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REMARKS ON q-EXPONENTS OF GENERALIZED MODULAR FUNCTIONS

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Abstract: We prove several multiplicity one theorems for generalized modular functions (GMF), in terms of their q-exponents, and make a precise statement about the nature of values that the prime q-exponents of a GMF can take. We shall also study the integrality of general q-exponents of a GMF and give an upper bound on the first sign change of these q-exponents.

Keywords: generalized modular functions, multiplicity one theorems, q-exponents, sign changes.

Determination of modular forms by their Fourier coefficients or by their Hecke eigenvalues has been an interesting topic of research in the theory of modular forms. For example, one can determine the Hecke eigenforms of integral weight by the central critical values of the corresponding L-functions twisted by certain Dirichlet characters [17] or by the central values of convolution of L-functions [7]. One can also determine the Hecke eigenforms by the eigenvalues of the Hecke operators T_p (p prime). In the literature, the latter one's are known as multiplicity one theorems.

In the integral weight case there are several multiplicity one theorems available in the literature, see [5], [19]. The case is similar for half-integral weight modular forms and Siegel modular forms, see [12], [20], [21]. In this note, we are interested in obtaining multiplicity one theorems for a class of generalized modular functions in terms of their q-exponents (for definition, see §1). Since there are no L-functions attached to generalized modular functions (GMF), we have not studied the analogous or the corresponding questions in this context.

In §2, after recalling the basic properties of GMF's, we prove several multiplicity one theorems for generalized modular functions, in terms of their q-exponents. In fact, we show that the signs of primes q-exponents itself determine the generalized modular function (up to a non-zero scalar). We improve this version by

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assuming the Sato-Tate conjecture (Conj. 2.7 in the text) for a pair of distinct non-CM Hecke eigenforms of integral weight, where CM means Complex Multiplication.

In [15, Thm. 1], Kohnen-Meher show that $c(p)$ (p prime) takes infinitely many distinct values. In §3, we sharpen this result by showing that $c(p)$ (p prime) takes infinitely many (real) distinct positive values and also infinitely many (real) distinct negative values, i.e., we provide more precise information about the nature of values that the prime q-exponents of a GMF can take. In the same section, we also show that the q-exponents $c(n)$ ($n \in \mathbb{N}$) are non-zero and integral only for finitely many n's, which generalizes [16, Thm. 5.1] to general q-exponents.

In the integral (or half-integral) weight modular forms, there are several results on producing an explicit upper bound on the first sign change of their Fourier coefficients (cf. [3], [4], [18]). In a similar flavor, in §4, we shall give an upper bound on the first sign change of general q-exponents of generalized modular functions.

1. Preliminaries

In this section, we will recall the definition of generalized modular functions and some basic results about them. We refer the reader to [11] for more details.

Definition 1.1. We say that f is a generalized modular function (GMF) of integral weight k on $\Gamma_0(N)$, if f is a holomorphic function on the upper half-plane $\mathbb{H}$ and

$$f\left(\frac{az+b}{cz+d}\right) = \chi(\gamma)(cz+d)^k f(z) \qquad \forall\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$$

for some (not necessarily unitary) character $\chi : \Gamma_0(N) \to \mathbb{C}^*$.

We will also suppose that $\chi(\gamma) = 1$ for all parabolic $\gamma \in \Gamma_0(N)$ of trace 2. We remark that in [11], a GMF in the above sense was called as a parabolic GMF (PGMF).

Let f be a non-zero generalized modular function of weight k on $\Gamma_0(N)$. Then f has an infinite product expansion

$$f(z) = c_0 q^h \prod_{n=1}^{\infty} (1-q^n)^{c(n)}, \tag{1.1}$$

where the product on the right-hand side of (1.1) is convergent in a small neighborhood of $q = 0$, where $q = e^{2\pi i z}$. Here c_0 is a non-zero constant, h is the order of f at infinity, and the $c(n)$ ($n \in \mathbb{N}$) are uniquely determined complex numbers [2], [6].

The following theorem is due to Knopp and Mason.

Theorem 1.2 ([11, Theorem 2]). *If f is a GMF of weight 0 on $\Gamma_0(N)$ with $\mathrm{div}(f) = 0$, then its logarithmic derivative*

$$g := \frac{1}{2\pi i}\frac{f'}{f} \in S_2(\Gamma_0(N), \chi_{\mathrm{triv}}). \tag{1.2}$$

Conversely, given any $g \in S_2(\Gamma_0(N), \chi_{\mathrm{triv}})$*, then there exists a GMF* f *of weight* 0 *on* $\Gamma_0(N)$ *with* $\mathrm{div}(f) = 0$ *such that* (1.2) *is satisfied and* f *is uniquely determined up to a non-zero scalar. Here,* $\mathrm{div}(f)$ *means the divisor of* f *as in [8, p. 131].*

Let f, g be as in the above theorem. Suppose that the Fourier expansion of $g(z)$ is given by

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n.$$

Let K_f (resp., K_g) be the field generated by the q-exponents $c(n)$ (resp., $b(n)$) of f (resp., of g) over $\mathbb{Q}$. Then, $K_f = K_g$, since for $n \geqslant 1$,

$$b(n) = -\sum\nolimits_{d|n} dc(d), \tag{1.3}$$

$$nc(n) = -\sum\nolimits_{d|n} \mu(d) b(n/d). \tag{1.4}$$

We finish this section with an observation which points out an extra feature that generalized modular forms have when compared with classical modular forms. The character that comes with the modular forms has to be unitary whereas for GMF's its not necessary. This feature allows us to have more modular forms. For example, there are no modular forms of weight 0 or with $\mathrm{div}(f) = 0$, whereas there are GMF's of weight 0 and with $\mathrm{div}(f) = 0$, which is clear from the Theorem 1.2 above.

2. Multiplicity one theorems

In this section, we state several multiplicity one theorems for generalized modular functions, in terms of their q-exponents. The basic idea of these proofs is motivated from the work of Inam and Wiese [9]. Now, we let us recall the notion of natural density/analytic density for subsets of $\mathbb{P}$ and the Sato-Tate measure.

Definition 2.1. Let S be a subset of $\mathbb{P}$. The set S has natural density $d(S)$ (resp., analytic density $d_{\mathrm{an}}(S)$), if the limit

$$\lim_{x \to \infty} \frac{\#\{p \leqslant x : p \in S\}}{\pi(x)} \qquad \left(\text{resp., } \lim_{s \to 1^+} \frac{\sum_{p \in S} \frac{1}{p^s}}{\log(\frac{1}{s-1})}\right) \tag{2.1}$$

exists and is equal to $d(S)$ (resp., is equal to $d_{\mathrm{an}}(S)$), where $\pi(x) := \#\{p \leqslant x : p \in \mathbb{P}\}$.

Remark 2.2. If a subset $S \subseteq \mathbb{P}$ has a natural density, then it also has an analytic density, and the two densities are the same. Observe, if $|S| < \infty$, then $d(S) = 0$ and hence $d_{\mathrm{an}}(S) = 0$.

Definition 2.3. The Sato-Tate measure μ_{ST} is the probability measure on $[-1, 1]$ given by $\frac{2}{\pi}\sqrt{1 - t^2} dt$.

The following multiplicity one theorem (Theorem 2.5 in the text) is an application of the Sato-Tate equidistribution theorem for Hecke eigenforms without complex multiplication (CM) [1] together with [18, Thm. 2]. For the reader's convenience, let us recall the Theorem in [loc. cit.]

Theorem 2.4 (Matomäki). *Let $k_1, k_2 \geqslant 2$ be even integers and let $N_1, N_2 \geqslant 1$ be integers. Let $f_1 = \sum_{n=1}^{\infty} a_{f_1}(n)q^n$ (resp., $f_2 = \sum_{n=1}^{\infty} a_{f_2}(n)q^n$) be a normalized Hecke eigenform without CM of weight k_1 and level N_1 (resp., of weight k_2 and level N_2). If $a_{f_1}(p)$ and $a_{f_2}(p)$ have same sign for every prime p except those in a set E with analytic density $d_{\mathrm{an}}(E) \leqslant 6/25$, then $f_1 = f_2$.*

Let f be a non-constant GMF of weight 0 on $\Gamma_0(N)$ with $\mathrm{div}(f) = 0$ and q-exponents $\{c(n)\}_{n\in\mathbb{N}}$ and $g = \sum_{n=1}^{\infty} b(n)q^n$ be the corresponding cuspform of weight 2 on $\Gamma_0(N)$ with trivial character (as in Theorem 1.2). We further assume that g is a normalized Hecke eigenform without CM. We know that $b(p) = 1 - pc(p)$. Hence, for $p \in \mathbb{P}$, we have

$$c(p) > 0 \Longleftrightarrow -1 \leqslant B(p) < \frac{1}{2\sqrt{p}}, \qquad c(p) < 0 \Longleftrightarrow \frac{1}{2\sqrt{p}} < B(p) \leqslant 1,$$

where $B(p) = \frac{b(p)}{2\sqrt{p}}$. We see that, if $c(p)$ (resp., $b(p)$) is positive, then it does not mean that $b(p)$ (resp., $c(p)$) is negative. So, the following theorem is not an immediate consequence of Theorem 2.4. However, we can still deduce our Theorem from there by a trick.

Theorem 2.5 (Multiplicity one theorem-I). *For $i = 1, 2$, let f_i be a non-constant GMF of weight 0 on $\Gamma_0(N_i)$ with $\mathrm{div}(f_i) = 0$, q-exponents $\{c_i(n)\}_{n\in\mathbb{N}}$. Let g_i's be the corresponding cuspforms in $S_2(\Gamma_0(N_i))$ (as in Theorem 1.2), resp. We assume that g_i's are normalized Hecke eigenforms without CM. If $c_1(p)$ and $c_2(p)$ have the same sign $\forall p \notin E$ where $E \subseteq \mathbb{P}$ with $d_{\mathrm{an}}(E) \leqslant 6/25$, then $f_1 = f_2$ (up to a non-zero scalar).*

Proof. Suppose that $g_i = \sum_{i=1}^{\infty} b_i(n)q^n$, for $i = 1, 2$. Let $B_i(p)$ denote $\frac{b_i(p)}{2\sqrt{p}}$. By Deligne's bound, we know that $|B_i(p)| \leqslant 1$. As explained above, we see that if $c_1(p)$ (resp., $b_1(p)$) is positive, then it does not mean that $b_1(p)$ (resp., $c_1(p)$) is negative. We now show that, except possibly for a natural density zero set of primes, the signs of $c_1(p)$ and $b_1(p)$ are exactly the opposite, i.e., the density of primes p for which $c_1(p)$ and $b_1(p)$ have the same sign is zero. To prove this, it is enough to show that

$$d\left(\left\{p \text{ prime} : p \nmid N_1, 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\right\}\right) = 0.$$

For any fixed (but small) $\epsilon > 0$, we have the following inclusion of sets

$$\{p \leqslant x : p \nmid N_1,\ B_1(p) \in [0, \epsilon]\} \supseteq \left\{p \leqslant x : p \nmid N_1,\ p > \frac{1}{4\epsilon^2},\ 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\right\}.$$

Hence, we have

$$\#\{p \leqslant x : p \nmid N_1,\ B_1(p) \in [0,\epsilon]\} + \pi\left(\frac{1}{4\epsilon^2}\right) \geqslant \#\left\{p \leqslant x : p \nmid N_1,\ 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\right\}.$$

Now divide the above inequality by $\pi(x)$

$$\frac{\#\{p \leqslant x : p \nmid N_1,\ B_1(p) \in [0,\epsilon]\}}{\pi(x)} + \frac{\pi\left(\frac{1}{4\epsilon^2}\right)}{\pi(x)} \geqslant \frac{\#\left\{p \leqslant x : p \nmid N_1,\ 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\right\}}{\pi(x)}.$$

The term $\frac{\pi(\frac{1}{4\epsilon^2})}{\pi(x)}$ tends to zero as $x \to \infty$ as $\pi(\frac{1}{4\epsilon^2})$ is finite. By the Sato-Tate equidistribution theorem ([1, Thm. B.]), we have

$$\frac{\#\{p \leqslant x : p \nmid N_1,\ B_1(p) \in [0,\epsilon]\}}{\pi(x)} \longrightarrow \mu_{\mathrm{ST}}([0,\epsilon]) \qquad \text{as } x \to \infty.$$

This implies that

$$\limsup_{x\to\infty} \frac{\{p \leqslant x : p \nmid N_1, 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\}}{\pi(x)} \leqslant \mu_{\mathrm{ST}}([0,\epsilon]). \tag{2.2}$$

Since the inequality (2.2) holds for all $\epsilon > 0$, we have that

$$\lim_{x\to\infty} \frac{\{p \leqslant x : p \nmid N_1, 0 \leqslant B_1(p) < \frac{1}{2\sqrt{p}}\}}{\pi(x)} = 0.$$

Now, if $c_1(p)$ and $c_2(p)$ have same sign for every prime $p \notin E$ with $d_{\mathrm{an}}(E) \leqslant 6/25$, then $b_1(p)$ and $b_2(p)$ also have same sign for every prime $p \notin E$, since the signs of $b_1(p)$ and $c_1(p)$ (resp., $b_2(p)$ and $c_2(p)$) are exactly the opposite for a natural density zero set of primes. This implies that, $b_1(p)$ and $b_2(p)$ have same sign for every prime $p \notin E$ with $d_{\mathrm{an}}(E) \leqslant 6/25$, hence $N_1 = N_2$ and $g_1 = g_2$ (by Theorem 2.4). Since g_i's determine f_i, up to a non-zero scalar, we are done with the proof. ∎

In particular, we have:

Corollary 2.6. *Let f_1, f_2, g_1, g_2 be as in the above Theorem. Suppose that $c_1(p)$ and $c_2(p)$ have the same sign for every prime p, then $f_1 = f_2$, up to a non-zero scalar, i.e., signs of the prime q-exponents determine the GMF, up to a non-zero scalar.*

2.1. Stronger version

Now, we shall state a stronger version of the multiplicity one theorem for GMF's, by assuming the pair Sato-Tate equidistribution conjecture for two distinct non-CM Hecke eigenforms of integral weight (Conjecture 2.7). Now, we let us recall the pair Sato-Tate equidistribution conjecture.

Let $g_1 = \sum_{n=1}^{\infty} b_1(n)q^n$ (resp., $g_2 = \sum_{n=1}^{\infty} b_2(n)q^n$) be a normalized cuspidal eigenform of weight $2k$ on $\Gamma_0(N)$. By Deligne's bound, for $i = 1, 2$, one has

$$|b_i(p)| \leqslant 2p^{k-\frac{1}{2}},$$

and we let

$$B_i(p) := \frac{b_i(p)}{2p^{k-\frac{1}{2}}} \in [-1, 1]. \tag{2.3}$$

We have the following pair Sato-Tate equidistribution conjecture for distinct cuspidal eigenforms g_1, g_2 without CM.

Conjecture 2.7. *Let $k \geqslant 1$ and let g_1, g_2 be distinct normalized cuspidal eigenforms of weight $2k$ on $\Gamma_0(N)$ without CM. For any two subintervals $I_1, I_2 \subseteq [-1, 1]$, we have*

$$\begin{aligned} d(S(I_1, I_2)) &= \lim_{x\to\infty} \frac{\#S(I_1, I_2)(x)}{\pi(x)} = \mu_{\mathrm{ST}}(I_1)\mu_{\mathrm{ST}}(I_2) \\ &= \frac{4}{\pi^2} \int_{I_1} \sqrt{1-s^2}ds \int_{I_2} \sqrt{1-t^2}dt, \end{aligned}$$

where

$$S(I_1, I_2) := \{p \in \mathbb{P} : p \nmid N, B_1(p) \in I_1, B_2(p) \in I_2\}$$

and

$$S(I_1, I_2)(x) := \{p \leqslant x : p \in S(I_1, I_2)\}.$$

In other words, the Fourier coefficients at primes are independently Sato-Tate distributed with respect to the measure μ_{ST}.

For the notational convenience, we let $\mathbb{P}_{<0}$ denote the set $\{p \in \mathbb{P} : p \nmid N, c_1(p)c_2(p) < 0\}$, and similarly $\mathbb{P}_{>0}$, $\mathbb{P}_{\leqslant 0}$, $\mathbb{P}_{\geqslant 0}$, and $\mathbb{P}_{=0}$.

Theorem 2.8. *For $i = 1, 2$, let f_i be a non-constant GMF of weight 0 on $\Gamma_0(N)$ with $\mathrm{div}(f_i) = 0$ and q-exponents $\{c_i(n)\}_{n\in\mathbb{N}}$. Let g_i's be the corresponding cuspforms in $S_2(\Gamma_0(N))$ (as in Theorem 1.2), resp. We assume that g_i's are two distinct normalized non-CM Hecke eigenforms. If the pair Sato-Tate equidistribution conjecture holds for (g_1, g_2) (Conjecture 2.7), then the product of q-exponents $c_1(p)c_2(p)$ (p prime), change signs infinitely often. Moreover, the sets*

$$\mathbb{P}_{>0}, \mathbb{P}_{<0}, \mathbb{P}_{\geqslant 0}, \mathbb{P}_{\leqslant 0}$$

have natural density 1/2, and $d(\mathbb{P}_{=0}) = 0$.

We let $\pi_{<0}(x)$ denote $\#\{p \leqslant x : p \nmid N, p \in \mathbb{P}_{<0}\}$, and similarly for $\pi_{>0}(x)$, $\pi_{\leqslant 0}(x)$, $\pi_{\geqslant 0}(x)$, and $\pi_{=0}(x)$.

Proof of Theorem 2.8. Since $b_i(p) = 1 - pc_i(p)$, we have

$$c_i(p) > 0 \Longleftrightarrow -1 \leqslant B_i(p) < \frac{1}{2\sqrt{p}}, \qquad c_i(p) < 0 \Longleftrightarrow \frac{1}{2\sqrt{p}} < B_i(p) \leqslant 1,$$

where $B_i(p) = \frac{b_i(p)}{2\sqrt{p}}$, by definition, for $i = 1, 2$. First, we shall show that

$$\liminf_{x\to\infty} \frac{\pi_{<0}(x)}{\pi(x)} \geqslant \mu_{\mathrm{ST}}([0,1]) = \frac{1}{2}.$$

For any fixed (but small) $\epsilon > 0$, we have the following inclusion of sets

$$\{p \leqslant x : p \nmid N,\ c_1(p)c_2(p) < 0\} \supseteq S_{\frac{1}{4\epsilon^2}}([\epsilon,1],[-1,0])(x) \cup S_{\frac{1}{4\epsilon^2}}([-1,0],[\epsilon,1])(x),$$

where $S_a(I_1, I_2)(x) := \{p \in S(I_1, I_2)(x) : p > a\}$, for any $a \in \mathbb{R}^+$. Hence, we have

$$\begin{aligned}\#\{p \leqslant x : p \nmid N,\ c_1(p)c_2(p) < 0\} + \pi\left(\frac{1}{4\epsilon^2}\right)& \\ \geqslant \#S([\epsilon,1],[-1,0])(x) + \#S([-1,0],[\epsilon,1])(x).&\end{aligned}$$

Now divide the above inequality by $\pi(x)$

$$\begin{aligned}\frac{\#\{p \leqslant x : p \nmid N,\ c_1(p)c_2(p) < 0\}}{\pi(x)} + \frac{\pi\left(\frac{1}{4\epsilon^2}\right)}{\pi(x)}& \\ \geqslant \frac{\#S([\epsilon,1],[-1,0])(x) + \#S([-1,0],[\epsilon,1])(x)}{\pi(x)}.&\end{aligned}$$

The term $\frac{\pi(\frac{1}{4\epsilon^2})}{\pi(x)}$ tends to zero as $x \to \infty$ as $\pi(\frac{1}{4\epsilon^2})$ is finite. By Conjecture 2.7, we have

$$\frac{\#S([\epsilon,1],[-1,0])(x) + \#S([-1,0],[\epsilon,1])(x)}{\pi(x)} \longrightarrow 2.\mu_{\mathrm{ST}}([\epsilon,1])\mu_{\mathrm{ST}}([-1,0])$$

as $x \to \infty$. This implies that

$$\liminf_{x\to\infty} \frac{\pi_{<0}(x)}{\pi(x)} \geqslant 2.\mu_{\mathrm{ST}}([\epsilon,1])\mu_{\mathrm{ST}}([-1,0]) = \mu_{\mathrm{ST}}([\epsilon,1]), \tag{2.4}$$

where $\pi_{<0}(x) = \#\{p \leqslant x : p \nmid N,\ c_1(p)c_2(p) < 0\}$ by definition. Since the inequality (2.4) holds for all $\epsilon > 0$, we have that

$$\liminf_{x\to\infty} \frac{\pi_{<0}(x)}{\pi(x)} \geqslant \mu_{\mathrm{ST}}([0,1]) = \frac{1}{2}.$$

A similarly proof shows that $\liminf_{x\to\infty} \frac{\pi_{\leqslant 0}(x)}{\pi(x)} \geqslant \frac{1}{2}$. Since $\pi_{>0}(x) = \pi(x) - \pi_{\leqslant 0}(x)$, we have that $\limsup_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \leqslant \frac{1}{2}$. Hence, the limit $\lim_{x\to\infty} \frac{\pi_{<0}(x)}{\pi(x)}$ exists and is equal to $\frac{1}{2}$. Therefore, the natural density of the set $\mathbb{P}_{<0}$ is $\frac{1}{2}$.

Similarly, one can also argue for the sets $\mathbb{P}_{>0}$, $\mathbb{P}_{\leqslant 0}$, and $\mathbb{P}_{\geqslant 0}$, and show that the natural densities of these sets are $\frac{1}{2}$. The claim for $\mathbb{P}_{=0}$ follows from the former statements. ■

Finally, we state a stronger version of Theorem 2.5 by assuming the Conjecture 2.7.

Corollary 2.9 (Multiplicity one theorem-II). *For $i = 1, 2$, let f_i be a non-constant GMF of weight* 0 *on* $\Gamma_0(N)$ *with* $\mathrm{div}(f_i) = 0$ *and q-exponents* $\{c_i(n)\}_{n\in\mathbb{N}}$. *Let g_i's be the corresponding cuspforms in $S_2(\Gamma_0(N))$ (as in Theorem 1.2), resp. We assume that g_i's are two normalized non-CM Hecke eigenforms. We further assume that the Conjecture 2.7 holds for the pair (g_1, g_2), if $g_1 \neq g_2$. If $c_1(p)$ and $c_2(p)$ have the same sign $\forall\ p \notin E \subseteq \mathbb{P}$ with $d_{\mathrm{an}}(E) < 1/2$, then $f_1 = f_2$ (up to a non-zero scalar).*

Proof. If $f_1 \neq cf_2$ for all $c \in \mathbb{C}^*$, then $g_1 \neq g_2$. Then the corollary follows immediately from the above theorem. ■

3. Nature of values taken by $c(p)$ (p prime)

Before we describe the results of this section, first let us a recall the product expansion of Ramanujan's Δ-function:

$$\Delta(z) = q \prod_{n\geqslant 1} (1 - q^n)^{24},$$

where $q = e^{2\pi iz}$ with $\mathrm{Im}(z) > 0$. It is known that the function $\Delta(z)$ is a holomorphic cusp form of weight 12 and level 1. Observe that the q-exponents $c(n)(= 24)$ of $\Delta(z)$ are non-zero, integral, and positive for all $n \in \mathbb{N}$. However, the behavior of q-exponents of GMF's differs quite drastically in every aspect from that of Δ.

Recall that Kohnen and Meher showed that q-exponents $c(p)$ (p prime) take infinitely many distinct values [15]. However, they did not mention anything about the nature of values that these exponents can take. By using the recent results of [16], one can have a precise information about the nature of values that $c(p)$ (p prime) can take.

Proposition 3.1. *Let f, g be modular forms as in Theorem 1.2. If g is a normalized Hecke eigenform, then $c(p)$ (p prime) take infinitely many (distinct) positive values. Moreover, almost all these positive values have to be non-integral, i.e., $c(p) \notin \mathcal{O}_{K_f}$.*

Proof. Since the character of g is trivial, K_g is totally real and all $c(p)$ (p prime) are real numbers. If $g = \sum_{n=1}^{\infty} b(n)q^n$, with $b(1) = 1$, has CM, then $b(p) = 0$ for infinitely many primes p. Hence, $c(p) = \frac{1}{p}$ for those primes p and this proves the proposition. Therefore, WLOG, we can assume that g is non-CM.

Now, suppose that the proposition is not true. Let $a_1, \ldots, a_r$ be the only positive real numbers taken by $c(p)$ (p prime). Observe that

$$\{p \in \mathbb{P} |\ c(p) > 0\} = \cup_{i=1}^{r} S_{a_i},$$

where $S_{a_i} = \{p \in \mathbb{P} |\ c(p) = a_i\}$. It follows that the set S_{a_i} is finite, since

$$\frac{1}{p} - \frac{2}{\sqrt{p}} \leqslant c(p) = a_i \leqslant \frac{1}{p} + \frac{2}{\sqrt{p}}.$$

This implies that the natural density of $\{p \in \mathbb{P} |\ p \nmid N, c(p) > 0\})$ is zero, which is a contradiction, by [16, Thm. 2.6]. Moreover, all these values have to be non-integral, i.e., they don't belong to $\mathcal{O}_{K_f}$, by [16, Thm. 5.1]. ■

Remark 3.2. By the same argument, one can also show that $c(p)$ (p prime) take infinitely many (distinct) negative values, which are almost all non-integral, if g does not have CM.

The above proposition answers the integrality of $c(p)$ (p prime). Even from this much of information, one may not be able to deduce the integrality or non-integrality of $c(n)$ ($n \in \mathbb{N}$). In this direction we have a weaker result. We show:

Theorem 3.3. *Let f be a non-constant GMF of weight 0 on $\Gamma_0(N)$ with $\mathrm{div}(f) = 0$ with **rational** q-exponents $c(n)$ ($n \in \mathbb{N}$). Suppose g is the corresponding cuspform in $S_2(\Gamma_0(N))$ (as in Theorem 1.2). If g is a normalized Hecke eigenform, then $c(n)$'s are non-zero and integral for only finitely many n, i.e., there exists a $N_0(f) \in \mathbb{N}$ such that $c(n)$ is non-integral, if non-zero, for all $n \geqslant N_0(f)$.*

Proof. By (1.3), we know that $nc(n)$'s are integral for all $n \in \mathbb{N}$. However, it may not be that $c(n)$'s are itself integral. We confirm this by showing that $c(n)$'s are integral only finitely many times, unless they are zero.

Suppose that the proposition is not true. Let S be an infinite subset of $\mathbb{N}$ such that $c(n) \in \mathbb{Z} - \{0\}$, for all $n \in S$. Recall that $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ with $b(1) = 1$. Therefore, n divides $b(n) + \sum_{d|n, d<n} dc(d)$ in $\mathbb{Z}$, for all n in S. Now, we show that this cannot happen. Let $\sigma_0(n)$ denote the number of divisors of n.

Since $|b(n)| \leqslant \sigma_0(n)\sqrt{n}$ and $nc(n) = -\sum_{d|n} \mu(d) b(n/d)$, we have

$$|nc(n)| \leqslant \sum_{d|n} |b(n/d)| \leqslant \sum_{d|n} \sigma_0(n/d)\sqrt{n/d} \leqslant \sqrt{n} \sum_{d|n} \sigma_0(n/d) \leqslant \sqrt{n}\sigma_0(n)^2. \quad (3.1)$$

If $c(n) \in \mathbb{Z} - \{0\}$, then n divides $b(n) + \sum_{d|n, d<n} dc(d)$ in $\mathbb{Z}$, then

$$n \leqslant |b(n) + \sum_{d|n, d<n} dc(d)| \leqslant \sigma_0(n)\sqrt{n} + \sum_{d|n, d<n} \sqrt{d}\sigma_0(d)^2 \leqslant 2\sqrt{n}\sigma_0(n)^3.$$

This implies that

$$n \leqslant 2\sqrt{n}\sigma_0(n)^3. \tag{3.2}$$

Since $\sigma_0(n) = o(n^\epsilon)$ for all $\epsilon > 0$, the above inequality can only hold for finitely many n's, therefore S is a finite set. Hence, $c(n)$ is non-zero and integral for only finitely many n's. ■

This above result can also be thought of as a generalization of Theorem 5.1 in [16]. The following corollary gives an another proof of [13, Thm. 1].

Corollary 3.4 (Kohnen-Mason). *Let $f = \sum_{n=0}^{\infty} a(n)q^n$ be a non-constant GMF of weight* 0 *on $\Gamma_0(N)$ with* $\mathrm{div}(f) = 0$, *q-exponents $c(n)$ $(n \in \mathbb{N})$ and g be the modular form as in Theorem 1.2. Assume that g is a normalized Hecke eigenform. If $a(0) = 1$ and $a(n) \in \mathbb{Z}$ for $n \in \mathbb{N}$ then $f = 1$ is constant.*

Proof. Assume that $f \neq 1$. Since $K_f = \mathbb{Q}$, one can show if $a(0) = 1$ and $a(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$, then $c(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$ (cf. the proof of [13, Thm. 1]). By [16, Theorems 2.6, 2.7], we see that $c(n)$ cannot be equal to 0, for all $n \gg 0$. Therefore, by Proposition 3.3, there exists $n \in \mathbb{N}$ such that $c(n)$ is non-integral, which is a contradiction. ■

Remark 3.5. In the above proof, we could have also used Proposition 3.1 to get the contradiction.

4. Remarks on q-exponents $c(n)$ $(n \in \mathbb{N})$ of GMF's

In spite of knowing the sign change results for $c(p)$ (p prime) in [16], we cannot give an upper bound on the first sign change of prime q-exponents. However, we can give an upper bound on the first sign change of general q-exponents of a GMF. Similar results have been considered in [4] for half-integral weight modular forms, in [3] for integral weight modular forms.

Proposition 4.1. *Let $N \in \mathbb{N}$ be a square-free integer and f be a non-constant GMF of weight* 0 *on $\Gamma_0(N)$ with* $\mathrm{div}(f) = 0$ *such that its q-exponents $c(n)$ $(n \in \mathbb{N})$ are real. Then there exists $d_1, d_2 \in \mathbb{N}$ with*

$$d_1, d_2 \ll N_0 := N^5 \log^{10}(N) \exp\left(c\frac{\log(N+1)}{\log\log(N+2)}\right) \max\{\psi_2(N), 4\sqrt{N}\log^{16}(2N)\}$$

such that $c(d_1) > 0, c(d_2) < 0$. Here, $c > 0$ is an absolute constant and $\psi_2(N) := \prod_{p|N} \frac{\log(2N)}{\log p}$.

Proof. Let g be the corresponding cuspform as in Theorem 1.2. Suppose the Fourier expansion of g is given by $\sum_{n=1}^{\infty} b(n)q^n$. The Fourier coefficients of g and q-exponents of f are related by:

$$b(n) = -\sum_{d|n} dc(d). \tag{4.1}$$

The main theorem of [3] implies that there exists $n_1, n_2 \in \mathbb{N}$ with $n_1, n_2 \ll N_0$ such that such that $b(n_1) < 0, b(n_2) > 0$. Hence, there exists a divisor d_1 (resp., d_2) of n_1 (resp., of n_2) such that $c(d_1) > 0$ (resp., $c(d_2) < 0$) by (4.1). Since, $d_i \leqslant n_i$ and $n_i \ll N_0$, we have that $d_i \ll N_0$ for $i = 1, 2$. ■

When the logarithmic derivative of f (as in (1.2)) is a normalized Hecke eigenform, the above bound can be improved.

Proposition 4.2. *Let f be a non-constant GMF of weight 0 on $\Gamma_0(N)$ with $\mathrm{div}(f) = 0$ and with q-exponents $c(n)$ $(n \in \mathbb{N})$ and g be the cuspform as in Theorem 1.2. Then there exists $1 < d_0$ with $d_0 \ll (4N)^{\frac{3}{8}}$ and $(d_0, N) = 1$ such that $c(d_0) > 0$, where the implied constant is absolute.*

Proof. By [11], we know that $g = \sum_{n=1}^{\infty} b(n)q^n \in S_2(\Gamma_0(N))$. The Fourier coefficients of g and q-exponents of f are related by:

$$b(n) = -\sum_{d|n} dc(d). \tag{4.2}$$

Observe $c(1) = -b(1) = -1 < 0$. By [18, Thm. 1], there exists $n_0 \in \mathbb{N}$ with $n_0 \ll (4N)^{\frac{3}{8}}$ such that $b(n_0) < 0$. Hence, there exists a divisor d_0 (> 1) of n_0 such that $c(d_0) > 0$, by (4.2). Since, $d_0 \leqslant n_0$ and $n_0 \ll (4N)^{\frac{3}{8}}$, we have that $d_0 \ll (4N)^{\frac{3}{8}}$. ■

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