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APPROXIMATION AND GENERALIZED GROWTH OF SOLUTIONS TO A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In the present paper, we study the approximation and growth of solutions to a class of elliptic partial differential equations. The characterizations of generalized order and generalized type of solutions to a class of elliptic partial differential equations have been obtained in terms of approximation errors.

Keywords: Helmholtz type equation, regular solution, analytic function, approximation errors, generalized order, generalized type.

1. Introduction

Following McCoy [4] , we first give some definitions. A Helmholtz type equation is given by

$$£[H] := [\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta} + F(r^2)]H(r, \theta) = 0. \tag{1.1}$$

Here (r, θ) are polar coordinates and F is an entire function. Let $H(r, \theta) = H(r, e^{i\theta})$ be a regular solution of (1.1) in some sufficiently small star-shaped neighborhood Ω about origin. Let R be the radius of convergence of this regular solution. Following Bergman [1], we have

$$H(r, \theta) = \mathbb{B}[f(z)] \ = \ \int_{-1}^{+1} E(r^2, t)\, f(\sigma)\, d\mu(t) \ ,$$

where $z = re^{i\theta} \in \Omega$,$\sigma = z(1 - t^2)/2$, $d\mu(t) = (1 - t^2)^{-1/2}dt$, and the associated function f is analytic for $2z \in \Omega$. The Taylor series expansion of the kernel $E(r^2, t)$ is given as

$$E(r^2, t) = 1 + \sum_{n=1}^{\infty} t^{2n} Q^{(2n)}(r^2).$$

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For a fixed $r \geqslant 0$, the kernel $E(r^2, t)$ is analytic for $t \in [-1, +1]$ and for every fixed $t \in [-1, +1]$, it is entire for $r \geqslant 0$. The Taylor coefficients $Q^{(2n)}(r^2)$ are entire function solutions of the system

$$\frac{\partial \left(Q^{(2)}(r^2)\right)}{\partial r^2} + 2F(r^2) = 0, \qquad Q^{(0)}(r^2) = 1,$$

$$(2n+1)\,\frac{\partial \left(Q^{(2n+2)}(r^2)\right)}{\partial r^2} + 2\frac{\partial \left(r^2 Q^{(2n)}(r^2)\right)}{\partial r^2} + F(r^2)Q^{(2n)}(r^2) - \, n\,\frac{\partial \left(Q^{(2n)}(r^2)\right)}{\partial r^2} = 0,$$

$$Q^{(2n+2)}(r^2)|_{r=0} = 0, \qquad n = 1, 2, 3 \ldots$$

McCoy [4] defined the basic set of particular solutions

$$\Phi_n(r, e^{i\theta}) = [r^n G_n(r^2)/R^n G_n(R^2)]e^{in\theta}$$

normalized by the conditions

$$\Phi_n(r, e^{i\theta}) = e^{in\theta}, \qquad n = 0, 1, 2, 3 \ldots$$

Here

$$G_n(r^2) = \int_{-1}^{+1} E(r^2, t)\,\left(1 - t^2\right)^n d\mu(t).$$

This set is complete relative to compact convergence on a disk $D_R = \{z : |z| < R\}$. Let $\mathrm{Im}(D_R)$ be the space of regular solutions of (1.1) on D_R. Then $H \in \mathrm{Im}(D_R)$ has the expansion in a uniformly convergent series

$$H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta}),$$

where $\{a_n\}$ is a sequence of real numbers. If $A(D_R)$ is the space of analytic functions on D_R, then $f \in A(D_R)$ has the Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n, \qquad z \in D_R.$$

McCoy [4] associated H with the analytic function f by defining an integral operator as given below:

$$H(r, e^{i\theta}) = T_\varepsilon[f(z)] = \frac{1}{2\pi i}\int_{|\zeta|=1-\varepsilon} K_R(\zeta) f(z/\zeta) d\zeta/\zeta, \qquad z = re^{i\theta}/R,$$

where $\varepsilon > 0$ is arbitrarily small. The kernel for this integral operator defined over the basis $\{\Phi_n\}$ is given by

$$K_R(\zeta) = \sum_{n=0}^{\infty} \zeta^n [G_n(r^2)/G_n(R^2)].$$

For $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that for all $n \geqslant N(\varepsilon)$, we have

$$(1-\varepsilon) \leqslant |G_n(r^2)/G_n(R^2)| \leqslant (1+\varepsilon).$$

Thus we can say that the kernel of this operator has uniformly convergent expansion. The above integral operator maps the function $f \in A(D_{R(1-\varepsilon)})$ onto regular solution $H \in \mathrm{Im}(D_{R(1-\varepsilon)})$ and the disk of regularity of H coincides with the disk of analyticity of f. The maximum modulus of H on D_r is given by

$$M(r,H) = \max\{|H(s,e^{i\theta})| : s \leqslant r\}.$$

Let H be regular on the closure Δ^* of the unit disk $\Delta = \{z : \ |z| \ < 1\}$ and define the norm of H as

$$||H|| = \begin{cases} ||H||_p = \left[\iint_{\Delta^*} |H|^p r dr d\theta\right]^{1/p}, & 1 \leqslant p < \infty, \\ ||H||_\infty = \lim\limits_{r\to 1^-} M(r,H). \end{cases}$$

The spaces of polynomial solutions of fixed degree $n = 0, 1, 2, \ldots$ are given by

$$\Pi_n = \left\{ P : P(r,e^{i\theta}) = \sum_{k=0}^{n} c_k \Phi_k(r,e^{i\theta}), \ c_k \in \mathbb{R} \right\}.$$

We define the approximation errors $E_n(H)$ (see [4]) by

$$E_n(H) = \inf_P \{||H-P|| : P \in \Pi_n\}, \qquad n = 0,1,2\ldots.$$

The definition of order and type for regular solution H are the same as those for the associated analytic function f (see [4], pp. 209). Hence the order ρ of regular solution H on D_R is given by

$$\rho = \lim_{r\to R^-} \sup \frac{\ln^+ \ln^+ M(r,H)}{\ln[R/(R-r)]},$$

where

$$\ln^+ x = \begin{cases} \ln x, & x > 1; \\ 0, & 0 < x \leqslant 1. \end{cases}$$

Further, for $0 < \rho < \infty$ the type σ of regular solution H on D_R is given by

$$\sigma = \lim_{r\to R^-} \sup \frac{\ln^+ M(r,H)}{[R/(R-r)]^\rho}.$$

McCoy [4] obtained the characterizations of order and type of function H in terms of approximation errors. Later, in [5], using the concept of index, McCoy studied the growth of entire solutions of the Helmoltz equation. Using the concept of (p, q) growth, Kumar [3] studied the relation between the growth and Chebyshev approximation of entire function solutions of Helmoltz equation. Srivastava and Kumar [7] obtained the characterizations of generalized growth of function H in terms of approximation errors and Taylor series coefficients It is clear from the above that the definition of σ is not valid if the order $\rho = \infty$. For such cases, following Janik [2] and Seremeta [6] we define the generalized order and generalized type of function H. Hence, let L^0 denote the class of functions h satisfying the following conditions:

(i) h is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and $h(x)$ tends to ∞ as $x \to \infty$,

(ii) $\lim_{x\to\infty} \frac{h\{(1+1/\psi(x))x\}}{h(x)} = 1$, for every function ψ such that $\psi(x) \to \infty$ as x tends to ∞.

(iii) let Λ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1, \qquad c > 0,$$

i.e., h is slowly increasing.

For $\alpha \in \Lambda$ and $\beta \in L^0$ we define the generalized order of H as

$$\rho(\alpha, \beta, H) = \lim_{r\to R^-} \sup \frac{\alpha[\ln^+ M(r, H)]}{\beta[R/(R-r)]}. \tag{1.2}$$

Further, for $\alpha, \beta, \gamma \in \Lambda$ and $0 < \rho < \infty$, we define the generalized type of H as

$$\sigma(\alpha, \beta, \gamma, H) = \lim_{r\to R^-} \sup \frac{\alpha[\ln^+ M(r, H)]}{\beta\{[\gamma\{R/(R-r)\}]^\rho\}}. \tag{1.3}$$

If $\rho(\alpha, \beta, H)$ defined as above is zero then the analytic function is of generalized order zero and $\sigma(\alpha, \beta, \gamma, H)$ is no longer defined. For such functions we give the modified definition of generalized order. Hence for $\alpha(x) \in \Lambda$, we define the generalized order $\rho(\alpha, H), (0 \leqslant \rho(\alpha, H) < \infty)$ of H on D_R as

$$\rho(\alpha, H) = \lim_{r\to R^-} \sup \frac{\alpha\left[\ln^+ M(r, H)\right]}{\alpha\left[\ln\{R/(R-r)\}\right]}. \tag{1.4}$$

Also for $\beta(x) \in L^0$ and $1 < \rho(\alpha, H) < \infty$, we define the generalized type $\sigma(\beta, \rho, H)$ of H on D_R as

$$\sigma(\beta, \rho, H) = \lim_{r\to R^-} \sup \frac{\beta\left[\ln^+ M(r, H)\right]}{(\beta\left[\ln\{R/(R-r)\}\right])^\rho}. \tag{1.5}$$

In the present paper we have obtained the characterizations of generalized order and type defined by (1.2) and (1.3). We have also obtained the characterizations of generalized slow growth of function H in terms of approximation errors.

2. Generalized (α, β)-growth

We now prove

Theorem 1. *Let H be a regular solution of (1.1) having the series expansion $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta})$. For $\alpha \in \Lambda, \beta \in L^0$ and positive numbers x and μ_1, set $U(x, \mu_1) = \beta^{-1}\{\mu_1 \alpha(x)\}$. Assume that $\alpha(x/U(x, \mu_1)) \cong [1 + o(x)]\alpha(x)$ as $x \to \infty$. Then H is the restriction of a solution H_1 whose disk of regularity is $D_R(R > 1)$ and having generalized order $\rho(0 < \rho < \infty)$ if and only if*

$$\rho = \rho(\alpha, \beta, H) = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left\{n/\ln^+(E_n(H)R^n)\right\}}. \tag{2.1}$$

Proof. Write

$$\eta_1 = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\beta\left\{n/\ln^+(E_n(H)R^n)\right\}}. \tag{2.2}$$

Now first we prove that $\eta_1 \leqslant \rho$. From (1.2), for $\mu_1 > \rho$ and r sufficiently close to R, we have

$$\log^+ M(r, H_1) \leqslant \alpha^{-1}[\mu_1 \beta\{R/(R - r)\}].$$

Let $\varepsilon > 0$ be arbitrary such that $\upsilon = (R^{-1} + \varepsilon) < 1$. Following McCoy ([4], pp.208), we have

$$E_k(H) \leqslant \frac{\pi K(\varepsilon)\upsilon^k}{1 - \upsilon}; \qquad k = n, n + 1, ...,$$

where $K(\varepsilon)$ is a finite positive number. Let us put $r = \upsilon^{-1}$. Then $1 < r < R$. For sufficiently small ε, r is close to R and $\pi K(\varepsilon) \leqslant M(r, H)$. Hence we have

$$E_k(H) \leqslant \frac{M(r, H)}{(r - 1)r^{k-1}} \leqslant \frac{M(r, H_1)}{(r - 1)r^{k-1}}, \qquad 1 < r < R,\ k \geqslant n. \tag{2.3}$$

Hence for every r sufficiently close to R and large n, we get

$$\ln^+(E_n(H)R^n) \leqslant O(1) - n \ln(r/R) + \alpha^{-1}[\mu_1 \beta\{R/(R - r)\}].$$

Putting

$$r = r_n = R\left[1 - 1/U\left(n/U(n, \mu_1^{-1}), \mu_1^{-1}\right)\right],$$

we get

$$\ln^+(E_n(H)R^n) \leqslant O(1) - n \ln\left[1 - 1/U\left(n/U(n, \mu_1^{-1}), \mu_1^{-1}\right)\right] + n/U(n, \mu_1^{-1}).$$

Now using the properties of logarithm and assumptions of the theorem for $\alpha(x)$ and $\beta(x)$, we get for sufficiently large values of n,

$$\ln^{+}\left(E_n(H)R^n\right) \leqslant C_1 \frac{n}{\beta^{-1}\left\{\mu_1^{-1}\alpha(n)\right\}},$$

where C_1 is a positive constant. Hence by using the properties of β, we get

$$\frac{\alpha(n)}{\beta\left\{n/\ln^{+}\left(E_n(H)R^n\right)\right\}} \leqslant \mu_1.$$

Now proceeding to limits as $n \to \infty$, we get $\eta_1 \leqslant \mu_1$. Since $\mu_1 > \rho$ is arbitrary, therefore we get $\eta_1 \leqslant \rho$.

Now we will prove that $\rho \leqslant \eta_1$. Let us assume that $0 \leqslant \eta_1 < \infty$ otherwise for $\eta_1 = \infty$, the inequality obviously holds. Therefore for a given $\varepsilon > 0$ there exists a positive integer n_0 such that for all $n > n_0$, we have

$$0 \leqslant \frac{\alpha(n)}{\beta\left\{n/\ln^{+}\left(E_n(H)R^n\right)\right\}} \leqslant \eta_1 + \varepsilon = \eta_1^*$$

or

$$E_n(H)r^n \leqslant r^n R^{-n} \exp\left[n/\beta^{-1}\left\{\left(\eta_1^*\right)^{-1}\alpha(n)\right\}\right]. \tag{2.4}$$

Now from the property of maximum modulus, we have

$$M(r,H) \leqslant \sum_{n=0}^{\infty} E_n(H)r^n$$

or

$$M(r,H) \leqslant \sum_{n=0}^{n_0} E_n(H)r^n + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left[n/\beta^{-1}\left\{\left(\eta_1^*\right)^{-1}\alpha(n)\right\}\right]$$

or

$$M(r,H) \leqslant A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left[n/\beta^{-1}\left\{\left(\eta_1^*\right)^{-1}\alpha(n)\right\}\right], \tag{2.5}$$

where A_1 is a positive real constant. We take

$$N(r) = \left[\alpha^{-1}\left(\eta_1^*\beta\left\{[\ln\{R/(N+1)r\}]^{-1}\right\}\right)\right],$$

where $[x]$ denotes the integer part of $x \geqslant 0$. Since $\alpha \in \Lambda$ and $\beta \in L^0$, the integer $N(r)$ is well defined. Now if r is sufficiently large, then from (2.4) we have

$$M(r,H) \leqslant A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1 \leqslant n \leqslant N(r)} R^{-n} \exp\left[n/\beta^{-1}\left\{\left(\eta_1^*\right)^{-1}\alpha(n)\right\}\right]$$
$$+ \sum_{n > N(r)} r^n R^{-n} \exp\left[n/\beta^{-1}\left\{\left(\eta_1^*\right)^{-1}\alpha(n)\right\}\right]$$

or

$$M(r,H) \leqslant A_1 r^{n_0} + r^{N(r)} \sum_{n=1}^{\infty} R^{-n} \exp\left[n/\beta^{-1}\left\{(\eta_1^*)^{-1}\alpha(n)\right\}\right] + \sum_{n>N(r)} r^n R^{-n} \exp\left[n/\beta^{-1}\left\{(\eta_1^*)^{-1}\alpha(n)\right\}\right]. \tag{2.6}$$

Now we have

$$\lim_{n\to\infty}\sup\left(R^{-n}\exp\left[n/\beta^{-1}\left\{(\eta_1^*)^{-1}\alpha(n)\right\}\right]\right)^{1/n} = \frac{1}{R} < 1.$$

Hence the first series on right hand side of (2.6) converges to a positive real constant A_2. So from (2.6) we get

$$M(r,H) \leqslant A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n R^{-n} \exp\left[n/\beta^{-1}\left\{(\eta_1^*)^{-1}\alpha(n)\right\}\right]$$

or

$$M(r,H) \leqslant A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n R^{-n} \exp[n\ln\{R/(N+1)r\}]$$

or

$$M(r,H) \leqslant A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} \left(\frac{1}{N+1}\right)^n$$

or

$$M(r,H) \leqslant A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^n. \tag{2.7}$$

It can be easily seen that the series in (2.7) converges to a positive real constant A_3. Therefore from (2.7), we get

$$M(r,H) \leqslant A_2 r^{N(r)}[1+o(1)]$$

or

$$\ln^+ M(r,H) \leqslant [1+o(1)]\left[\alpha^{-1}\left(\overline{\eta_1}\beta\left\{[\ln\{R/(N+1)r\}]^{-1}\right\}\right)\right]\ln r$$

or

$$\ln^+ M(r,H) \leqslant [1+o(1)]\alpha^{-1}\left[\{\eta_1^*+\delta_1\}\beta\left\{[\ln\{R/(N+1)r\}]^{-1}\right\}\right],$$

where $\delta_1 > 0$ is suitably small. Hence

$$\alpha[\ln^+ M(r,H)] \leqslant \{\eta_1^*+\delta_1\}\beta\left\{[1+o(1)]^{-1}[\ln(R/r)]^{-1}\right\}.$$

Thus for r sufficiently close to R, we get

$$\frac{\alpha[\ln^+ M(r,H)]}{\beta\left\{[1+o(1)]^{-1}[R/(R-r)]\right\}} \leqslant \eta_1^* + \delta_1.$$

Proceeding to limits as $r \to R$ and using the property of β, we get

$$\lim_{r\to R^-} \sup \frac{\alpha[\ln^+ M(r,H)]}{\beta\{R/(R-r)\}} \leqslant \eta_1^* + \delta_1.$$

Since ε and δ_1 are arbitrarily small, therefore finally we get $\rho \leqslant \eta_1$. Combining this with the earlier inequality obtained, we get $\rho = \eta_1$.

Now from (2.2), for every $\lambda_1 > \eta_1$ and for sufficiently large n, we have

$$\frac{\alpha(n)}{\beta\left\{n/\ln^+(E_n(H)R^n)\right\}} \leqslant \lambda_1$$

or

$$E_n(H)R^n \leqslant \exp\left[n/\beta^{-1}\left\{\lambda_1^{-1}\alpha(n)\right\}\right].$$

Hence proceeding to limits as $n \to \infty$, we get

$$\lim_{n\to\infty} \sup(E_n(H)R^n)^{1/n} \leqslant 1.$$

Since $\eta_1 > 0$, the sequence $(E_n(H)R^n)_{n\in\mathbb{N}}$ is unbounded, whence

$$\lim_{n\to\infty} \sup(E_n(H)R^n)^{1/n} \geqslant 1.$$

Hence finally we get

$$\lim_{n\to\infty} \sup(E_n(H)R^n)^{1/n} = 1.$$

Thus following McCoy ([4], Theorem 1) we claim that the regular solution H can be continuously extended to a regular solution whose disk of regularity is $D_R(R > 1)$.

Let us put

$$H_1(r, e^{i\theta}) = \sum_{n=0}^{\infty} E_n(H)\Phi_n(r, e^{i\theta}).$$

Now we show that H_1 is the required continuation of H and $\rho(\alpha, \beta, H_1) = \eta_1$. For every $\lambda_1 > \eta_1$ and for sufficiently large n, we have

$$E_n(H)R^n \leqslant \exp\left[n/\beta^{-1}\left\{\lambda_1^{-1}\alpha(n)\right\}\right].$$

Now as in the proof of this theorem (see (2.4) to (2.7) above), we claim that

$$\rho(\alpha, \beta, H_1) \leqslant \lambda_1.$$

Since $\lambda_1 > \eta_1$ is arbitrary, so we get

$$\rho(\alpha, \beta, H_1) \leqslant \eta_1.$$

Also following the proof of first part given above, we get

$$\eta_1 \leqslant \rho(\alpha, \beta, H_1).$$

Hence finally we get $\rho(\alpha, \beta, H_1) = \eta_1$. This completes the proof of Theorem 1. ∎

Next we prove

Theorem 2. *Let H be a regular solution of* (1.1) *and have the series expansion $H(r,e^{i\theta})=\sum_{n=0}^{\infty}a_n\Phi_n(r,e^{i\theta})$. For positive x,μ_2 and ρ, we set*

$$V(x,\mu_2,\rho)=\gamma^{-1}\{[\beta^{-1}(\mu_2\alpha(x))]^{1/\rho}\}.$$

Assume that for $\alpha(x)$, $\beta(x),\gamma(x)\in\Lambda$,

$$V\left(\frac{n(\rho+1)}{\rho V(n/\rho,1/\mu_2,\rho+1)},\frac{1}{\mu_2},\rho\right)\cong[1+o(n)]V\left(n/\rho,1/\mu_2,\rho+1\right)\qquad\text{as } x\to\infty.$$

Then H is the restriction of a solution H_1 whose disk of regularity is $D_R(R>1)$ and having generalized type $\sigma(0<\sigma<\infty)$ if and only if

$$\sigma=\sigma(\alpha,\beta,\gamma,H_1)=\lim_{n\to\infty}\sup\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\ln^{+}(E_n(H)R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}}.$$

Proof. Write

$$\eta_2=\lim_{n\to\infty}\sup\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\ln^{+}(E_n(H)R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}}.\tag{2.8}$$

Now first we prove that $\eta_2\leqslant\sigma$. From (1.3), for $\mu_2>\sigma$ and r sufficiently close to R, we have

$$\ln^{+}M(r,H_1)\leqslant\alpha^{-1}[\mu_2\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}].$$

Thus as in the proof of Theorem 1, here we have

$$\ln^{+}(E_n(H)R^n)\leqslant O(1)-n\ln(r/R)+\alpha^{-1}[\mu_2\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}].$$

Putting

$$r=r_n=R\left[1-\left\{V\left(\frac{n(\rho+1)}{\rho V(n/\rho,1/\mu_2,\rho+1)},\frac{1}{\mu_2},\rho\right)\right\}^{-1}\right],$$

we get

$$\ln^{+}(E_n(H)R^n)\leqslant O(1)-n\ln\left[1-\left\{V\left(\frac{n(\rho+1)}{\rho V(n/\rho,1/\mu_2,\rho+1)},\frac{1}{\mu_2},\rho\right)\right\}^{-1}\right]$$
$$+n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu_2^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.$$

Now using the properties of logarithm and assumptions of theorem, we get for sufficiently large values of n

$$\ln^+ (E_n(H)R^n) \leqslant C_2 n \frac{\rho+1}{\rho} \left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu_2^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1},$$

where C_2 is a positive constant. Hence by using the properties of α, β and γ, we get

$$\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\ln^+ (E_n(H)R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \leqslant \mu_2.$$

Now proceeding to limits as $n \to \infty$ we get $\eta_2 \leqslant \mu_2$. Since $\mu_2 > \sigma$ is arbitrary, therefore finally we get $\eta_2 \leqslant \sigma$. Now we will prove that $\sigma \leqslant \eta_2$. If $\eta_2 = \infty$, then there is nothing to prove. So let us assume that $0 \leqslant \eta_2 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \leqslant \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log^+ (E_n(H)R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \leqslant \eta_2 + \varepsilon = \eta_2^*$$

or

$$E_n(H)R^n \leqslant \exp\left\{n\tfrac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\} \tag{2.9}$$

or

$$E_n(H)r^n \leqslant r^n R^{-n} \exp\left\{n\tfrac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.$$

Now from the property of maximum modulus, we have

$$\begin{aligned} M(r,H) &\leqslant \sum_{n=0}^{\infty} E_n(H)r^n \\ &\leqslant \sum_{n=0}^{n_0} E_n(H)r^n \\ &\quad + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\} \end{aligned}$$

or

$$\begin{aligned} M(r,H) \leqslant B_1 r^{n_0} + &\sum_{n=n_0+1}^{\infty} r^n R^{-n} \\ &\times \exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}, \end{aligned} \tag{2.10}$$

where B_1 is a positive real constant. We take

$$N(r) = \left[\rho\alpha^{-1}\left\{\eta_2^*\beta\left([\gamma\{(\rho+1)[\rho\ln\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right],$$

where $[x]$ denotes the integer part of $x \geqslant 0$. Since $\alpha(x), \beta(x), \gamma(x) \in \Lambda$, the integer $N(r)$ is well defined. Now if r is sufficiently close to R, then from (2.10) we have

$$\begin{aligned}
&M(r,H) \leqslant B_1 r^{n_0}\\
&+ r^{N(r)} \sum_{n_0+1\leqslant n\leqslant N(r)} R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}\\
&+ \sum_{n>N(r)} r^n R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}
\end{aligned}$$

or

$$\begin{aligned}
M(r,H) &\leqslant B_1 r^{n_0}\\
&\quad + r^{N(r)} \sum_{n=1}^{\infty} R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}\\
&\quad + \sum_{n>N(r)} r^n R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.
\end{aligned} \tag{2.11}$$

Now we have

$$\limsup_{n\to\infty}\left(R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}\right)^{1/n} = \frac{1}{R} < 1.$$

Hence the first series in (2.11) converges to a positive real constant B_2. Hence from (2.11), we get

$$\begin{aligned}
M(r,H) &\leqslant B_1 r^{n_0} + B_2 r^{N(r)}\\
&\qquad + \sum_{n>N(r)} r^n R^{-n}\exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\eta_2^*)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}
\end{aligned}$$

or

$$M(r,H) \leqslant B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{n>N(r)} r^n R^{-n}\exp[n\ln\{R/(N+1)r\}]$$

or

$$M(r,H) \leqslant B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{n>N(r)}\left(\frac{1}{N+1}\right)^n$$

or

$$M(r,H) \leqslant B_1 r^{n_0} + B_2 r^{N(r)} + \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^n. \tag{2.12}$$

It can be easily seen that the series in (2.12) converges to a positive real constant B_3. Therefore from (2.12), we get

$$M(r,H) \leqslant B_1 r^{n_0} + B_2 r^{N(r)} + B_3 \leqslant B_2 r^{N(r)}[1+o(1)]$$

or

$$\begin{aligned}\ln^+ M(r,H) &\leqslant [1+o(1)] \\ &\times \left[\rho\alpha^{-1}\left\{\eta_2^*\beta\left([\gamma\{(\rho+1)[\rho\ln\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right]\ln r,\end{aligned}$$

or

$$\begin{aligned}\ln^+ M(r,H) &\leqslant [1+o(1)] \\ &\times \left[\alpha^{-1}\left\{(\eta_2^*+\delta_2)\,\beta\left([\gamma\{(\rho+1)[\rho\ln\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right)\right\}\right],\end{aligned}$$

where $\delta_2 > 0$ is suitably small. Hence

$$\alpha[\ln^+ M(r,H)] \leqslant (\eta_2^*+\delta_2)\,\beta\left([\gamma\{(\rho+1)[\rho\ln\{R/(N+1)r\}]^{-1}\}]^{(\rho+1)}\right).$$

When r is sufficiently close to R, then by using properties of β and γ, we get

$$\frac{\alpha[\ln^+ M(r,H)]}{\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}} \leqslant \eta_2^* + \delta_2.$$

Since ε and δ_2 are arbitrarily small, proceeding to limits as $r \to R^-$, we get

$$\sigma \leqslant \eta_2. \tag{2.13}$$

Now as in Theorem 1 we can similarly prove that the regular solution H can be continuously extended to a regular solution whose disk of regularity is $D_R (R > 1)$. Let us put

$$H_1(r, e^{i\theta}) = \sum_{n=0}^{\infty} E_n(H)\Phi_n(r, e^{i\theta}).$$

Now we claim that H_1 is the required continuation of H and $\sigma(\alpha,\beta,\gamma,H_1) = \eta_2$. From (2.8), for every $\lambda_2 > \eta_2$ and for sufficiently large n, we have

$$E_n(H)R^n \leqslant \exp\left\{n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\lambda_2)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.$$

Now as in the proof of this theorem (see (2.9) to (2.13)), we claim that

$$\sigma(\alpha,\beta,\gamma,H_1) \leqslant \lambda_2.$$

Since $\lambda_2 > \eta_2$ is arbitrary, so finally we get

$$\sigma(\alpha,\beta,\gamma,H_1) \leqslant \eta_2.$$

Also following the proof of first part given above, we get

$$\eta_2 \leqslant \sigma(\alpha,\beta,\gamma,H_1).$$

Hence finally we get $\sigma(\alpha,\beta,\gamma,H_1) = \eta_2$. This completes the proof of Theorem 2. ■

3. Functions of generalized slow growth

In this section we give the characterizations of generalized order and type for functions of slow growth. We have

Theorem 3. *Let H be a regular solution of* (1.1) *and have the series expansion $H(r,e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r,e^{i\theta})$. Then for $\alpha(x) \in \Lambda, H$ is a restriction of a solution H_1 whose disk of regularity is $D_R(R>1)$ and having generalized order $\rho(\alpha,H_1)$ if and only if*

$$\rho(\alpha,H_1) = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\left[\log^+\left\{n/\ln^+\left(E_n(H)R^n\right)\right\}\right]}.$$

Proof. First we assume that H has an extension H_1 whose disk of regularity is $D_R(R>1)$ and is of generalized order $\rho(\alpha,H_1)$. We write $\rho(\alpha,H_1)=\rho$ and

$$\zeta_1 = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\left[\log^+\left\{n/\ln^+\left(E_n(H)R^n\right)\right\}\right]}. \tag{3.1}$$

First we prove that $\zeta_1 \leqslant \rho$. As shown above, from (2.3) we have

$$E_k(H) \leqslant \frac{M(r,H)}{(r-1)r^{k-1}}, \qquad 1<r<R,\ k \geqslant n \tag{3.2}$$

Also using (1.4), for arbitrarily small $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$M(r,H) \leqslant \exp\left(\alpha^{-1}\left\{\rho^*\alpha\left[\ln\left\{R/\left(R-r\right)\right\}\right]\right\}\right), \tag{3.3}$$

where $\rho^* = \rho + \varepsilon$. From (3.2) and (3.3), we get

$$\ln^+\left(E_n(H)R^n\right) \leqslant -\ln^+(r-1) - n\ln^+(r/R) + \alpha^{-1}\left\{\rho^*\alpha\left[\ln\left\{R/\left(R-r\right)\right\}\right]\right\}.$$

Write $F(x,c_1) = \alpha^{-1}\{c_1\alpha(x)\}$, where x and c_1 are positive real numbers. Now putting $r = r_n$, where

$$r_n = R\left(1 - \left[\exp\left\{F\left(n/\exp\left\{F\left(n,(\rho^*)^{-1}\right)\right\},(\rho^*)^{-1}\right)\right\}\right]^{-1}\right),$$

we get

$$\begin{aligned}\ln^+(E_n(H)R^n) \leqslant & -\ln^+(r_n-1)\\ & -n\ln^+\left(1-\left[\exp\left\{F\left(n/\exp\left\{F\left(n,(\rho^*)^{-1}\right)\right\},(\rho^*)^{-1}\right)\right\}\right]^{-1}\right)\\ & +n/\exp\left\{F\left(n,(\rho^*)^{-1}\right)\right\}.\end{aligned}$$

Now using the properties of logarithm, we get for sufficiently large value of n

$$\ln^+(E_n(H)R^n) \leqslant \{1+o(1)\}\left[n/\exp\left\{F\left(n,(\rho^*)^{-1}\right)\right\}\right].$$

From the above inequality, we get

$$\alpha^{-1}\left\{(\rho^*)^{-1}\alpha(n)\right\} \leqslant \{1+o(1)\}\ln^+\left\{n/\ln^+(E_n(H)R^n)\right\}$$

or

$$\alpha(n) \leqslant \rho^*\alpha\left[\{1+o(1)\}\ln^+\left\{n/\ln^+(E_n(H)R^n)\right\}\right]$$

or

$$\frac{\alpha(n)}{\alpha\left[\ln^+\left\{n/\ln^+(E_n(H)R^n)\right\}\right]} \leqslant \rho^*\frac{\alpha\left[\{1+o(1)\}\ln^+\left\{n/\ln^+(E_n(H)R^n)\right\}\right]}{\alpha\left[\ln^+\left\{n/\ln^+(E_n(H)R^n)\right\}\right]}.$$

Proceeding to limits as $n\to\infty$ and using the properties of $\alpha(x)$, we get $\zeta_1 \leqslant \rho^*$. Since $\varepsilon > 0$ is arbitrarily small, we finally get $\zeta_1 \leqslant \rho$.

Now we will prove that $\rho \leqslant \zeta_1$. If $\zeta_1 = \infty$, then there is nothing to prove. So let us assume that $0 \leqslant \zeta_1 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \leqslant \frac{\alpha(n)}{\alpha\left[\log^+\left\{n/\log^+(E_n(H)R^n)\right\}\right]} \leqslant \zeta_1+\varepsilon = \zeta_1^*$$

or

$$E_n(H)R^n \leqslant \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\} \tag{3.4}$$

or

$$E_n(H)r^n \leqslant r^nR^{-n}\exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\}.$$

Now from the property of maximum modulus, we have

$$\begin{aligned}M(r,H_1) \leqslant \sum_{n=0}^{\infty}E_n(H)r^n \leqslant & \sum_{n=0}^{n_0}E_n(H)r^n\\ & +\sum_{n=n_0+1}^{\infty}r^nR^{-n}\exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\}\end{aligned}$$

or

$$M(r,H_1) \leqslant A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\}, \quad (3.5)$$

where A_1 is a positive real constant. We take

$$W_1(r) = \left[\alpha^{-1}\left\{\zeta_1^* \alpha\left[\ln\left\{\ln\left[R/(\delta_1+1)r\right]\right\}^{-1}\right]\right\}\right],$$

where $\delta_1 > 0$ is arbitrarily small and $[x]$ denotes the integer part of $x \geqslant 0$. Since $\alpha(x) \in \Lambda$, the integer $W_1(r)$ is well defined. Now if r is sufficiently large, then from (3.5), we have

$$\begin{aligned} M(r,H_1) \leqslant{} & A_1 r^{n_0} + r^{W_1(r)} \\ & \times \sum_{n_0+1 \leqslant n \leqslant W_1(r)} R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\} \\ & + \sum_{n > W_1(r)} r^n R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\} \end{aligned}$$

or

$$\begin{aligned} M(r,H_1) \leqslant{} & A_1 r^{n_0} + r^{W_1(r)} \sum_{n=1}^{\infty} R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\} \\ & + \sum_{n > W_1(r)} r^n R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\}. \end{aligned} \quad (3.6)$$

Now we have

$$\lim_{n\to\infty} \sup \left(R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\}\right)^{1/n} = \frac{1}{R} < 1.$$

Hence the first series in (3.6) converges to a positive real constant A_2. So from (3.6), we get

$$\begin{aligned} M(r,H_1) \leqslant{} & A_1 r^{n_0} + A_2 r^{W_1(r)} \\ & + \sum_{n > W_1(r)} r^n R^{-n} \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\zeta_1^*)^{-1}\alpha(n)\right\}\right]\right\} \\ \leqslant{} & A_1 r^{n_0} + A_2 r^{W_1(r)} + \sum_{n > W_1(r)} r^n R^{-n} \exp[n \ln\{R/(\delta_1+1)r\}] \\ \leqslant{} & A_1 r^{n_0} + A_2 r^{W_1(r)} + \sum_{n > W_1(r)} [1/(\delta_1+1)]^n \end{aligned}$$

or

$$M(r,H_1) \leqslant A_1 r^{n_0} + A_2 r^{W_1(r)} + \sum_{n=1}^{\infty} [1/(\delta_1+1)]^n. \quad (3.7)$$

It can be easily seen that the series in (3.7) converges to a positive real constant A_3. Therefore from (3.7), we get

$$M(r, H_1) \leqslant A_1 r^{n_0} + A_2 r^{W_1(r)} + A_3 \leqslant A_2 r^{W_1(r)}[1 + o(1)]$$

or

$$\begin{aligned}\ln^+ M(r, H_1) &\leqslant [1 + o(1)] W_1(r) \ln r \\ &\leqslant [1 + o(1)] \left[\alpha^{-1}\left\{\zeta_1^* \alpha \left[\ln \{\ln [R/ (\delta_1 + 1) r]\}^{-1}\right]\right\}\right] \ln r \\ &\leqslant O(1) \left[\alpha^{-1}\left\{\zeta_1^* \alpha \left[\ln \{\ln [R/ (\delta_1 + 1) r]\}^{-1}\right]\right\}\right].\end{aligned}$$

Since $\delta_1 > 0$ is arbitrarily small, for r sufficiently close to R, we get

$$\ln^+ M(r, H_1) \leqslant O(1) \left[\alpha^{-1} \{\zeta_1^* \alpha [\ln \{R/(R - r)\}]\}\right]$$

or

$$\alpha \left[\ln^+ M(r, H_1)\right] \leqslant \zeta_1^* \alpha [\ln \{R/(R - r)\}] + O(1)$$

Thus for r sufficiently close to R, we get

$$\frac{\alpha \left[\ln^+ M(r, H_1)\right]}{\alpha [\ln \{R/(R - r)\}]} \leqslant \zeta_1^* + o(1).$$

Proceeding to limits as $r \to R^-$, we get

$$\rho \leqslant \zeta_1^*.$$

Since $\varepsilon > 0$ is arbitrarily small, therefore finally we get

$$\rho \leqslant \zeta_1. \tag{3.8}$$

Now from (3.1), for every $\lambda_1 > \zeta_1$ and for sufficiently large value of n, we have

$$\frac{\alpha(n)}{\alpha \left[\log^+ \left\{n/ \log^+ (E_n(H) R^n)\right\}\right]} \leqslant \lambda_1$$

or

$$E_n(H) R^n \leqslant \exp\left\{n/ \exp\left[\alpha^{-1}\left\{(\lambda_1)^{-1} \alpha(n)\right\}\right]\right\}.$$

Now for sufficiently large value of n, we get

$$[E_n(H) R^n]^{1/n} \leqslant 1.$$

Proceeding to limits as $n \to \infty$, we get

$$\lim_{n \to \infty} \sup [E_n(H) R^n]^{1/n} \leqslant 1.$$

Since $\eta_1 > 0$, the sequence $(E_n(H)R^n)_{n\in\mathbb{N}}$ is unbounded, whence

$$\lim_{n\to\infty}\sup\,[E_n(H)R^n]^{1/n} \geqslant 1.$$

Hence finally we get

$$\lim_{n\to\infty}\sup\,[E_n(H)R^n]^{1/n} = 1.$$

Thus following McCoy ([4], Theorem 1) we claim that the regular solution H can be continuously extended to a regular solution whose disk of regularity is $D_R(R > 1)$. Let us put

$$H_1(r, e^{i\theta}) = \sum_{n=0}^{\infty} E_n(H)\Phi_n(r, e^{i\theta}).$$

Now we claim that H_1 is the required continuation of H and $\rho(\alpha, H_1) = \zeta_1$. For every $\lambda_1{>}\zeta_1$ and for sufficiently large value of n, we have

$$E_n(H)R^n \leqslant \exp\left\{n/\exp\left[\alpha^{-1}\left\{(\lambda_1)^{-1}\alpha(n)\right\}\right]\right\}.$$

Now as in the proof of this theorem [(3.4) to (3.8)], we claim that

$$\rho(\alpha, H_1) \leqslant \lambda_1.$$

Since $\lambda_1{>}\zeta_1$ is arbitrary, so we get

$$\rho(\alpha, H_1) \leqslant \zeta_1.$$

Also following the proof of first part given above, we get

$$\zeta_1 \leqslant \rho(\alpha, H_1).$$

So finally we get

$$\rho(\alpha, H_1) = \zeta_1.$$

This completes the proof of Theorem 3. ■

Next we have

Theorem 4. *Let H be a regular solution of* (1.1) *and have the series expansion $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n\Phi_n(r, e^{i\theta})$. Then for $1 < \rho < \infty$ and $\beta(x) \in L^0, H$ is a restriction of a solution H_1 whose disk of regularity is $D_R(R > 1)$ and having generalized type $\sigma(\beta, \rho, H_1)$ if and only if*

$$\sigma(\beta, \rho, H_1) = \lim_{n\to\infty}\sup\frac{\beta(n)}{\left(\beta\left[\log^+\left\{n/\ln^+\left(E_n(H)R^n\right)\right\}\right]\right)^{\rho}}.$$

Proof. The proof of the above theorem follows on the lines of proof of Theorem 2 and Theorem 3. Hence we omit the proof. ■

Next we have

Theorem 5. *Let H be a regular solution of* (1.1) *and have the series expansion* $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta})$. *Then for* $\alpha(x) \in \Lambda$ *the generalized order* $\rho(\alpha, H)$ $(0 \leqslant \rho(\alpha, H) < \infty)$ *of H is given by*

$$\rho(\alpha, H) = \lim_{n\to\infty} \sup \frac{\alpha(n)}{\alpha\left[\ln^{+}\left\{n/\ln^{+}\left(|a_n|R^n\right)\right\}\right]}.$$

Proof. The proof is similar to Theorem 3 above and ([7], Theorem 2.1). Hence the proof is omitted. ■

Lastly we have

Theorem 6. *Let H be a regular solution of* (1.1) *and have the series expansion* $H(r, e^{i\theta}) = \sum_{n=0}^{\infty} a_n \Phi_n(r, e^{i\theta})$. *Then for* $1 < \rho < \infty$ *and* $\beta(x) \in L^0$ *the generalized type* $\sigma(\beta, \rho, H)$ *of H is given by*

$$\sigma(\beta, \rho, H) = \lim_{n\to\infty} \sup \frac{\beta(n)}{\left(\beta\left[\ln^{+}\left\{n/\ln^{+}\left(|a_n|R^n\right)\right\}\right]\right)^{\rho}}.$$

Proof. The proof is similar to Theorem 2 above and ([7], Theorem 2.2). Hence the proof is omitted. ■

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