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MEAN SQUARE OF THE ERROR TERM IN THE ASYMMETRIC MULTIDIMENSIONAL DIVISOR PROBLEM

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Abstract: Let $\boldsymbol{a} = (a_1, \cdots, a_k)$ denote a k-tuple of positive integers such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$. We put $d(\boldsymbol{a}; n) = \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} 1$ and let $\Delta(\boldsymbol{a}; x)$ be the error term of the corresponding asymptotic formula for the summatory function of $d(\boldsymbol{a}; n)$. In this paper we show an asymptotic formula of the mean square of $\Delta(\boldsymbol{a}; x)$ under a certain condition. Moreover, when k equals 2 or 3, we give unconditional asymptotic formulas for these mean squares.

Keywords: asymmetric multidimensional divisor problem, mean square of the error term, Dirichlet series, functional equation, the Tong-type representation.

1. Introduction and the statement of results

Let k be a fixed positive integer and $x \geqslant 1$. We put $\boldsymbol{a} := (a_1, \ldots, a_k)$, where a_j $(j = 1, \ldots, k)$ are positive integers such that $a_1 \leqslant \cdots \leqslant a_k$. By $d(\boldsymbol{a}; n)$ we denote the number of representations of an integer n in the form $n = n_1^{a_1} \cdots n_k^{a_k}$, namely,

$$d(\boldsymbol{a}; n) = \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} 1. \tag{1.1}$$

We define

$$\Delta(\boldsymbol{a}; x) := \sum_{n \leqslant x}{}' d(\boldsymbol{a}; n) - H(\boldsymbol{a}; x),$$

where $H(\boldsymbol{a}; x)$ is the main term of the summatory function of $d(\boldsymbol{a}; n)$ given by the sum of residues of $\prod_{j=1}^{k} \zeta(a_j s) \frac{x^s}{s}$, and $'$ in the summation symbol means that the last term $d(\boldsymbol{a}; x)$ should be counted with weight $1/2$ when x is an integer. The

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asymmetric multidimensional divisor problem (or the general divisor problem) is to study the behaviour of $\Delta(\boldsymbol{a};x)$. See also Ivić [7] and Krätzel [10], or the survey paper [9].

When $a_1 = a_2 = 1$, $d(1,1;n) = \sum_{d|n} 1$, $\Delta(1,1;x) = \sum_{n\leqslant x} d(1,1,;n) - x(\log x + 2\gamma - 1)$, ($\gamma$ is the Euler constant), the above problem is the classical Dirichlet divisor problem. Dirichlet proved $\Delta(1,1;x) = O(x^{1/2})$ by his famous hyperbola method. The exponent 1/2 was later improved by many researchers. The latest result is

$$\Delta(x) = O(x^{131/416}(\log x)^{26947/8320})$$

due to Huxley [6]. For the lower bounds, it is known that

$$\Delta(1,1;x) = \Omega_+\left(x^{\frac14}(\log x)^{\frac14}(\log\log x)^{\frac{3+\log 4}{4}}\exp(-c\sqrt{\log\log\log x})\right) \qquad (c>0)$$

and

$$\Delta(1,1;x) = \Omega_-\left(x^{\frac14}\exp(c'(\log\log x)^{\frac14}(\log\log\log x)^{-\frac34})\right) \qquad (c'>0),$$

which are due to Hafner [5] and Corrádi and Kátai [3], respectively. Many corresponding upper bounds and Ω-results for the asymmetric multidimensional divisor problem can be found in [7] and [10].

The mean square estimate is one of the main topics in the theory of divisor problem. Let $R(T)$ be the error term defined by the following formula

$$R(T) = \int_1^T \Delta^2(1,1;x)dx - cT^{3/2},$$

where $c = \frac{1}{6\pi^2}\sum_{n=1}^\infty \frac{d(1,1;n)^2}{n^{3/2}}$ is a positive constant. Cramér [4] first proved that

$$R(T) = O(T^{5/4+\varepsilon}).$$

Cramér's estimate of $R(T)$ was improved to

$$R(T) = O(T\log^5 T) \tag{1.2}$$

by Tong [12] and recently to $R(T) = O(T\log^3 T\log\log T)$ by Lau and Tsang [11]. Tong's method of proving (1.2) is the initial motivation of our previous paper [2].

Ivić [8] studied the upper bound and Ω-result of the mean square of $\Delta(\boldsymbol{a};x)$ for general k. As for the upper bound, he proved that if

$$\int_1^T \Delta^2(\boldsymbol{a};x)dx \ll T^{1+2\beta_k} \qquad (\beta_k \geqslant 0)$$

then $\beta_k \geqslant g_k$, where

$$g_k = \frac{r-1}{2(a_1+\cdots+a_r)}$$

and r is the largest integer such that

$$(r-2)a_r \leqslant a_1 + \cdots + a_{r-1} \qquad (2 \leqslant r \leqslant k)$$

[8, (1.5)]. Moreover, he showed that if the estimate

$$\int_1^T |\zeta(1/2+it)|^{2k-2} dt \ll T^{1+\varepsilon}$$

holds, then $\beta_k = g_k$. In particular, $\beta_k = g_k$ holds for $k = 2$ and 3. For the lower bound, he showed that

$$\int_1^T \Delta^2(\boldsymbol{a}; x) dx = \Omega(T^{1+2g_k} \log^A T)$$

with some constant $A \geqslant 0$. Inspired by these facts, Ivić conjectured that the asymptotic formula

$$\int_1^T \Delta^2(\boldsymbol{a}; x) dx = (E_k + o(1)) T^{1+2g_k} \log^{A_k} T \tag{1.3}$$

holds for general $k \geqslant 2$ with some constants $E_k > 0$ and $A_k \geqslant 0$ [8, (5.7)].

When $k = 2$, Ivić's conjecture (1.3) was confirmed by Cao and Zhai [13]. More precisely they proved that

$$\int_1^T \Delta^2(\boldsymbol{a}; x) dx = c(\boldsymbol{a}) T^{\frac{1+a_1+a_2}{a_1+a_2}} + O\left(T^{\frac{1+a_1+a_2}{a_1+a_2} - \frac{a_1}{2a_2(a_1+a_2)(a_1+a_2-1)}} \log^{\frac{7}{2}} T \right), \tag{1.4}$$

where a_1 and a_2 are integers such that $1 \leqslant a_1 \leqslant a_2$, $\boldsymbol{a} = (a_1, a_2)$ and $c(\boldsymbol{a})$ is some constant. Their method is based on the transformation formula of the exponential sum and the Chowla and Walum type representation of $\Delta(\boldsymbol{a}; x)$ (see also [1]). When $a_1 = a_2 = 1$, the error term in (1.4) becomes $O(T^{\frac{5}{4}} \log^{\frac{7}{2}} T)$. Hence (1.4) is an analogue of Cramér's result for $\Delta(1, 1; x)$.

In this paper we shall study the mean square estimate of the error term $\Delta(\boldsymbol{a}; x)$ more closely by means of the Tong method [2, 12]. For this purpose, we need an auxiliary divisor function defined by

$$\hat{d}(\boldsymbol{a}; n) = \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} n_1^{a_1 - 1} \cdots n_k^{a_k - 1}, \tag{1.5}$$

which is a dual function of $d(\boldsymbol{a}; n)$. For convenience, we write

$$b(n) = \pi^{2\alpha - k/2} \hat{d}(\boldsymbol{a}; n) \qquad \text{and} \qquad \mu_n = \pi^{2\alpha} n,$$

where

$$\alpha := (a_1 + \cdots + a_k)/2.$$

From (1.1) and (1.5), we have

$$\varphi(s) := \sum_{n=1}^{\infty} \frac{d(\boldsymbol{a};n)}{n^s} = \prod_{j=1}^{k} \zeta(a_j s) \qquad (\operatorname{Re} s > 1/a_1)$$

and

$$\begin{aligned}\psi(s) &:= \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s} = \pi^{2\alpha-k/2-2\alpha s} \sum_{n=1}^{\infty} \frac{\hat{d}(\boldsymbol{a};n)}{n^s} \\ &= \pi^{2\alpha-k/2-2\alpha s} \prod_{j=1}^{k} \zeta(a_j s - a_j + 1) \qquad (\operatorname{Re} s > 1). \end{aligned} \tag{1.6}$$

Let $1/2 \leqslant \sigma^* < 1$ be a real number defined by

$$\sigma^* := \inf \left\{ \sigma \;\middle|\; \int_0^T |\psi(\sigma + it)|^2 dt \ll T^{1+\varepsilon} \right\}. \tag{1.7}$$

From (1.6) it is easy to check that

$$\sigma^* \geqslant 1 - \frac{1}{2a_k}. \tag{1.8}$$

In this paper we assume that σ^* satisfies the condition

$$\sigma^* < 1 - \frac{k-1}{4\alpha}. \tag{1.9}$$

This condition plays an important role in Tong's method. From (1.8), we note that (1.9) implies, as a necessary condition, that

$$(k-2)a_k < a_1 + \cdots + a_{k-1}. \tag{1.10}$$

We first prove a conditional asymptotic formula of the mean square of $\Delta(\boldsymbol{a}, x)$.

Theorem 1. *Suppose that* (1.9) *and* (1.10) *hold. Then we have*

$$\int_1^T \Delta^2(\boldsymbol{a};x)dx = c(\boldsymbol{a})T^{1+\frac{k-1}{2\alpha}} + O\left(T^{1+\frac{k-1}{2\alpha}-\eta(\boldsymbol{a})+\varepsilon}\right), \tag{1.11}$$

where $c(\boldsymbol{a})$ *is a certain positive constant and*

$$\eta(\boldsymbol{a}) := \frac{2(1-\sigma^*) - \frac{k-1}{2\alpha}}{2\alpha(3 - 2\sigma^* - \frac{1}{a_k}) - 1} > 0. \tag{1.12}$$

It is an important problem to determine the exact value of σ^*. Generally it is a very difficult problem, but it is easy to see that if the Lindelöf hypothesis for $\zeta(s)$ is true, then $\sigma^* = 1 - 1/2a_k$. Hence from Theorem 1 we have

Corollary 1. *Suppose that* (1.10) *holds. If the Lindelöf hypothesis is true, then we have*

$$\int_1^T \Delta^2(\boldsymbol{a};x)dx = c(\boldsymbol{a})T^{1+\frac{k-1}{2\alpha}} + O\left(T^{1+\frac{k-1}{2\alpha}-\frac{2\alpha-(k-1)a_k}{2\alpha(2\alpha-1)a_k}+\varepsilon}\right),$$

where $c(\boldsymbol{a})$ *is a certain positive constant.*

When $k=2$, we find that $\sigma^* = 1-1/2a_2$ holds unconditionally, which is a consequence of the fourth power moment of $\zeta(s)$ on the critical line. Hence (1.11) gives

Theorem 2. *Suppose* $a_1 \leqslant a_2$. *Then we have*

$$\int_1^T \Delta^2(a_1,a_2;x)dx = c_2 T^{1+\frac{1}{a_1+a_2}} + O\left(T^{1+\frac{1}{a_1+a_2}-\frac{a_1}{a_2(a_1+a_2)(a_1+a_2-1)}+\varepsilon}\right), \quad (1.13)$$

where c_2 *is a certain positive constant.*

Theorem 2 improves the error term of (1.4). We note that if we take $a_1 = a_2 = 1$, the error term in (1.13) is $O(T^{1+\varepsilon})$. So (1.13) is an analogue of (1.2) modulo term T^ε.

Another interesting case is $k=3$. In this case we can prove the following Theorem 3.

Theorem 3. *Let* $k=3$. *If* $a_1 \leqslant a_2 \leqslant a_3$ *and* $a_3 < a_1+a_2$, *then we have*

$$\int_1^T \Delta^2(a_1,a_2,a_3;x)dx = c_3 T^{1+\frac{2}{a_1+a_2+a_3}} + O(T^{1+\frac{2}{a_1+a_2+a_3}-\eta_3+\varepsilon}),$$

where

$$\eta_3 = \begin{cases} \frac{1}{(a_1+a_2+a_3)(3+2(a_1+a_2+a_3)(1-1/a_3))} & \text{if } 3(a_2+a_3) \leqslant 7a_1, \\ \frac{4a_1a_3}{(a_1+a_2+a_3)\big((a_1+a_2+a_3)(a_1+3a_2+3a_3)(a_3-1)+a_3(5a_1+3a_2+3a_3)\big)} & \\ \qquad \text{if } 3(a_2+a_3) > 7a_1, 3a_3+a_1 \leqslant 5a_2 \text{ and } 3a_3 < a_1+3a_2, & \\ \frac{a_1+a_2-a_3}{a_3(a_1+a_2+a_3)(a_1+a_2+a_3-1)} & \text{otherwise,} \end{cases}$$

and c_3 *is a certain positive constant.*

We shall prove Theorem 3 in Section 4.

2. The truncated Tong-type formula of $\Delta(\boldsymbol{a};x)$

In [12], Tong studied the mean square of $\Delta(\underbrace{1,\ldots,1}_{k};x)$. By using the functional equation of $\zeta^k(s)$ he derived a very useful formula of $\Delta(1,\ldots,1;x)$, which we call the truncated Tong-type formula, where the first finite sum is the same as that of the truncated Voronoï formula, while its error term is represented by the integrals like (2.6) below.

In our case, using the functional equation of the Riemann zeta function

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

we find easily that the functional equation of $\varphi(s)$ and $\psi(s)$ has a form

$$\Delta_1(s)\varphi(s)=\Delta_2(1-s)\psi(1-s), \tag{2.1}$$

where

$$\Delta_1(s):=\prod_{j=1}^{k}\Gamma\left(\frac{a_j s}{2}\right) \tag{2.2}$$

and

$$\Delta_2(s):=\prod_{j=1}^{k}\Gamma\left(\frac{a_j s-a_j+1}{2}\right). \tag{2.3}$$

Note that $\hat{d}(\boldsymbol{a};n)$ does not satisfy the Ramanujan conjecture and also the gamma factors on the left and right hand side of (2.1) are not the same for general $\boldsymbol{a}$, so the pair of Dirichlet series $\varphi(s)$ and $\psi(s)$ is not contained in the so-called Selberg class. In our previous paper [2], we developed the theory of the truncated Tong-type formula of the error term for such a pair of Dirichlet series. Obviously $\varphi(s)$ and $\psi(s)$ satisfy the conditions therein.

In order to write the truncated Tong-type formula for $\Delta(\boldsymbol{a};x)$ in the present case, we use the same notations as in [2]. From (2.2) and (2.3), we have (we repeat the definion of α for its importance)

$$\alpha=\frac{a_1+\cdots+a_k}{2}, \qquad r=1,$$

$$\mu=\frac{1-k}{2}, \qquad \mu'=\sum_j\left(-\frac{a_j}{2}\right)+\frac{1}{2}=-\alpha+\frac{1}{2},$$

$$\nu=-\frac{1}{2}\sum_j\log a_j, \qquad \nu'=-\frac{1}{2}\sum_j a_j\log a_j,$$

$$\lambda=\sum_j a_j\log a_j=\lambda', \qquad h=2\alpha e^{-\frac{\lambda+\lambda'}{2\alpha}}=2\alpha\prod_{j=1}^{k}a_j^{-a_j/\alpha}$$

and

$$\theta_\varrho=\frac{r}{2}-\frac{1}{4\alpha}+\varrho\left(1-\frac{1}{2\alpha}\right)+\frac{\mu'-\mu}{2\alpha}.$$

In this paper we only consider the case $\varrho=0$, hence

$$\theta_0=\frac{1}{2}-\frac{1}{4\alpha}+\frac{\mu'-\mu}{2\alpha}=\frac{k-1}{4\alpha}. \tag{2.4}$$

We also put

$$\lambda_0 = \theta_0 + \frac{1}{2\alpha} - r - 1 = \frac{k+1}{4\alpha} - 2. \tag{2.5}$$

In Tong's theory, it is important to approximate $\Delta(\boldsymbol{a}; x)$ by the K-th averaging integral

$$\int_{\mathbf{E}_K} \Delta(\boldsymbol{a}; \tilde{y}) d\mathrm{Y}_K,$$

where we use the notation

$$\int_{\mathbf{E}_K} g(\tilde{y}) d\mathrm{Y}_K = \int_0^1 \cdots \int_0^1 g(\tilde{y}) dy_1 \cdots dy_K,$$

with

$$\tilde{y} = y + \frac{1}{x}(y_1 + \cdots + y_K)$$

for an integrable function $g(y)$. Let $\hat{\Delta}(\boldsymbol{a}; x)$ be the error term of the asymptotic formula of summatory function of $\hat{d}(\boldsymbol{a}; n)$, which is defined mutatis mutandis as for $\Delta(\boldsymbol{a}; x)$. Then the averaging integral can be expressed by the function defined by

$$I(\lambda, M, N, y) = 2\pi i \int_M^N u^{\lambda} \hat{\Delta}(\boldsymbol{a}; u) \exp\left(-ih(uy)^{\frac{1}{2\alpha}}\right) du. \tag{2.6}$$

The next lemma gives the truncated Tong-type formula of $\Delta(\boldsymbol{a}; y)$. Applying Theorem 5 of [2] directly we get

Lemma 1. *Let* $1 \leqslant x \leqslant y \leqslant (1+\delta)x$, $N = [x^{4\alpha-1-\varepsilon}]$ *and* $J = [(4\alpha^2 r + 4\alpha)\varepsilon^{-1}]$, *where* δ *is a small positive constant. In every subinterval* $[t, t + Bt^{1-1/2\alpha}] \subset [1, \sqrt{N}]$, *there exists* $M \neq \mu_n$ *such that the following Tong-type formula holds:*

$$\Delta(\boldsymbol{a}; y) = \sum_{j=1}^{7} R_j(y),$$

where

$$\begin{aligned}
R_1(y) &= \kappa_0 y^{\theta_0} \sum_{\mu_n \leqslant M} \frac{b(n)}{\mu_n^{1-\theta_0}} \cos(h(y\mu_n)^{1/2\alpha} + c_0\pi) \\
&= \kappa_0 \pi^{2\alpha(\theta_0-1)} y^{\theta_0} \sum_{n \leqslant M'} \frac{b(n)}{n^{1-\theta_0}} \cos(h\pi(yn)^{1/2\alpha} + c_0\pi) \\
&= \kappa_0 \pi^{2\alpha\theta_0 - k/2} y^{\theta_0} \sum_{n \leqslant M'} \frac{\hat{d}(\boldsymbol{a}; n)}{n^{1-\theta_0}} \cos(h\pi(yn)^{1/2\alpha} + c_0\pi), \\
R_2(y) &= y^{\theta_0 + \frac{1}{2\alpha}} \operatorname{Re}\{c_{00} I(\lambda_0, M, N, y)\}, \\
R_3(y) &= \sum_{\substack{l=0 \\ l+m>0}}^{J} \sum_{m=0}^{J} \operatorname{Re}\left\{c_{lm} I\left(\lambda_0 + \frac{l-m}{2\alpha}, M, N, y\right)\right\} x^{-l} y^{-l+\theta_0+\frac{1}{2\alpha}+\frac{l-m}{2\alpha}},
\end{aligned}$$

$$
\begin{aligned}
R_4(y) &= \sum_{j=0}^{K}\sum_{m=0}^{K} \mathrm{Re}\left\{ c'_{jm} I\left(\lambda_0 - \frac{K+m}{2\alpha}, N, \infty, y + \frac{j}{x}\right)\right\} \\
&\quad \times x^K \left(y + \frac{j}{x}\right)^{K+\theta_0+\frac{1}{2\alpha}-\frac{K+m}{2\alpha}}, \\
R_5(y) &= x^{\frac{k-3}{4\alpha}} M^{\max(\frac{k-3}{4\alpha},0)+\varepsilon} + x^{\frac{k+1}{4\alpha-2}} M^{\frac{k+1}{4\alpha}+\varepsilon} + x^{\frac{k-1}{4\alpha}-\frac{1}{2}} M^{\omega_1-\frac{3}{2}+\frac{k-1}{4\alpha}} \\
&\quad + x^{(4\alpha-1)(1+\omega_1)-2K+\frac{k}{2\alpha}+\frac{2K}{\alpha}-6\alpha}, \\
R_6(y) &= 0, \\
R_7(y) &= \Delta(\boldsymbol{a}; y) - \int_{\mathbf{E}_K} \Delta(\boldsymbol{a}; \tilde{y}) d\mathrm{Y}_K,
\end{aligned}
$$

where $M' = M/\pi^{2\alpha}$ *and* $\kappa_0 \neq 0, c_{00}, c_{lm}, c'_{jm}$ *are certain constants,* K *is a suitably large integer and* $\omega_1 < 1$ *is a certain constant.*

We need one remark on $R_6(y)$. In fact in [2] $R_6(y)$ is given by

$$
R_6(y) \ll \begin{cases} 0 & \text{if } b(n) \geqslant 0, \\ x^{\theta_0} M^{\omega_0 - 1 + \frac{k-1}{4\alpha}}, & \text{if } b(n) \ll n^{\omega_0}. \end{cases}
$$

In our case we can take $R_6(y) = 0$ since $b(n) = \pi^{2\alpha - k/2}\hat{d}(\boldsymbol{a}, n)$ is always non-negative.

We recall important estimates of the integral of $I(\lambda, M, N, y)$ which we will need in the next section.

Lemma 2. *Let* $M < N < x^A$*, where* A *is a fixed positive number,* w *be a real number and* $0 < \mu < \frac{M}{2}$*. Then we have*

$$
\int_x^{(1+\delta)x} I(\lambda, M, N, y) y^w \cos(h(\mu y)^{1/2\alpha} + c_0\pi) dy \ll x^{w+1-3/4\alpha+\varepsilon} \max_{M \leqslant P \leqslant N} P^{\lambda+\sigma^*+1-3/4\alpha}.
$$

Lemma 3. *Let* $2(\lambda+\sigma^*) \neq -1,\ M < N < x^A$*, where* A *is a fixed positive number, and* $\delta > 0$ *with* $(1+\delta)^{1/\alpha} - 1 < 1/4$*. Then we have*

$$
\int_x^{(1+\delta)x} |I(\lambda, M, N, y)|^2 dy \ll x^{1-1/\alpha+\varepsilon} \max_{M \leqslant P \leqslant N} P^{2(\lambda+\sigma^*+1)-1/\alpha}.
$$

Lemma 4. *Let* $2(\lambda+\sigma^*) \neq -1, 2(\lambda+\sigma^*+1) < 1/\alpha, M \geqslant 1$ *and* $\delta > 0$ *with* $(1+\delta)^{1/\alpha} - 1 < 1/4$*. Then we have*

$$
\int_x^{(1+\delta)x} |I(\lambda, M, \infty, y)|^2 dy \ll x^{1-1/\alpha+\varepsilon} M^{2(\lambda+\sigma^*+1)-1/\alpha}.
$$

These lemmas are Lemmas 8, 9 and 10 of [2], respectively. See [2] for details.

3. Mean square of $\boldsymbol{\Delta(a, x)}$

In the asymmetric multidimensional divisor problem, the number $(\mu' - \mu)/2 = -\alpha + k/2$ plays an important role. Although the proof of Theorem 1 is similar to that of Theorem 1 in [2], we shall give all details for the sake of completeness.

Let

$$K_1(y) = R_1(y) + R_2(y)$$

and

$$K_2(y) = \sum_{j=3}^{7} R_j(y).$$

It is sufficient to evaluate the integral $\int_x^{(1+\delta)x} (K_1(y) + K_2(y))^2 dy$ for $1 \leqslant x < T$, where δ is some fixed small positive number.

We need the upper bound of the summatory function of $\hat{d}^{\,2}(\boldsymbol{a}, n)$. Moreover, we have

Lemma 5. *Let $x > 1$. Then we have*

$$x^{2-1/a_k} \ll \sum_{n \leqslant x} \hat{d}^{\,2}(\boldsymbol{a}; n) \ll x^{2-1/a_k+\varepsilon}. \tag{3.1}$$

Proof. By Cauchy's inequality we get

$$\begin{aligned}
\hat{d}^{\,2}(\boldsymbol{a}; n) &= \Bigg(\sum_{n_1^{a_1} \cdots n_k^{a_k} = n} n_1^{a_1-1} \cdots n_k^{a_k-1} \Bigg)^2 \\
&\leqslant \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} 1 \times \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} n_1^{2(a_1-1)} \cdots n_k^{2(a_k-1)} \\
&\ll n^{\varepsilon} c(\boldsymbol{a}; n),
\end{aligned}$$

where $c(\boldsymbol{a}; n) = \sum_{n_1^{a_1} \cdots n_k^{a_k} = n} n_1^{2(a_1-1)} \cdots n_k^{2(a_k-1)}$. We also note that $\hat{d}^{\,2}(\boldsymbol{a}; n) \geqslant c(\boldsymbol{a}; n)$. It is easy to see that the generating Dirichlet series of $c(\boldsymbol{a}; n)$ has the form

$$\sum_{n=1}^{\infty} \frac{c(\boldsymbol{a}; n)}{n^s} = \prod_{j=1}^{k} \zeta(a_j s - 2(a_j - 1)), \qquad \mathrm{Re}(s) > 2 - 1/a_k.$$

This Dirichlet series has poles at points $2 - 1/a_j \quad (j = 1, \dots, k)$, hence

$$\sum_{n \leqslant x} c(\boldsymbol{a}; n) = c x^{2-1/a_k} \log^{A-1} x \cdot (1 + o(1))$$

where c is some constant and A is the number of j such that $a_j = a_k$. Therefore Lemma 5 follows. ∎

Let σ^* be the number defined by (1.7) which satisfies (1.9). The inequality (1.9) is equivalent to

$$2(\lambda_0 + \sigma^* + 1) < \frac{1}{\alpha}, \tag{3.2}$$

where λ_0 was defined by (2.5).

3.1. Evaluation of $\int_x^{(1+\delta)x} K_1^2(y)dy$

Let $\kappa_0' = \kappa_0 \pi^{2\alpha(\theta_0 - 1)}$ for simplicity. By using the identity

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

we get

$$\begin{aligned} R_1(y)^2 &= \frac{{\kappa_0'}^2}{2} y^{\frac{k-1}{2\alpha}} \sum_{n\leqslant M'} \sum_{m\leqslant M'} \frac{b(n)b(m)}{(nm)^{1-\frac{k-1}{4\alpha}}} \Big(\cos(h\pi y^{1/2\alpha}(n^{1/2\alpha} - m^{1/2\alpha})) \\ &\quad + \cos(h\pi y^{1/2\alpha}(n^{1/2\alpha} + m^{1/2\alpha}) + 2c_0\pi)\Big) \\ &= \frac{{\kappa_0'}^2}{2} \left(W_1(y) + W_2(y) + W_3(y)\right), \end{aligned}$$

where

$$\begin{aligned} W_1(y) &= y^{\frac{k-1}{2\alpha}} \sum_{n\leqslant M'} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}}, \\ W_2(y) &= y^{\frac{k-1}{2\alpha}} \sum\sum_{\substack{n,m\leqslant M' \\ n\neq m}} \frac{b(n)b(m)}{(nm)^{1-\frac{k-1}{4\alpha}}} \cos(h\pi y^{1/2\alpha}(n^{1/2\alpha} - m^{1/2\alpha})), \\ W_3(y) &= y^{\frac{k-1}{2\alpha}} \sum\sum_{n,m\leqslant M'} \frac{b(n)b(m)}{(nm)^{1-\frac{k-1}{4\alpha}}} \cos(h\pi y^{1/2\alpha}(n^{1/2\alpha} + m^{1/2\alpha}) + 2c_0\pi). \end{aligned}$$

For the integral of $W_1(y)$, we have

$$\int_x^{(1+\delta)x} W_1(y)dy = \sum_{n\leqslant M'} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} \int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}} dy.$$

Since (1.10) is equivalent to $\frac{k-1}{2\alpha} < \frac{1}{a_k}$, we find that the series $\sum_n^\infty \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}}$ is convergent. So from (3.1), we have

$$\sum_{n\leqslant M'} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} = \sum_{n=1}^\infty \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} + O(M^{\frac{k-1}{2\alpha} - \frac{1}{a_k} + \varepsilon}).$$

Hence

$$\int_x^{(1+\delta)x} W_1(y)dy = \sum_{n=1}^{\infty} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} \int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}} dy + O(x^{1+\frac{k-1}{2\alpha}} M^{\frac{k-1}{2\alpha}-\frac{1}{a_k}+\varepsilon}). \quad (3.3)$$

By the first derivative test, we have

$$\int_x^{(1+\delta)x} W_2(y)dy \ll x^{\frac{k-1}{2\alpha}+1-\frac{1}{2\alpha}} \sum_{\substack{m,n\leqslant M' \\ m\neq n}}\sum \frac{b(n)b(m)}{(nm)^{1-\frac{k-1}{4\alpha}}} \frac{1}{|n^{1/2\alpha}-m^{1/2\alpha}|}$$
$$= x^{\frac{k-2}{2\alpha}+1} \left\{\Sigma_1 + \Sigma_2\right\},$$

where the summation conditions of Σ_1 and Σ_2 are given by

$$SC(\Sigma_1) : |n^{1/2\alpha} - m^{1/2\alpha}| > \frac{1}{10}(nm)^{1/4\alpha}$$

and

$$SC(\Sigma_2) : |n^{1/2\alpha} - m^{1/2\alpha}| \leqslant \frac{1}{10}(nm)^{1/4\alpha},$$

respectively. It is not hard to see that

$$\Sigma_1 \ll \sum_{\substack{n,m\leqslant M' \\ |n^{1/2\alpha}-m^{1/2\alpha}|>\frac{1}{10}(nm)^{1/4\alpha}}}\sum \frac{b(n)b(m)}{(nm)^{1-\frac{k-1}{4\alpha}}} \frac{1}{(nm)^{\frac{1}{4\alpha}}}$$
$$\ll \left(\sum_{n\leqslant M'} \frac{b(n)}{n^{1-\frac{k-2}{4\alpha}}}\right)^2 \ll M^{\frac{k-2}{2\alpha}+\varepsilon},$$

where we used the trivial estimate $\sum_{n\leqslant x} b(n) \ll x^{1+\varepsilon}$. Next we consider Σ_2. By Lagrange's mean value theorem we have

$$n^{1/2\alpha} - m^{1/2\alpha} = \frac{1}{2\alpha} u_0^{1/2\alpha-1}(n-m)$$

for some u_0 between n and m. Since $n \asymp m$ by $SC(\Sigma_2)$, we find

$$|n^{1/2\alpha} - m^{1/2\alpha}| \geqslant (nm)^{1/4\alpha-1/2}|n-m|,$$

thus we get

$$\Sigma_2 \ll \sum_{\substack{n,m\leqslant M' \\ n\neq m}}\sum \frac{b(n)b(m)}{(nm)^{\frac{1}{2}-\frac{k-2}{4\alpha}}} \frac{1}{|n-m|}$$
$$\ll \sum_{\substack{n,m\leqslant M' \\ n\neq m}}\sum \left\{ \left(\frac{b(n)}{n^{\frac{1}{2}-\frac{k-2}{4\alpha}}}\right)^2 + \left(\frac{b(m)}{m^{\frac{1}{2}-\frac{k-2}{4\alpha}}}\right)^2 \right\} \frac{1}{|n-m|}.$$

By the symmetry of n and m and then using Lemma 5 we obtain

$$\Sigma_2 \ll \sum\sum_{\substack{n,m\leqslant M' \\ n\neq m}} \frac{b(n)^2}{n^{1-\frac{k-2}{2\alpha}}} \frac{1}{|n-m|} \ll M^{1-\frac{1}{a_k}+\frac{k-2}{2\alpha}+\varepsilon}.$$

Here we note that the exponent of M is $1-1/a_k+(k-2)/2\alpha \geqslant 0$ and Σ_2 is greater than Σ_1. Hence

$$\int_x^{(1+\delta)x} W_2(y)dy \ll x^{\frac{k-2}{2\alpha}+1} M^{1-\frac{1}{a_k}+\frac{k-2}{2\alpha}+\varepsilon}. \tag{3.4}$$

It is easy to see that $\int_x^{(1+\delta)x} W_3(y)dy$ is absorbed into the right hand side of (3.4).

From (3.3) and (3.4), we get

$$\int_x^{(1+\delta)x} R_1^2(y)dy = \frac{{\kappa_0'}^2}{2} \sum_{n=1}^{\infty} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} \int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}} dy$$
$$+ O\left(x^{\frac{k-1}{2\alpha}+1+\varepsilon} M^{\frac{k-1}{2\alpha}-\frac{1}{a_k}}\right) + O\left(x^{\frac{k-2}{2\alpha}+1+\varepsilon} M^{\frac{k-2}{2\alpha}+1-\frac{1}{a_k}}\right). \tag{3.5}$$

Now we consider the mean square of $R_2(y)$. By Cauchy's inequality and Lemma 3, we have

$$\int_x^{(1+\delta)x} R_2^2(y)dy \ll x^{\frac{k-1}{2\alpha}+\frac{1}{\alpha}} \int_x^{(1+\delta)x} |I(\lambda_0, M, N, y)|^2 dy$$
$$\ll x^{\frac{k-1}{2\alpha}+\frac{1}{\alpha}} x^{1-\frac{1}{\alpha}+\varepsilon} \max_{M\leqslant P\leqslant N} P^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}}.$$

From (2.5) and assumption (1.9), we have

$$2(\lambda_0+\sigma^*+1) - 1/\alpha < -1/a_k + (k-1)/2\alpha < 0.$$

Therefore

$$\int_x^{(1+\delta)x} R_2^2(y)dy \ll x^{\frac{k-1}{2\alpha}+1+\varepsilon} M^{2\sigma^*-2+\frac{k-1}{2\alpha}}. \tag{3.6}$$

Finally we consider $\int_x^{(1+\delta)x} R_1(y)R_2(y)dy$. From definitions of $R_1(y)$ and $R_2(y)$, we have

$$\int_x^{(1+\delta)x} R_1(y)R_2(y)dy$$
$$= \mathrm{Re}\,\kappa_0' c_{00} \int_x^{(1+\delta)x} y^{\frac{k}{2\alpha}} I(\lambda_0, M, N, y) \sum_{n\leqslant M'} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} \cos(h\pi(ny)^{1/2\alpha} + c_0\pi)dy$$
$$= Re\kappa_0' c_{00}(I_1 + I_2),$$

where

$$I_1 = \int_x^{(1+\delta)x} y^{\frac{k}{2\alpha}} I(\lambda_0, M, N, y) \sum_{n \leqslant M'/2} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} \cos(h\pi(ny)^{1/2\alpha} + c_0\pi) dy$$

and

$$I_2 = \int_x^{(1+\delta)x} y^{\frac{k}{2\alpha}} I(\lambda_0, M, N, y) \sum_{M'/2 < n \leqslant M'} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} \cos(h\pi(ny)^{1/2\alpha} + c_0\pi) dy.$$

By Lemma 2 we have

$$I_1 \ll \sum_{n \leqslant M'} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} x^{\frac{k}{2\alpha}+1-\frac{3}{4\alpha}+\varepsilon} \max_{M \leqslant P \leqslant N} P^{\lambda_0+\sigma^*+1-\frac{3}{4\alpha}}.$$

By assumption (1.9), the exponent of P in the above estimate is negative. Hence by using $\sum_{n \leqslant x} b(n) \ll x^{1+\varepsilon}$ again, we get

$$\begin{aligned} I_1 &\ll x^{\frac{2k-3}{4\alpha}+1+\varepsilon} M^{\lambda_0+\sigma^*+1-3/4\alpha} \sum_{n \leqslant M'/2} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} \\ &\ll x^{\frac{2k-3}{4\alpha}+1+\varepsilon} M^{\sigma^*-1+\frac{2k-3}{4\alpha}}. \end{aligned} \tag{3.7}$$

By applying Cauchy's inequality to I_2, we have

$$I_2 \ll x^{\frac{k}{2\alpha}} (V_1 V_2)^{1/2}, \tag{3.8}$$

where

$$V_1 = \int_x^{(1+\delta)x} |I(\lambda_0, M.N, y)|^2 dy$$

and

$$V_2 = \int_x^{(1+\delta)x} \left| \sum_{M'/2 < n \leqslant M'} \frac{b(n)}{n^{1-\frac{k-1}{4\alpha}}} \cos(h\pi(ny)^{1/2\alpha} + c_0\pi) \right|^2 dy.$$

Applying Lemma 3 to V_1 we get

$$V_1 \ll x^{1-\frac{1}{\alpha}+\varepsilon} M^{2\sigma^*-2+\frac{k-1}{2\alpha}}. \tag{3.9}$$

The value of V_2 can be bounded by the same approach as the mean square of $R_1(y)$ and we get

$$V_2 \ll x M^{\frac{k-1}{2\alpha}-\frac{1}{a_k}+\varepsilon} + x^{1-\frac{1}{2\alpha}+\varepsilon} M^{1-\frac{1}{a_k}+\frac{k-2}{2\alpha}}. \tag{3.10}$$

By (3.8), (3.9) and (3.10) we get

$$I_2 \ll x^{1+\frac{k-1}{2\alpha}+\varepsilon} M^{\sigma^*-1+\frac{k-1}{2\alpha}-\frac{1}{2a_k}} + x^{1+\frac{2k-3}{4\alpha}+\varepsilon} M^{\sigma^*-\frac{1}{2}+\frac{2k-3}{4\alpha}-\frac{1}{2a_k}}. \tag{3.11}$$

From the estimates (3.5), (3.6), (3.7) and (3.11) we get

$$\int_x^{(1+\delta)x} K_1^2(y)dy = \frac{{\kappa_0'}^2}{2}\sum_{n=1}^{\infty}\frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}}\int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}}dy + O\left(x^{\frac{k-2}{2\alpha}+1+\varepsilon}M^{\frac{k-2}{2\alpha}+1-\frac{1}{a_k}}\right) + O\left(x^{\frac{k-1}{2\alpha}+1+\varepsilon}M^{2\sigma^*-2+\frac{k-1}{2\alpha}}\right), \tag{3.12}$$

where we used the facts $1-1/2a_k \leqslant \sigma^*$ and

$$x^{1+\frac{2k-3}{4\alpha}}M^{\sigma^*-\frac{1}{2}+\frac{2k-3}{4\alpha}-\frac{1}{2a_k}} = \left(x^{\frac{k-2}{2\alpha}+1}M^{\frac{k-2}{2\alpha}+1-\frac{1}{a_k}}\right)^{1/2}\left(x^{\frac{k-1}{2\alpha}+1}M^{2\sigma^*-2+\frac{k-1}{2\alpha}}\right)^{1/2}.$$

All error terms in (3.5), (3.7) and (3.11) are bounded by the two error terms in (3.12).

3.2. Evaluation of $\int_x^{(1+\delta)x} K_2^2(y)dy$

We first give the upper bounds of $\int_x^{(1+\delta)x} R_j^2(y)dy$ $(j = 3,\ldots,7)$. By Cauchy's inequality and Lemma 3, we have

$$\begin{aligned}\int_x^{(1+\delta)x} R_3^2(y)dy &\ll \sum_{\substack{0\leqslant l,m\leqslant J\\ l+m>0}}\sum x^{-4l+\frac{k+1}{2\alpha}+\frac{l-m}{\alpha}}\int_x^{(1+\delta)x}\left|I(\lambda_0+\frac{l-m}{2\alpha},M,N,y)\right|^2 dy\\ &\ll \sum_{\substack{0\leqslant l,m\leqslant J\\ l+m>0}}\sum x^{-4l+\frac{k+1}{2\alpha}+\frac{l-m}{\alpha}}x^{1-\frac{1}{\alpha}+\varepsilon}\max_{M\leqslant P\leqslant N}P^{2(\lambda_0+\frac{l-m}{2\alpha}+\sigma^*+1)-\frac{1}{\alpha}}\\ &= \Sigma_3+\Sigma_4,\end{aligned}$$

where the summation conditions are

$$SC(\Sigma_3): 0\leqslant l\leqslant m\leqslant J,\ l+m>0 \qquad \text{and} \qquad SC(\Sigma_4): 0\leqslant m< l\leqslant J,$$

respectively. Since we have assumed $2(\lambda_0+\sigma^*+1)<1/\alpha$, we have

$$\begin{aligned}\Sigma_3 &\ll \sum_{\substack{0\leqslant m\leqslant l\leqslant J\\ l+m>0}} x^{-4l+\frac{k-1}{2\alpha}+\frac{l-m}{\alpha}+1+\varepsilon}M^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}+\frac{l-m}{\alpha}}\\ &= x^{\frac{k-1}{2\alpha}+1+\varepsilon}M^{2(\sigma^*-1)+\frac{k-1}{2\alpha}}\sum_{\substack{0\leqslant m\leqslant l\leqslant J\\ l+m>0}} x^{-4l+\frac{l-m}{\alpha}}M^{\frac{l-m}{\alpha}}.\end{aligned}$$

The sum over l and m in the above formula is bounded by

$$\ll (xM)^{-1/\alpha}+x^{-4}\ll (xM)^{-1/\alpha}.$$

So we have

$$\Sigma_3 \ll x^{\frac{k-3}{2\alpha}+1+\varepsilon}M^{2(\sigma^*-1)+\frac{k-3}{2\alpha}}. \tag{3.13}$$

Next we treat Σ_4. Since

$$2(\lambda_0+\sigma^*+\frac{l-m}{2\alpha}+1)-\frac{1}{\alpha}\geqslant\frac{(a_k-a_1)+\cdots+(a_k-a_{k-1})+a_k}{(a_1+\cdots+a_k)a_k}>0,$$

we have

$$\begin{aligned}\Sigma_4&\ll\sum_{0\leqslant m<l\leqslant J}x^{-4l+\frac{k-1}{2\alpha}+1+\frac{l-m}{\alpha}+\varepsilon}N^{2(\lambda_0+\sigma^*+\frac{l-m}{2\alpha}+1)-\frac{1}{\alpha}}\\&=x^{\frac{k-1}{2\alpha}+1+\varepsilon}N^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}}\sum_{0\leqslant m<l\leqslant J}x^{-4l+\frac{l-m}{\alpha}}N^{\frac{l-m}{\alpha}}.\end{aligned}$$

Having in mind that $N=[x^{4\alpha-1-\varepsilon}]$, the sum over l and m is $O(1)$. So

$$\Sigma_4\ll x^{\frac{k-1}{2\alpha}+1+\varepsilon}N^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}}. \tag{3.14}$$

From (3.13), (3.14) and assumption $M\leqslant\sqrt{N}$ we get

$$\int_x^{(1+\delta)x}R_3^2(y)dy\ll x^{\frac{k-3}{2\alpha}+1+\varepsilon}M^{2(\sigma^*-1)+\frac{k-3}{2\alpha}}+x^{\frac{k-1}{2\alpha}+1+\varepsilon}M^{4(\sigma^*-1)+\frac{k-1}{\alpha}}. \tag{3.15}$$

By Lemma 4 we have

$$\begin{aligned}\int_x^{(1+\delta)x}R_4^2(y)dy&\ll\sum_{j,m=0}^{K}x^{4K+\frac{k-1}{2\alpha}+\frac{1}{\alpha}-\frac{K+m}{\alpha}}\\&\quad\times\int_x^{(1+\delta)x}\left|I\left(\lambda_0-\frac{K+m}{2\alpha},N,\infty,y+\frac{j}{x}\right)\right|^2dy\\&\ll\sum_{j,m=0}^{K}x^{4K+\frac{k-1}{2\alpha}+\frac{1}{\alpha}-\frac{K+m}{\alpha}}x^{1-\frac{1}{\alpha}+\varepsilon}N^{2(\lambda_0-\frac{K+m}{2\alpha}+\sigma^*+1)-\frac{1}{\alpha}}\\&=x^{4K+\frac{k-1}{2\alpha}+1-\frac{K}{\alpha}+\varepsilon}N^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}-\frac{K}{\alpha}}\sum_{j,m=0}^{K}(xN)^{-m/\alpha}.\end{aligned}$$

Since the sum over j and m is bounded, we get by the definition of N that

$$\begin{aligned}\int_x^{(1+\delta)x}R_4^2(y)dy&\ll x^{4K+\frac{k-1}{2\alpha}+1-\frac{K}{\alpha}+\varepsilon}N^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}}x^{-(4\alpha-1-\varepsilon)\frac{K}{\alpha}}\\&\ll x^{\frac{k-1}{2\alpha}+1-\frac{K}{\alpha}+\varepsilon}N^{2(\lambda_0+\sigma^*+1)-\frac{1}{\alpha}}.\end{aligned} \tag{3.16}$$

Now consider $R_5(y)$. By taking K large, we have

$$R_5(y)\ll x^{\frac{k-3}{2\alpha}}M^{\max(\frac{k-3}{4\alpha},0)+\varepsilon}+x^{\frac{k+1}{4\alpha}-2}M^{\frac{k+1}{4\alpha}+\varepsilon}+x^{\frac{k-1}{4\alpha}-\frac{1}{2}}M^{-\frac{1}{2}+\frac{k-1}{4\alpha}}.$$

It is easy to see that

$$R_5(y)\ll\begin{cases}x^{-1/4\alpha}&\text{if }k=2\\x^{\frac{k-3}{4\alpha}}M^{\frac{k-3}{4\alpha}}&\text{if }k\geqslant3\text{ and }M\ll x^{2\alpha-1}.\end{cases}$$

Hence

$$\int_x^{(1+\delta)x} R_5^2(y)dy \ll \begin{cases} x^{1-1/2\alpha} & \text{if } k=2 \\ x^{1+\frac{k-3}{2\alpha}} M^{\frac{k-3}{2\alpha}} & \text{if } k \geqslant 3 \text{ and } M \ll x^{2\alpha-1}. \end{cases} \tag{3.17}$$

By the choice of M, $R_6(y)=0$, so its mean square is bounded trivially.

By the same method as in [2], we have

$$\int_x^{(1+\delta)x} R_7^2(y)dy \ll x^{\varepsilon}. \tag{3.18}$$

The first error term in the right hand side of (3.15) is clearly bounded by the term in the right hand side of (3.17). Hence from (3.15), (3.16), (3.17) and (3.18) we get

$$\int_x^{(1+\delta)x} K_2^2(y)dy \ll x^{\frac{k-1}{2\alpha}+1+\varepsilon} M^{4(\sigma^*-1)+\frac{k-1}{\alpha}} + \begin{cases} x^{1-1/2\alpha} & \text{if } k=2 \\ x^{1+\frac{k-3}{2\alpha}} M^{\frac{k-3}{2\alpha}} & \text{if } k \geqslant 3 \text{ and } M \ll x^{2\alpha-1}. \end{cases} \tag{3.19}$$

3.3. Proof of Theorem 1

Choose M such that two error terms in (3.12) are of the same order, namely,

$$x^{\frac{k-2}{2\alpha}+1} M^{\frac{k-2}{2\alpha}+1-\frac{1}{a_k}} \asymp x^{\frac{k-1}{2\alpha}+1} M^{2(\sigma^*-1)+\frac{k-1}{2\alpha}}. \tag{3.20}$$

The above formula gives

$$M \asymp x^{\frac{1}{2\alpha(3-2\sigma^*-1/a_k)-1}}. \tag{3.21}$$

Clearly M satisfies $M \ll x^{2\alpha-1} \ll \sqrt{N}$. Therefore (3.12) becomes

$$\int_x^{(1+\delta)x} K_1^2(y)dy = \frac{{\kappa_0'}^2}{2} \sum_{n=1}^{\infty} \frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}} \int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}} dy + O\left(x^{1+\frac{k-1}{2\alpha}-\eta(\boldsymbol{a})+\varepsilon}\right), \tag{3.22}$$

where $\eta(\boldsymbol{a})$ is given by (1.12).

By Cauchy's inequality, formula (3.22) and bound (3.19) we have

$$\begin{aligned} \int_x^{(1+\delta)x} K_1(y)K_2(y)dy &\ll \left(\int_x^{(1+\delta)x} K_1^2(y)dy\right)^{1/2} \left(\int_x^{(1+\delta)x} K_2^2(y)dy\right)^{1/2} \\ &\ll x^{1+\frac{k-1}{2\alpha}+\varepsilon} M^{2(\sigma^*-1)+\frac{k-1}{2\alpha}} + \begin{cases} x & \text{if } k=2 \\ x^{1+\frac{k-2}{2\alpha}} M^{\frac{k-3}{4\alpha}} & \text{if } k \geqslant 3 \end{cases} \\ &\ll x^{1+\frac{k-1}{2\alpha}-\eta(\boldsymbol{a})+\varepsilon}, \end{aligned} \tag{3.23}$$

where in the last step we have used (3.20).

We also have

$$\int_x^{(1+\delta)x} K_2^2(y)dy \ll x^{1+\frac{k-1}{2\alpha}-\eta(\boldsymbol{a})+\varepsilon}. \tag{3.24}$$

Consider the first error term in (3.19) first. Since the exponent of M is negative, it is bounded by the term in the right hand side of (3.24). Next consider the second error term of (3.19). For $k=2$ there is nothing to prove. For $k \geqslant 3$, it is enough to show that

$$\frac{1}{\alpha} - \frac{k-3}{2\alpha} \cdot \frac{1}{2\alpha(3-2\sigma^* - 1/a_k) - 1} > \eta(\boldsymbol{a}),$$

or equivalently $2 - 1/a_k > \sigma^*$. This is true under assumption (1.9).

From (3.22)-(3.24) we get immediately that

$$\int_x^{(1+\delta)x} \Delta^2(\mathbf{a};y)dy = \frac{{\kappa_0'}^2}{2}\sum_{n=1}^{\infty}\frac{b(n)^2}{n^{2-\frac{k-1}{2\alpha}}}\int_x^{(1+\delta)x} y^{\frac{k-1}{2\alpha}}dy + O\left(x^{1+\frac{k-1}{2\alpha}-\eta(\boldsymbol{a})+\varepsilon}\right),$$

which implies Theorem 1 by a splitting argument. This completes the proof of Theorem 1.

4. Proof of Theorem 3

In order to prove Theorem 3 we need some preparations. Define $m(\sigma)$ (for $1/2 \leqslant \sigma < 1$) as the supremum of all numbers m such that

$$\int_1^T |\zeta(\sigma+it)|^m dt \ll T^{1+\varepsilon}.$$

It is known that $m(\sigma) \geqslant 4$ for $\sigma \geqslant 1/2$, $m(7/12) \geqslant 6$ and $m(5/8) \geqslant 8$. Ivić studied $m(\sigma)$ in great detail. Without loss of generality we can assume that $m(\sigma)$ is a continuous function of σ. One can find a lower bound of $m(\sigma)$ in [7, Theorem 8.4]. Especially we have the following simpler but a little weaker form:

$$m(\sigma) \geqslant \begin{cases} \frac{4}{3-4\sigma} & \text{if } \frac{1}{2} \leqslant \sigma \leqslant \frac{5}{8} \\ \frac{3}{1-\sigma} & \text{if } \frac{5}{8} \leqslant \sigma < 1. \end{cases} \tag{4.1}$$

The following lemma is used essentially in Ivić's argument [8].

Lemma 6. *Let a_j $(1 \leqslant j \leqslant k)$ be positive integers such that $a_1 \leqslant \cdots \leqslant a_k$ and let $\psi(s)$ and σ^* be defined by (1.6) and (1.7), respectively. Define the function $H(\sigma)$ by*

$$H(\sigma) = \sum_{j=1}^{k} \frac{1}{m(a_j\sigma - a_j + 1)}.$$

If

$$H(\sigma) \leqslant 1/2$$

for some σ, we have $\sigma^* \leqslant \sigma$.

Proof. We write $\sigma_j = a_j\sigma - a_j + 1$ for simplicity. Suppose that

$$\sum_{j=1}^{k} \frac{1}{m(\sigma_j)} \leqslant \frac{1}{2}.$$

Then by Hölder's inequality, we have

$$\begin{aligned}\int_1^T |\psi(s)|^2 dt &= \int_1^T \prod_{j=1}^{k} |\zeta(\sigma_j + ia_j t)|^2 dt \\ &\leqslant \prod_{j=1}^{k} \left(\int_1^T |\zeta(\sigma_j + ia_j t)|^{m(\sigma_j)} dt \right)^{\frac{2}{m(\sigma_j)}} \left(\int_1^T 1 dt \right)^{1-\sum_{j=1}^{k} \frac{2}{m(\sigma_j)}} \\ &\ll T^{1+\varepsilon}.\end{aligned}$$

Hence from the definition of σ^*, we have $\sigma^* \leqslant \sigma$. ■

We remark that since $H(\sigma)$ is decreasing, if

$$H\left(1 - \frac{k-1}{4\alpha}\right) < \frac{1}{2},$$

then Theorem 1 holds.

Lemma 7. *Let $k = 3$, $a_1 \leqslant a_2 \leqslant a_3$ and $a_3 < a_1 + a_2$. Let σ^* be defined by (1.7). Then we have*

$$\sigma^* \begin{cases} \leqslant 1 - \frac{5}{4(a_1+a_2+a_3)} & \textit{if } 3(a_2+a_3) \leqslant 7a_1, \\ \leqslant 1 - \frac{3}{a_1+3a_2+3a_3} & \textit{if } 3(a_2+a_3) > 7a_1,\ 3a_3 + a_1 \leqslant 5a_2 \textit{ and } 3a_3 < a_1 + 3a_2, \\ = 1 - \frac{1}{2a_3} & \textit{otherwise.} \end{cases} \tag{4.2}$$

Proof. Let $a_1 \leqslant a_2 \leqslant a_3$ and $a_1 + a_2 > a_3$. By Lemma 6 we shall find σ such that

$$1 - \frac{1}{2a_3} \leqslant \sigma < 1 - \frac{1}{a_1 + a_2 + a_3}, \qquad H(\sigma) \leqslant 1/2.$$

For the sake of simplicity we put $\sigma_j = a_j\sigma - a_j + 1$ $(j = 1, 2, 3)$ for $\sigma \in [\frac{1}{2}, 1]$ as before. It is easy to see that $\frac{1}{2} \leqslant \sigma_3 \leqslant \sigma_2 \leqslant \sigma_1 < 1$.

We shall use the weak version (4.1).

Case 1: We first consider the case $3(a_2 + a_3) \leqslant 7a_1$ and we put

$$\sigma := 1 - \frac{5}{4(a_1 + a_2 + a_3)}.$$

Clearly $\sigma < 1 - 1/(a_1 + a_2 + a_3)$. Since $3a_3 \leqslant 7a_1 - 3a_2 \leqslant (2a_1 + 5a_2) - 3a_2 = 2(a_1 + a_2)$, we have $\sigma \geqslant 1 - \frac{1}{2a_3}$ and $\sigma_1 \leqslant \frac{5}{8}$. By (4.1) we have

$$H(\sigma) = \sum_{j=1}^{3} \frac{1}{m(\sigma_j)} \leqslant \frac{3 - 4\sigma_1}{4} + \frac{3 - 4\sigma_2}{4} + \frac{3 - 4\sigma_3}{4} = \frac{1}{2}.$$

Hence we get $\sigma^* \leqslant \sigma$.

Case 2: When $3(a_2 + a_3) > 7a_1$, $3a_3 + a_1 \leqslant 5a_2$ and $3a_3 < a_1 + 3a_2$, we put

$$\sigma := 1 - \frac{3}{a_1 + 3a_2 + 3a_3}.$$

It is clear that $\sigma < 1 - 1/(a_1 + a_2 + a_3)$ and $\sigma > 1 - \frac{1}{2a_3}$ by the last condition. One can check that the first two conditions imply that $\frac{5}{8} < \sigma_1 < 1$ and $\frac{1}{2} \leqslant \sigma_3 \leqslant \sigma_2 \leqslant \frac{5}{8}$. Hence

$$H(\sigma) = \sum_{j=1}^{3} \frac{1}{m(\sigma_j)} \leqslant \frac{1 - \sigma_1}{3} + \frac{3 - 4\sigma_2}{4} + \frac{3 - 4\sigma_3}{4} = \frac{1}{2}.$$

Hence we get $\sigma^* \leqslant \sigma$.

Case 3: We consider the case $3(a_2 + a_3) > 7a_1$, $3a_3 + a_1 \leqslant 5a_2$ and $3a_3 \geqslant a_1 + 3a_2$. In this case we put

$$\sigma := 1 - \frac{1}{2a_3}.$$

Note that this is the best possible choice. Using the last condition we easily check that

$$3a_3 \geqslant a_1 + 3a_2 \geqslant 4a_1$$

and hence

$$\sigma_1 = a_1 \left(1 - \frac{1}{2a_3}\right) - a_1 + 1 = 1 - \frac{a_1}{2a_3} \geqslant \frac{5}{8}.$$

Now we consider two cases.

(i) If $3a_3 \leqslant 4a_2$, then $\sigma_2 \leqslant \frac{5}{8}$. By the third condition we get

$$H(\sigma) = \sum_{j=1}^{3} \frac{1}{m(\sigma_j)} \leqslant \frac{1 - \sigma_1}{3} + \frac{3 - 4\sigma_2}{4} + \frac{1}{4} = \frac{a_1 + 3a_2}{6a_3} \leqslant \frac{1}{2}.$$

(ii) If $3a_3 > 4a_2$, then $\sigma_2 > \frac{5}{8}$. By the third condition we have $3a_3 \geqslant a_1 + 3a_2 \geqslant 2(a_1 + a_2)$. Hence

$$\begin{aligned} H(\sigma) = \sum_{j=1}^{3} \frac{1}{m(\sigma_j)} &\leqslant \frac{1 - \sigma_1}{3} + \frac{1 - \sigma_2}{3} + \frac{1}{4} \\ &= \frac{1}{4} + \frac{a_1 + a_2}{6a_3} \leqslant \frac{1}{4} + \frac{1}{6} \cdot \frac{3}{2} = \frac{1}{2}. \end{aligned}$$

Combining the two cases (i) and (ii), we have $\sigma^* = \sigma = 1 - 1/(2a_3)$.

Case 4: Finally we consider the case $3(a_2+a_3)>7a_1$, $3a_3+a_1>5a_2$, where we put

$$\sigma := 1-\frac{1}{2a_3}.$$

In this case, using the second condition, we easily check that

$$3a_3>5a_2-a_1\geqslant 4a_2$$

and hence

$$\sigma_2 = a_2\left(1-\frac{1}{2a_3}\right)-a_2+1 = 1-\frac{a_2}{2a_3}>\frac{5}{8}.$$

We have $1>\sigma_2>\frac{5}{8}, 1>\sigma_1>\frac{5}{8}$, and $\sigma_3=\frac{1}{2}$. Hence

$$H(\sigma)=\sum_{j=1}^{3}\frac{1}{m(\sigma_j)}\leqslant\frac{1-\sigma_1}{3}+\frac{1-\sigma_2}{3}+\frac{1}{4}\leqslant\frac{1}{8}+\frac{1}{8}+\frac{1}{4}=\frac{1}{2}.$$

Therefore we have $\sigma^*=\sigma=1-1/(2a_3)$. ■

Proof of Theorem 3. Now the proof of Theorem 3 is immediate by substituting each value on the right hand side of (4.2) to (1.12). ■

Remark. From Lemma 7 we have

$$\sigma^*(3,4,5)=\frac{9}{10},\qquad \sigma^*(2,3,4)=\frac{7}{8},$$

which are the best possible results . By Theorem 8.4 of Ivić[7] we also note the following slightly better results

$$\sigma^*(4,5,6)\leqslant\frac{214}{233},\qquad \sigma^*(1,2,2)\leqslant\frac{41761}{54522}=0.765948\ldots.$$

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