtained by summing both sides of (1.1) over orthogonal bases of the corresponding spaces of cusp forms.

Theorem 1.2. *Let $k \geqslant 2$ and N be positive integers. Let $D < 0$ be a fundamental discriminant with $(D, N) = 1$ and $D \equiv r^2(4N)$ for some $r \in \mathbb{Z}$, and let $m > 0$ be a positive integer with $(m, N) = 1$. Then it holds*

$$\sum_{\phi \in J_{k+1,N}^{\mathrm{cusp}}} \frac{c_\phi(D, r)\overline{c_{\phi|T_m}(D, r)}}{\langle \phi, \phi \rangle}$$

$$= \frac{(k-1)!}{2^{2k-1}\pi^k N^{k-1}} |D|^{k-1/2} \sum_{f \in S_{2k}(N)} \frac{L(f, D, k)}{\langle f, f \rangle} \overline{a_f(m)}$$

where the sums run over orthogonal bases of the spaces of Jacobi cusp forms of weight $k+1$ and index N and elliptic cusp forms of weight $2k$ and level N, and T_m denotes the m-th Hecke operator on $J_{k+1,N}^{\mathrm{cusp}}$.

Iwaniec [Iwa87] gave another averaged version of Waldspurger's Theorem which is very different from our formula, e.g. it involves averaging over forms of different levels.

We remark that the formula does not mention the Shimura correspondence, that is, the forms ϕ and f need not be related to each other. Further, the forms ϕ and f in the sums are not required to be newforms or Hecke eigenforms. We see that the left-hand side does not depend on the choice of r with $D \equiv r^2(4N)$, although the coefficient $c_\phi(D, r)$ of ϕ is in general not independent of r.

For the proof of the theorem we look at the Fourier expansion of the kernel function $R_{2k,N,1}\left(\tau, s, \left(\frac{D}{\cdot}\right)\right)$ at the point $s = k$. We show that it equals the image of the (D, r)-th Jacobi Poincaré series $P_{k+1,N,(D,r)}^J(\tau, z) \in J_{k+1,N}^{\text{cusp}}$ (see (3.1)) under the (D, r)-th Shimura-type lifting map $\mathcal{S}_{D,r}$ from $J_{k+1,N}^{\text{cusp}}$ to $S_{2k}(N)$ (see (3.3)). Using the Petersson coefficient formulas (2.1) and (3.2) for the corresponding Poincaré series in $S_{2k}(N)$ and $J_{k+1,N}^{\text{cusp}}$ we obtain the equation

$$\frac{N^{k-1}\Gamma(k-1/2)}{2\pi^{k-1/2}|D|^{k-1/2}} \sum_{\phi \in J_{k+1,N}^{\text{cusp}}} \frac{\overline{c_\phi(D,r)}}{\langle \phi, \phi \rangle} \mathcal{S}_{D,r}(\phi) = \mathcal{S}_{D,r}\left(P_{k+1,N,(D,r)}^J\right)$$
$$= R_{2k,N,1}\left(\tau, k, \left(\tfrac{D}{\cdot}\right)\right)$$
$$= \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \sum_{f \in S_{2k}(N)} \frac{\overline{L(f,D,k)}}{\langle f, f \rangle} f,$$

and taking the m-th Fourier coefficient yields the result.

2. The kernel function for the twisted L-function of a cusp form

In the following let k and N be positive integers with $k \geqslant 4$. Further, let ψ be a Dirichlet character mod N with $\psi(-1) = (-1)^k$ and let χ be a primitive Dirichlet character mod h with $(N, h) = 1$. We write $S_k(N, \psi)$ for the space of cusp forms of weight k for $\Gamma_0(N)$ wit character ψ. Throughout we write $e(z) := e^{2\pi i z}$ for $z \in \mathbb{C}$.

Inspired by [Koh97] we define the function

$$R_{k,N,\psi}(\tau, s, \chi) = \sum_{m=1}^{\infty} \overline{\chi(m)} m^{s-1} P_{k,N,\psi,m}(\tau)$$

where

$$P_{k,N,\psi,m}(\tau) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\psi(d)}(c\tau + d)^{-k} e\left(m \frac{a\tau + b}{c\tau + d}\right)$$

with $\Gamma_\infty = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ is the m-th Poincaré series in $S_k(N, \psi)$. For $1 < \Re(s) < k - 1$ the function $R_{k,N,\psi}(\tau, s, \chi)$ defines a cusp form in $S_k(N, \psi)$. Using the Petersson coefficient formula

$$\langle f, P_{k,N,\psi,m} \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_f(m) \tag{2.1}$$

for $f = \sum_{m=1}^{\infty} a_f(m) e(m\tau) \in S_k(N, \psi)$ we obtain

$$\langle f, R_{k,N,\psi}(\cdot, s, \chi) \rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} L(f, \chi, k - \overline{s}). \tag{2.2}$$

We now compute the Fourier expansion of $R_{k,N,\psi}(\tau, s, \chi)$.

Proposition 2.1. *For* $1 < \Re(s) < k - 1$ *the* m*-th Fourier coefficient of* $R_{k,N,\psi}(\tau, s, \chi)$ *is given by*

$$\overline{\chi(m)}m^{s-1} + \delta_{N,1}\chi(-1)i^{-k}h^{2s-k}(2\pi)^{k-2s}\frac{\Gamma(s)}{\Gamma(k-s)}\frac{G(\overline{\chi})}{G(\chi)}\chi(m)m^{k-s-1}$$

$$+ \frac{1}{2}i^{-k}(2\pi)^{k-s}\frac{\Gamma(s)}{\Gamma(k)}m^{k-1}\frac{h^s}{G(\chi)}\sum_{\ell(h)}\chi(\ell)\sum_{\substack{a,c\in\mathbb{Z},(a,c)=1,N|c \\ (ha+\ell c)c>0}}\psi(a)c^{-k}\left(\frac{c}{ha+\ell c}\right)^s$$

$$\times\left(e^{\pi is/2}e^{2\pi im\bar{a}/c}\,{}_1F_1\left(s,k;-\frac{2\pi imh}{(ha+\ell c)c}\right)\right.$$

$$\left.+ \psi(-1)\chi(-1)e^{-\pi is/2}e^{-2\pi im\bar{a}/c}\,{}_1F_1\left(s,k;\frac{2\pi imh}{(ha+\ell c)c}\right)\right),$$

where $\delta_{N,1}$ *is the Kronecker delta,* $\bar{a}$ *is any integer with* $a\bar{a} \equiv 1(c)$, $G(\chi) = \sum_{\ell(h)}\chi(\ell)e(\ell/h)$ *denotes the Gauss sum of* χ *and* ${}_1F_1(a,b,z)$ *is Kummer's confluent hypergeometric function (denoted by* $M(a,b,z)$ *in [AS72], 13.1.2).*

Proof. The proof generalises that of Lemma 2 in [Koh97]. We will frequently use the Lipschitz summation formula

$$\sum_{m=-\infty}^{\infty}(m+\tau)^{-s} = \frac{(-2\pi i)^s}{\Gamma(s)}\sum_{m=1}^{\infty}m^{s-1}e(m\tau) \qquad (2.3)$$

which is valid for $\tau \in \mathbb{H}$ and $\Re(s) > 1$. Since χ is primitive, we have

$$\overline{\chi(m)} = \frac{1}{G(\chi)}\sum_{\ell(h)}\chi(\ell)e(\ell m/h). \qquad (2.4)$$

The Lipschitz formula (2.3) together with (2.4) shows that $R_{k,N,\psi}(\tau, s, \chi)$ can be written as

$$\sum_{\left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)}\overline{\psi(d)}(c\tau+d)^{-k}\sum_{m=1}^{\infty}\overline{\chi(m)}m^{s-1}e\left(m\frac{a\tau+b}{c\tau+d}\right)$$

$$= \frac{\Gamma(s)}{G(\chi)(-2\pi i)^s}\sum_{\left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)}\overline{\psi(d)}(c\tau+d)^{-k}$$

$$\times\sum_{\ell(h)}\chi(\ell)\sum_{m=-\infty}^{\infty}\left(\frac{\ell}{h}+m+\frac{a\tau+b}{c\tau+d}\right)^{-s}$$

$$= \frac{1}{2}\frac{\Gamma(s)}{G(\chi)(-2\pi i)^s}\sum_{\ell(h)}\chi(\ell)\sum_{\left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)}\overline{\psi(d)}\tau^{-s}\bigg|_k\begin{pmatrix}1 & \frac{\ell}{h} \\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix}$$

where $f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d} \right)$ denotes the usual weight k slash operator. We first compute the Fourier coefficients of the inner sum over $\Gamma_0(N)$ for fixed ℓ by splitting it into three pieces corresponding to $c = 0, a + \frac{\ell}{h}c = 0$ and $(a + \frac{\ell}{h}c)c \neq 0$.

For $c = 0$ we have to sum over the matrices $\pm \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right)$ with $b \in \mathbb{Z}$:

$$\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N) \\ c=0}} \overline{\psi(d)} \tau^{-s} \Big|_k \begin{pmatrix} 1 & \frac{\ell}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2 \sum_{b=-\infty}^{\infty} \left(b + \tau + \frac{\ell}{h} \right)^{-s}$$

$$= 2 \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{m=1}^{\infty} m^{s-1} e(m\tau) e\left(\frac{m\ell}{h} \right)$$

where we again used the Lipschitz formula and the fact that two matrices with opposite sign give the same contribution.

Now suppose we have a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$ with $a + \frac{\ell}{h}c = 0$. We may assume that ℓ and h are coprime, since otherwise $\chi(\ell) = 0$. Then the only integral solutions of $a + \frac{\ell}{h}c = 0$ with $(a, c) = 1$ are $a = \pm\ell, c = \mp h$. This is only possible if $N \mid h$, and as $(N, h) = 1$, it is only possible for $N = 1$. In this case we write

$$\begin{pmatrix} 1 & \frac{\ell}{h} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{h} \\ -h & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\bar{\ell}}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\ell} & \frac{\ell\bar{\ell}-1}{h} \\ h & \ell \end{pmatrix}$$

where $\bar{\ell}$ is an integer with $\ell\bar{\ell} = 1(h)$. Note that $\left(\begin{smallmatrix} \bar{\ell} & \frac{\ell\bar{\ell}-1}{h} \\ h & \ell \end{smallmatrix} \right) \in \Gamma_0(N) = \mathrm{SL}_2(\mathbb{Z})$. So if $N = 1$ the character ψ is trivial and we obtain

$$\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N) \\ a+\frac{\ell}{h}c=0}} \tau^{-s} \Big|_k \begin{pmatrix} 1 & \frac{\ell}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (-1)^{-s} h^{2s-k} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N) \\ a+\frac{\ell}{h}c=0}} \tau^{s-k} \Big|_k \begin{pmatrix} 1 & -\frac{\bar{\ell}}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\ell} & \frac{\ell\bar{\ell}-1}{h} \\ h & \ell \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (-1)^{-s} h^{2s-k} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N) \\ c=0}} \tau^{s-k} \Big|_k \begin{pmatrix} 1 & -\frac{\bar{\ell}}{h} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now the same calculation as in the case $c = 0$ but with s replaced by $k - s$ shows that for $\Re(s) < k - 1$ the last expression equals

$$2(-1)^{-s} h^{2s-k} \frac{(-2\pi i)^{k-s}}{\Gamma(k-s)} \sum_{m=1}^{\infty} m^{k-s-1} e(m\tau) e\left(-\frac{m\bar{\ell}}{h} \right).$$

For $(a + \frac{\ell}{h}c)c \neq 0$ we have to compute the Fourier integral

$$\int_{iC}^{iC+1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \\ (a+\frac{\ell}{h}c)c \neq 0}} \overline{\psi(d)}(c\tau + d)^{-k} \left(\frac{(a + \frac{\ell}{h}c)\tau + b + \frac{\ell}{h}d}{c\tau + d} \right)^{-s} e(-m\tau)d\tau$$

with $C > 0$. The calculation can be done in the same way as in the case $ac \neq 0$ in the proof of Lemma 2 in [Koh97], so we omit the details. The result for the part with $(a + \frac{\ell}{h}c)c > 0$ is

$$i^{-k}\frac{(2\pi)^k}{\Gamma(k)}m^{k-1} \sum_{\substack{(a,c)=1,N|c \\ (a+\frac{\ell}{h}c)c>0}} \psi(a)c^{-k}e(m\bar{a}/c) \left(\frac{c}{(a + \frac{\ell}{h}c)c} \right)^s {}_1F_1\left(s, k; -\frac{2\pi i m}{(a + \frac{\ell}{h}c)c} \right)$$

where $a\bar{a} = 1(c)$. If we replace a by $-a$ and b by $-b$ in the part with $(a+\frac{\ell}{h}c)c < 0$ in the Fourier integral, the sum runs over all integral matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $ad - bc = -1$, $N \mid c$ and $(a - \frac{\ell}{h}c)c > 0$. Now this part can be computed in the same way as the part for $(a + \frac{\ell}{h}c)c > 0$ and gives the contribution

$$i^{-k}(-1)^{-s}\frac{(2\pi)^k}{\Gamma(k)}m^{k-1} \sum_{\substack{(a,c)=1,N|c \\ (a-\frac{\ell}{h}c)c>0}} \psi(-a)c^{-k}e(-m\bar{a}/c)$$

$$\times \left(\frac{c}{(a - \frac{\ell}{h}c)c} \right)^s {}_1F_1\left(s, k; \frac{2\pi i m}{(a - \frac{\ell}{h}c)c} \right).$$

We now put everything together. In the pieces corresponding to $(a + \frac{\ell}{h}c)c = 0$ and $(a + \frac{\ell}{h}c)c < 0$ we sum over $-\bar{\ell}$ and $-\ell$ instead of ℓ, respectively, giving a factor $\overline{\chi(-1)} = \chi(-1)$ in both cases. Taking into account (2.4) and $(-i)^{-s} = e^{\pi i s/2}, (-1)^{-s} = e^{-\pi i s}$, we obtain the stated Fourier expansion. ∎

Proof of Theorem 1.1. We follow the arguments of [Koh97]: Let χ be a primitive Dirichlet character mod h as above. From (2.2) it follows that

$$\sum_{f \in S_k(N,\psi)} \frac{\overline{L(f,\chi,k-\bar{s})}}{\langle f, f \rangle}f(\tau) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}R_{k,N,\psi}(\tau, s, \chi), \tag{2.5}$$

where the sum is taken over an orthogonal basis of $S_k(N,\psi)$. Suppose that $s = k/2 - \sigma - it$ with $\varepsilon < \sigma < \frac{1}{2}$ and $-T < t < T$ is a zero of the m-th Fourier coefficient (with $(m,h) = 1$) of the left-hand side of (2.5). Then also the m-th

coefficient of $R_{k,N,\psi}(\tau,s,\chi)$ vanishes:

$$-\overline{\chi}(m)m^{k/2-\sigma-it-1}$$

$$= \delta_{N,1}\chi(-1)i^{-k}(2\pi/h)^{2\sigma+2it}\frac{\Gamma(k/2-\sigma-it)}{\Gamma(k/2+\sigma+it)}\frac{G(\overline{\chi})}{G(\chi)}\chi(m)m^{k/2+\sigma+it-1}$$

$$+ \frac{1}{2}i^{-k}(2\pi)^{k/2+\sigma+it}\frac{\Gamma(k/2-\sigma-it)}{\Gamma(k)}m^{k-1}\frac{h^{k/2-\sigma-it}}{G(\chi)}$$

$$\times \sum_{\ell(h)}\chi(\ell)\sum_{\substack{a,c\in\mathbb{Z},(a,c)=1,N|c\\(ha+\ell c)c>0}}\psi(a)c^{-k/2-\sigma-it}(ha+\ell c)^{-k/2+\sigma+it}$$

$$\times \left(e^{\pi i(k/2-\sigma-it)/2}e^{2\pi i\bar{a}/c}\,_1F_1\left(k/2-\sigma-it,k;-\frac{2\pi imh}{(ha+\ell c)c}\right)\right.$$

$$\left.+ \psi\chi(-1)e^{-\pi i(k/2-\sigma-it)/2}e^{-2\pi i\bar{a}/c}\,_1F_1\left(k/2-\sigma-it,k;\frac{2\pi imh}{(ha+\ell c)c}\right)\right).$$

For $\Re(\beta)>\Re(\alpha)>0$ the integral representation

$$_1F_1(\alpha,\beta;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 e^{uz}u^{\alpha-1}(1-u)^{\beta-\alpha-1}du$$

from [AS72], 13.2.1, gives the estimate

$$|_1F_1(\alpha,\beta;2\pi ix)| \leqslant \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \tag{2.6}$$

for $\Re(\alpha)>1,\Re(\beta-\alpha)>1$ and $x\in\mathbb{R}$.

We now take absolute values on both sides of the big formula and divide everything by $m^{k/2-1-\sigma}$. Using $|G(\chi)|=|G(\overline{\chi})|=\sqrt{h}$, the estimate above and some elementary calculations we obtain

$$1 \leqslant \delta_{N,1}(2\pi/h)^{2\sigma}m^{2\sigma}\frac{|\Gamma(k/2-\sigma-it)|}{|\Gamma(k/2+\sigma+it)|} \tag{2.7}$$

$$+ \frac{4\cosh(\pi t/2)(2\pi m)^{k/2+\sigma}h^{k/2-\sigma+1/2}\zeta(k/2+\sigma)\zeta(k/2-\sigma)}{N^{k/2+\sigma}|\Gamma(k/2+\sigma+it)|}.$$

We want to show that the right-hand side goes to 0 for fixed N and $k\to\infty$ or fixed k and $N\to\infty$. We have

$$\frac{\Gamma(k/2-\sigma-it)}{\Gamma(k/2+\sigma+it)}(k/2)^{2\sigma+2it} \to 1 \qquad (k\to\infty)$$

uniformly in σ and t by [AS72], 6.1.47, hence the first summand in (2.7) tends to 0 as $k\to\infty$. For fixed level N the second summand in (2.7) goes to 0 for $k\to\infty$ as well, hence we get a contradiction for large k.

If we fix the weight k, the first summand in (2.7) is 0 for $N\geqslant 2$, and the second summand tends to 0 for $N\to\infty$, also giving a contradiction.

For $s=k/2+\sigma-it$ the statement follows for $N=1$ by the functional equation of (the completion of) $L(f,\chi,s)$ and for $N>1$ by the same arguments as above. ∎

3. Waldspurger's Theorem on average

Now we take $\psi = 1$ and $\chi = \left(\frac{D}{\cdot}\right)$ where $D < 0$ is a negative fundamental discriminant such that $(D, N) = 1$ and D is a square mod $4N$. Note that for a negative fundamental discriminant D the Kronecker symbol $\left(\frac{D}{\cdot}\right)$ is a primitive quadratic Dirichlet character mod $|D|$ with $\left(\frac{D}{-1}\right) = -1$ and Gauss sum $G\left(\left(\frac{D}{\cdot}\right)\right) := \sum_{\ell(D)} \left(\frac{D}{\ell}\right) e(\ell/|D|) = i\sqrt{|D|}$.

Let $e(x) := e^{2\pi i x}$ and $e^m(x) := e^{2\pi i m x}$ for $x \in \mathbb{C}$ and $m \in \mathbb{Z}$. Recall from [EZ85] that a Jacobi cusp form of weight $k + 1$ and index N is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ which is invariant under the slash operation

$$\phi\Big|_{k+1,N} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu] \right] (\tau, z)$$

$$= (c\tau + d)^{-k-1} e^N \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu \right)$$

$$\times \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right)$$

of the group $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and which has a Fourier expansion of the form $\phi(\tau, z) = \sum_{D < 0} \sum_{r^2 \equiv D(4N)} c_\phi(D, r) e\left(\frac{r^2 - D}{4N}\tau + rz \right)$ with coefficients $c_\phi(D, r) \in \mathbb{C}$. The space $J^{\mathrm{cusp}}_{k+1,N}$ consisting of all these forms is spanned by the Poincaré series

$$P^J_{k+1,N,(D,r)}(\tau, z) = \sum_{\gamma \in \Gamma^J_\infty \backslash \Gamma^J} e\left(\frac{r^2 - D}{4N}\tau + rz \right) \Big|_{k+1,N} \gamma, \qquad (3.1)$$

where $\Gamma^J_\infty = \{ [\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right), [0, \mu]] : n, \mu \in \mathbb{Z} \}$ and (D, r) varies over all $D < 0$ and $r \in \mathbb{Z}$ with $D \equiv r^2(4N)$. The Poincaré series $P^J_{k+1,N,(D,r)}$ can also be characterised by the Petersson coefficient formula

$$\langle \phi, P^J_{k+1,N,(D,r)} \rangle = \frac{N^{k-1}\Gamma(k - 1/2)}{2\pi^{k-1/2}} |D|^{-k+1/2} c_\phi(D, r) \qquad (3.2)$$

for all $\phi \in J^{\mathrm{cusp}}_{k+1,N}$ (see [GKZ87], Proposition II.1).

It was shown in [GKZ87] that for each fundamental discriminant $D < 0$ and $r \in \mathbb{Z}$ with $D \equiv r^2(4N)$ there is a Hecke-equivariant lifting map

$$\mathcal{S}_{D,r}(\phi)(w) = \sum_{m=1}^\infty \left(\sum_{d|m} \left(\frac{D}{d} \right) d^{k-1} c_\phi \left(\frac{m^2}{d^2} D, \frac{m}{d} r \right) \right) e(mw) \qquad (3.3)$$

from $J^{\mathrm{cusp}}_{k+1,N}$ to $S_{2k}(N)$. By comparing Fourier coefficients we will show that

$$R_{2k,N}\left(\tau, k, \left(\tfrac{D}{\cdot} \right) \right) = \mathcal{S}_{D,r} \left(P^J_{k+1,N,(D,r)} \right) \qquad (3.4)$$

for $k \geqslant 2$. Note that we dropped $\psi = 1$ from the notation.

Proposition 3.1. *The m-th Fourier coefficient of $R_{2k,N}\left(\tau, k, \left(\frac{D}{\cdot}\right)\right)$ is given by*

$$(1 \pm \delta_{N,1}) \left(\frac{D}{m}\right) m^{k-1} + i^{k+1} \pi \sqrt{2} m^{k-1/2} \sum_{\substack{n \geqslant 1 \\ N|n}} n^{-1/2} K_{N,n}^{\pm}(m, D) J_{k-1/2}\left(\frac{\pi}{n} m|D|\right)$$

where $\pm 1 = (-1)^{k+1}$, $\delta_{N,1}$ is the Kronecker delta, $J_{k-1/2}$ is the Bessel function of order $k - 1/2$,

$$K_{N,n}(m, D) = \sum_{\ell(D)} \left(\frac{D}{\ell}\right) \sum_{\substack{(a,c) \in \mathbb{Z}^2 \\ (a,c)=1, c>0, N|c \\ (|D|a+\ell c)c=n}} e_{2n}\left(m\left(|D| - 2(|D|a + \ell c)\bar{a}\right)\right)$$

with $a\bar{a} \equiv 1(c)$ is a finite exponential sum, $K_{N,n}^{\pm}(m, D) = K_{N,n}(m, D) \pm K_{N,n}(-m, D)$ and $e_{2n}(z) = e^{2\pi i z/2n}$ for $z \in \mathbb{C}$.

Proof. The hypergeometric function appearing in the Fourier expansion of $R_{2k,N}\left(\tau, k, \left(\frac{D}{\cdot}\right)\right)$ is related to the Bessel function of order $k - 1/2$:

$${}_1F_1\left(k, 2k; \frac{2\pi i m|D|}{(|D|a+\ell c)c}\right) = \Gamma(k + 1/2)\left(\frac{1}{2}\frac{\pi m|D|}{(|D|a+\ell c)c}\right)^{-k+1/2}$$
$$\times e\left(\frac{m|D|}{2(|D|a+\ell c)c}\right) J_{k-1/2}\left(\frac{\pi m|D|}{(|D|a+\ell c)c}\right),$$

see [AS72], 13.6.1. Using the Kummer transformation

$${}_1F_1(a, b; z) = e^z \, {}_1F_1(b - a, b; -z),$$

see [AS72], 13.1.27, we get

$${}_1F_1\left(k, 2k; -\frac{2\pi i m|D|}{(|D|a+\ell c)c}\right)$$
$$= \Gamma(k + 1/2)\left(\frac{1}{2}\frac{\pi m|D|}{(|D|a+\ell c)c}\right)^{-k+1/2} e\left(-\frac{m|D|}{(|D|a+\ell c)c}\right)$$
$$\times e\left(\frac{m|D|}{2(|D|a+\ell c)c}\right) J_{k-1/2}\left(\frac{\pi m|D|}{(|D|a+\ell c)c}\right).$$

The formula now follows by a straightforward calculation using the Legendre duplication formula $\Gamma(k)\Gamma(k + 1/2) = 2^{1-2k}\sqrt{\pi}\Gamma(2k)$. ∎

In order to prove the identity (3.4) we compute the Fourier expansion of the right-hand side. To this end, we need the Fourier expansion of the Poincaré series $P_{k+1,N,(D,r)}^{J} \in J_{k+1,N}^{\mathrm{cusp}}$.

Proposition 3.2 ([GKZ87], Proposition II.2). *The Jacobi Poincaré series* $P^J_{k+1,N,(D,r)} \in J^{\mathrm{cusp}}_{k+1,N}$ *has the expansion*

$$P^J_{k+1,N,(D,r)}(\tau, z) = \sum_{\substack{D'<0, r' \in \mathbb{Z} \\ r'^2 \equiv D'(4N)}} g^{\pm}_{k+1,N,(D,r)}(D',r') e\left(\frac{r'^2 - D'}{4N}\tau + r'z\right)$$

where $\pm 1 = (-1)^{k+1}$,

$$g^{\pm}_{k+1,N,(D,r)}(D',r') = g_{k+1,N,(D,r)}(D',r') \pm g_{k+1,N,(D,r)}(D',-r')$$

and

$$g_{k+1,N,(D,r)}(D',r') = \delta_N(D,r,D',r') + i^{k+1}\pi\sqrt{2}N^{-1/2}(D'/D)^{\frac{k}{2}-\frac{1}{4}}$$
$$\times \sum_{n \geqslant 1} H_{N,n}(D,r,D',r') J_{k-1/2}\left(\frac{\pi}{Nn}\sqrt{D'D}\right)$$

where

$$\delta_N(D,r,D',r') = \begin{cases} 1, & \text{if } D' = D \text{ and } r' \equiv r(2N), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$H_{N,n}(D,r,D',r')$$
$$= n^{-3/2} \sum_{\substack{\rho(n)^* \\ \lambda(n)}} e_n\left(\left(N\lambda^2 + r\lambda + \frac{r^2 - D}{4N}\right)\bar{\rho} + \frac{r'^2 - D'}{4N}\rho + r'\lambda\right) e_{2Nn}(rr')$$

with $\rho\bar{\rho} \equiv 1(n)$ *is a Kloosterman-type sum.*

Proposition 3.3. *The m-th Fourier coefficient of* $\mathcal{S}_{D,r}\left(P^J_{k+1,N,(D,r)}\right)$ *is given by*

$$(1 \pm \delta_{N,1})\left(\frac{D}{m}\right)m^{k-1} + i^{k+1}\pi\sqrt{2}m^{k-1/2}\sum_{\substack{n \geqslant 1 \\ N|n}} n^{-1/2} S^{\pm}_{N,n}(m,D) J_{k-1/2}\left(\frac{\pi}{n}m|D|\right)$$

where $\pm 1 = (-1)^{k+1}$, $\delta_{N,1}$ *is the Kronecker delta, $J_{k-1/2}$ is the Bessel function of order $k - 1/2$,*

$$S_{N,n}(m,D) = \sum_{\substack{b(2n) \\ b^2 \equiv D^2(4n) \\ b \equiv D(2N)}} \chi_D\left(\begin{pmatrix} n & b/2 \\ b/2 & \frac{b^2-D^2}{4n} \end{pmatrix}\right) e_{2n}(bm)$$

with the genus character χ_D defined in [GKZ87], $S^{\pm}_{N,n}(m,D) = S_{N,n}(m,D) \pm S_{N,n}(-m,D)$ and $e_{2n}(z) = e^{2\pi i z/2n}$ for $z \in \mathbb{C}$.

Proof. The m-th coefficient of $\mathcal{S}_{D,r}\left(P^J_{k+1,N,(D,r)}\right)$ is

$$\sum_{d|m}\left(\frac{D}{d}\right)d^{k-1}g^{\pm}_{k+1,N,(D,r)}\left(\frac{m^2}{d^2}D,\frac{m}{d}r\right)$$

where $g^{\pm}_{k+1,N,(D,r)}(D',r')$ denotes the (D',r')-th coefficient of $P^J_{k+1,N,(D,r)}$.

The δ_N-part only gives a contribution for $d=m$, and in this case we have $\delta_N(D,r,D,r)=1$. Further, $\delta_N(D,r,D,-r)$ is 1 if and only if $N\mid r$ which is equivalent to $N\mid D$ since D is a fundamental discriminant and $r^2\equiv D(4N)$. Since $(D,N)=1$ by assumption, this is only possible for $N=1$.

The Kloosterman sum part of $g_{k+1,N,(D,r)}$ gives the contribution

$$i^{k+1}\pi\sqrt{2}N^{-1/2}m^{k-1/2}\sum_{n\geqslant 1}\sum_{d|(m,n)}\left(\frac{D}{d}\right)d^{-1/2}$$

$$\times H_{N,n/d}\left(D,r,\frac{m^2}{d^2}D,\frac{m}{d}r\right)J_{k-1/2}\left(\frac{\pi}{Nn}m|D|\right).$$

The proof will be finished by the following lemma from [GKZ87]. ∎

Lemma 3.4 ([GKZ87], Lemma II.3). *For all $m\geqslant 1, n\geqslant 0, r\in\mathbb{Z}$ with $D=r^2-4Nn<0$ we have*

$$S_{N,Nn}(m,D)=\sum_{d|(m,n)}\left(\frac{D}{d}\right)(n/d)^{1/2}H_{N,n/d}\left(D,r,\frac{m^2}{d^2}D,\frac{m}{d}r\right).$$

Comparing the Fourier coefficients of the kernel function $R_{2k,N}\left(\tau,k,\left(\frac{D}{\cdot}\right)\right)$ with those of $\mathcal{S}_{D,r}(P^J_{k+1,N,(D,r)})$, we see that it suffices to show

$$S_{N,n}(m,D)=K_{N,n}(m,D)$$

for all $n\geqslant 1$ with $N\mid n, (D,N)=1$, and all $m\in\mathbb{Z}$. This immediatly follows from the next lemma:

Lemma 3.5. *Let $N\mid n$ and $(D,N)=1$. A set of representatives for $b(2n)$ with $b^2\equiv D^2(4n)$ and $b\equiv D(2N)$ is given by*

$$b=|D|-2(|D|a+\ell c)\bar{a}$$

where a,c run through $\mathbb{Z}$ with $(a,c)=1$, $c>0$ and $N\mid c$, and ℓ runs mod D such that $(|D|a+\ell c)c=n$. Here $\bar{a}$ is any fixed integer with $a\bar{a}\equiv 1(c)$. If b is of this form, we have

$$\chi_D\left(\begin{pmatrix}n & b/2 \\ b/2 & \frac{b^2-D^2}{4n}\end{pmatrix}\right)=\left(\frac{D}{\ell}\right).$$

Proof. Recall that for every class $\rho \bmod 2N$ the group $\Gamma_0(N)$ acts on the set of quadratic forms

$$\mathcal{Q}_{N,D,\rho} = \left\{ Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : N \mid a, \ b^2 - 4ac = D^2, \ b \equiv \rho(2N) \right\}$$

by $Q \circ M = M^t QM$. The elements $b(2n)$ with $b^2 \equiv D^2(4n)$ and $b \equiv D(2N)$ can be thought of as the upper right entries of a full system of $\Gamma_0(N)$-inequivalent quadratic forms in $\mathcal{Q}_{N,D^2,D}$ with fixed upper left entry n. A system of $\Gamma_0(N)$-representatives for the whole set $\mathcal{Q}_{N,D^2,D}$ is given by $\begin{pmatrix} 0 & D/2 \\ D/2 & \ell \end{pmatrix}$ where ℓ runs mod D (here the condition $(D,N) = 1$ is used). Hence we need to find all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ such that the matrices

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & D/2 \\ D/2 & \ell \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} (\alpha D + \gamma \ell)\gamma & \frac{-D + 2(\alpha D + \gamma \ell)\delta}{2} \\ \frac{-D + 2(\alpha D + \gamma \ell)\delta}{2} & (\beta D + \delta \ell)\delta \end{pmatrix} \quad (3.5)$$

have upper left entry n and are inequivalent mod $\Gamma_0(N)$. Setting $a = -\alpha, c = \gamma, \bar{a} = -\delta$ and changing a and c to $-a$ and $-c$ if $c < 0$ it is now easy to see that we can take $b = |D| - 2(|D|a + \ell c)\bar{a}$ with a, c, ℓ as stated in the lemma.

The genus character χ_D of the matrix (3.5) and the Kronecker symbol $\left(\frac{D}{\ell}\right)$ both are 0 for $(D, \ell) > 1$. For $(D, \ell) = 1$ the quadratic form (3.5) properly represents ℓ, so χ_D applied to the matrix (3.5) equals $\left(\frac{D}{\ell}\right)$ by definition of the genus character. ∎

Proof of Theorem 1.2. As in (2.5) we write

$$R_{2k,N}\left(\tau, s, \left(\tfrac{D}{\cdot}\right)\right) = \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \sum_{f \in S_{2k}(N)} \frac{\overline{L(f, D, 2k - \bar{s})}}{\langle f, f \rangle} f(\tau),$$

with f running through an orthogonal basis of $S_{2k}(N)$. On the other hand, the Petersson coefficient formula (3.2) gives

$$\mathcal{S}_{D,r}\left(P_{k+1,N,(D,r)}^J\right)(\tau) = \frac{N^{k-1}\Gamma(k-1/2)}{2\pi^{k-1/2}|D|^{k-1/2}} \sum_{\phi \in J_{k+1,N}^{\mathrm{cusp}}} \frac{\overline{c_\phi(D,r)}}{\langle \phi, \phi \rangle} \mathcal{S}_{D,r}(\phi)(\tau),$$

where ϕ runs through an orthogonal basis of $J_{k+1,N}^{\mathrm{cusp}}$ and $c_\phi(D,r)$ is the (D,r)-th coefficient of ϕ. As we have shown above, both expressions are equal at $s = k$. By [EZ85], Theorem 4.5, the m-th Fourier coefficient of $\mathcal{S}_{D,r}(\phi)$ is just the (D,r)-th coefficient of $\phi|T_m$, so taking the m-th coefficient (and complex conjugation) yields

$$\frac{\Gamma(2k-1)}{(4\pi)^{2k-1}} \sum_{f \in S_{2k}(N)} \frac{L(f, D, s)}{\langle f, f \rangle} \overline{a_f(m)}$$

$$= \frac{N^{k-1}\Gamma(k-1/2)}{2\pi^{k-1/2}|D|^{k-1/2}} \sum_{\phi \in J_{k+1,N}^{\mathrm{cusp}}} \frac{c_\phi(D,r)}{\langle \phi, \phi \rangle} \overline{c_{\phi|T_m}(D,r)}.$$

Now the Legendre duplication formula $\Gamma(k - 1/2)\Gamma(k) = 2^{2-2k}\sqrt{\pi}\Gamma(2k - 1)$ completes the proof of Theorem 1.2. ∎

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