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ON THE INVOLUTIONS OF THE RIORDAN GROUP

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Abstract: We give an algebraic description of involutions in the Riordan group.
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1. The Riordan group

Let F be a field of characteristic 0 and $F[[x]]$ the formal power series ring over F. The Riordan group is first introduced in [6]. We recall its definition. Let $N = F[[x]]^{\times}$ be the set of invertible formal power series over F. The set N forms a commutative group under multiplication. Also let $H = xF[[x]]^{\times}$ be the set of formal power series whose constant term is zero and whose coefficient of x is non-zero. The set H forms a non-commutative group under composition [1, Chapter 4 §4.7]. The identity element of H is x. The opposite group H^{op} of H acts on N from the left by substitution: namely, for $g(x) \in N$ and $f(x) \in H^{\mathrm{op}}$, we have

$$^{f}g(x) = g(f(x))$$

and if $f_1(x), f_2(x) \in H^{\mathrm{op}}$ and $\circ$ denotes the multiplication in H^{op}, then

$$^{f_1 \circ f_2}g(x) = g(f_2(f_1(x))).$$

By this action, we glue N and H^{op} together and form a left semi-direct product

$$\mathscr{R} = N \rtimes H^{\mathrm{op}}.$$

The group $\mathscr{R}$ is called the Riordan group and the multiplication of $(g_1(x), f_1(x))$, $(g_2(x), f_2(x)) \in \mathscr{R}$ is given by

$$\begin{aligned}(g_1(x), f_1(x))(g_2(x), f_2(x)) &= (g_1(x)\ {}^{f_1}g_2(x), f_2(f_1(x))) \\ &= (g_1(x)g_2(f_1(x)), f_2(f_1(x))).\end{aligned}$$

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The identity element of $\mathscr{R}$ is $(1, x)$. The inverse of $(g_1(x), f_1(x)) \in \mathscr{R}$ is $\left(\frac{1}{g_1(\bar{f}_1(x))}, \bar{f}_1(x)\right)$, where $\bar{f}_1(x)$ is the compositional inverse of $f_1(x)$ in H^{op}, namely a power series satisfying $f_1(\bar{f}_1(x)) = \bar{f}_1(f_1(x)) = x$.

Although the Riordan group is usually defined by means of certain infinite matrices, we do not need such a description.

Recently the Riordan group has been used to obtain sequence identities. For example, in [7] the authors rewrote a combinatorially interesting element $(g, f) \in \mathscr{R}$ by a product of two elements to obtain sequence identities.

In this paper, we are interested in the action of $\mathscr{R}$ on $F[[x]]$. Let $G(x) \in F[[x]]$ be a formal power series and $(g(x), f(x)) \in \mathscr{R}$. We define

$$(g(x), f(x))G(x) = g(x)G(f(x)).$$

Hence elements of order 2 in the group $\mathscr{R}$ acts as involutions on $F[[x]]$. We call such an element of order 2 simply an involution in $\mathscr{R}$.

In his paper [5], Shapiro raised some problems on the involutions in $\mathscr{R}$, which can be stated in our notation as follows.

Problem 1.1. Is every involution a conjugate to $(1, -x)$?

Problem 1.2. Let $(f(x), g(x)) \in \mathscr{R}$ be an involution. Is there a simple condition for $g(x)$ in terms of $f(x)$?

These problems are solved by Cheon and Kim [2] in the category of analytic functions. In fact, they used a result on nonlinear functional equations. The aim of this paper is to give formal algebraic solutions to these problems (Propositions in the next section). The result and its proof are even simpler than Cheon and Kim's.

2. Involutions in the Riordan group

Let $\mathscr{R} = N \rtimes H = F[[x]]^\times \rtimes xF[[x]]^\times$ be the Riordan group as defined in the first section.

An easy computation shows that an element $(g(x), f(x)) \in \mathscr{R}$ has order 2 if and only if the following two identities hold:

$$f(f(x)) = x, \tag{2.1}$$

$$g(x)g(x)^f = 1. \tag{2.2}$$

The following description of $f(x)$ satisfying (2.1) is due to O'Farrell.

Lemma 2.1 ([4, Lemma 22]). *Let $f(x) \in H$. Suppose that $f(x) \neq x$. If $f(f(x)) = x$, then $f(x)$ is conjugate to $-x$ in H.*

Next we consider (2.2). Let $F((x))$ be the field of formal Laurent power series, which is a quotient field of $F[[x]]$ (see [1, Chapter 4 §4.9]).

The following proposition gives an answer to Problem 1.2.

Proposition 2.2. *Suppose that $f(x) \in H$ satisfies $f(f(x)) = x$ and $f(x) \neq x$. Then $g(x) \in N$ satisfies* (2.2) *if and only if there exists a non-zero formal Laurent power series $w(x) \in F((x))$ such that $g(x) = w(x)/w(f(x))$.*

Proof. First of all, note that the group H acts also on $F((x))^\times$ by substitution. Moreover an element of H defines an automorphism of the field $F((x))$. Hence, if $f(x) \in H$, then by Galois theory $F((x))$ is a quadratic extension over the fixed field $F((x))^{\langle f \rangle}$. An element $g(x)$ satisfying (2.2) is nothing but an element whose norm is 1 in the field extension $F((x))/F((x))^{\langle f \rangle}$. By Hilbert's theorem [1, Chapter 5 §11 Theorem 3], we have $g(x) = w(x)/w(x)^f$ for some $w(x) \in F((x))^\times$. The converse is obvious. ■

It is easy to see that two $w(x), w'(x) \in F((x))^\times$ give the same $g(x)$ if and only if they differ by an element in $F((x))^{\langle f \rangle}$. Hence there are infinitely possible $g(x)$ for a given $f(x)$.

The following proposition answers to Problem 1.1.

Proposition 2.3. *Assume that $f(x) \neq x$. An element $(g(x), f(x)) \in \mathscr{R}$ has order* 2 *if and only if it is conjugate to* $(1, -x)$ *in $\mathscr{R}$.*

Proof. Suppose that $(g(x), f(x)) \in \mathscr{R}$ has order 2. By Lemma 2.1, there exists $u(x) \in H = xF[[x]]^\times$ such that $f(x) = \bar{u}(-u(x))$. Also by Proposition 2.2, $g(x)$ can be written as $g(x) = w(x)/w(f(x))$ with some $w(x) \in F((x))^\times$. Consider an element

$$a = \left(\frac{1}{w(\bar{u}(x))}, \bar{u}(x) \right).$$

Then we have $a^{-1} = (w(x), u(x))$ and

$$\begin{aligned} a^{-1}(1, -x)a &= (w(x), -u(x)) \left(\frac{1}{w(\bar{u}(x))}, \bar{u}(x) \right) \\ &= (w(x)/w(f(x)), f(x)) \\ &= (g(x), f(x)) \end{aligned}$$

as desired. The converse is obvious. ■

Example 2.4. In our previous paper [3], by an analogy of modular form, we define an action of a lower triangular matrix

$$A_c = \begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix} \in \mathrm{GL}_2(F)$$

on $G(x) \in F[[x]]$ of weight $k \in \mathbb{Z}$ by

$$G|_{[A]_k}(x) = (cx + 1)^{-k} G\left(\frac{-x}{cx + 1} \right). \tag{2.3}$$

This action can be interpreted in terms of involutions in $\mathscr{R}$. In fact, we have

$$[A_c]_k = \left(\frac{1}{(1+cx)^k}, \frac{-x}{1+cx}\right) \in \mathscr{R}.$$

For this $[A_c]_k$, we may take $u(x) = \frac{2x}{2+cx}$ and $w(x) = \left(1 + \frac{c}{2}x\right)^k$. There are many other choices.

Let $(g(x), f(x))$ be an involution in $\mathscr{R}$. A formal power series $G(x)$ is called an *eigenseries* of $(g(x), f(x))$ if it satisfies

$$(g(x), f(x))G(x) = \pm G(x).$$

The explicit description of involutions given in Proposition 2.3 enables us to prove an interesting result: any power series is an eigenseries of infinitely many involutions in $\mathscr{R}$.

Proposition 2.5. *Let $G(x)$ be any formal power series in $F[[x]]$. For any element $f(x) \in H$ of order 2 in H, there exist infinitely many $g(x) \in N$ such that $g(x)g(f(x)) = 1$ and that $G(x)$ is an eigenseries of involutions $(g(x), f(x)) \in \mathscr{R}$.*

Proof. By Lemma 2.1 we can write $f(x) = \bar{u}(-u(x))$ using some $u(x) \in H$. Let $e^+(x)$ (resp. $e^-(x)$) be any even (resp. odd) formal power series. We consecutively define

$$v(x) = e^{\pm}(u(x)), \qquad w(x) = G(x)/v(x), \qquad g(x) = w(x)/w(f(x)).$$

Then $g(x)$ clearly satisfies $g(x)g(f(x)) = 1$ and it is obvious that there are infinitely many such $g(x)$. Moreover we have

$$\begin{aligned} g(x)G(f(x)) &= \frac{w(x)}{w(f(x))} w(f(x))v(f(x)) \\ &= w(x)e^{\pm}(u(\bar{u}(-u(x)))) \\ &= w(x)e^{\pm}(-u(x)) \\ &= \pm w(x)e^{\pm}(u(x)) \\ &= \pm w(x)v(x) \\ &= \pm G(x). \end{aligned}$$

This completes the proof. ■

In [3] we used the involutions $[A_c]_k$ to produce identities involving their eigenseries. These involutions have very rich eigenseries such as the generating functions of Bernoulli numbers, Fibonacci numbers, certain orthogonal polynomials and so on. While the above proposition indicates a possibility of extending our results in [3] to any series (or sequences), finding good simple involutions $(g(x), f(x)) \in \mathscr{R}$ seems to be inevitable to have a good theory. Our involutions $[A_c]_k$ in Example 2.4 are surely of this kind.

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