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SUBDIFFERENTIABILITY OF INFIMAL CONVOLUTION ON BANACH COUPLES

Natan Kruglyak, Japhet Niyobuhungiro

Abstract: We use duality in convex analysis and particularly the famous Attouch-Brezis theorem to prove subdifferentiability of infimal convolution on Banach couples.

Keywords: real interpolation, infimal convolution, subdifferentiability.

1. Introduction

Let (X_0, X_1) be a regular Banach couple, i.e. $X_0 \cap X_1$ is dense in both X_0 and X_1, and let $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and proper functions (see Section 2 for used definitions from convex analysis) and let

$$\overline{\varphi}_i(u) = \begin{cases} \varphi_i(u) & \text{if } u \in X_i \\ +\infty & \text{if } u \in (X_0 + X_1) \backslash X_i \end{cases} \qquad i = 0, 1 \tag{1.1}$$

be functions defined on the sum $X_0 + X_1$. Then the K-, L- and E-functionals (see [4, 5]) are particular cases of infimal convolution of functions $\overline{\varphi}_0$ and $\overline{\varphi}_1$ defined as follows:

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \inf_{x = x_0 + x_1} (\varphi_0(x_0) + \varphi_1(x_1)). \tag{1.2}$$

The infimal convolution (1.2) is called *exact* at a point $x \in (X_0 + X_1)$ if the infimum is achieved, i.e., $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \min_{x = x_0 + x_1} \{\varphi_0(x_0) + \varphi_1(x_1)\}$. Suppose that $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x)$ is *finite and exact*. Then the decomposition $x = x_0 + x_1$, on which the infimum is attained will be called *optimal* and denoted as $x = x_{0,opt} + x_{1,opt}$.

Usually, calculation of optimal decomposition is a difficult extremal problem and only near-optimal decomposition can be constructed (see [8]). However for applications, for example in image processing (see [9], [1] and [6]), exact optimal decomposition is required.

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In [7] the following dual characterization of optimal decomposition was obtained:

Theorem 1.1. *Let $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex proper functions. Suppose also that $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ is subdifferentiable for a given element $x \in dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$. Then the decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ if and only if there exists $y_* \in X_0^* \cap X_1^*$ such that it is dual to both $x_{0,opt}$ and $x_{1,opt}$ with respect to φ_0 and φ_1, respectively, i.e.*

$$\begin{cases} \varphi_0 \left(x_{0,opt}\right) = \langle y_*, x_{0,opt} \rangle - \varphi_0^* \left(y_*\right) \\ \varphi_1 \left(x_{1,opt}\right) = \langle y_*, x_{1,opt} \rangle - \varphi_1^* (y_*). \end{cases} \tag{1.3}$$

Note that to use Theorem 1.1 we need to check subdifferentiability of the function $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ for a given $x \in dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$, which is often not trivial problem. In this paper we develop an approach based on Attouch-Brezis theorem that provides sufficient conditions for subdifferentiability of infimal convolution defined on a Banach couple. Important feature of this result is that it works also for boundary points of the set $dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$. Moreover, we show how these conditions can be verified for the K, L and E-functionals.

Remark 1.1. We would like to note that all the Banach spaces considered throughout this paper are real.

2. Some definitions and results from convex analysis

Below we present some definitions and results from convex analysis that are needed for the proofs of our main results. Throughout, E will denote a Banach space with the norm $\|\cdot\|_E$ and E^* will denote its dual space. In this section, by $F : E \to \mathbb{R} \cup \{+\infty\}$ we will denote a convex function on E. The *effective domain* or simply *domain* of a function F is a convex set $domF$, defined by

$$domF = \{x \in E : F\left(x\right) < +\infty\}.$$

A function F is said to be *proper* if $domF \neq \emptyset$. If the *epigraph* of F, i.e. the set

$$epiF = \{(x, \alpha) \in E \times \mathbb{R} : F\left(x\right) \leqslant \alpha\},$$

is closed then the function F is called *lower semicontinuous* (l.s.c.). Equivalently, this can be expressed as

$$F\left(x\right) \leqslant \liminf_{y \to x} F(y),$$

i.e. for every $x \in domF$ and for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{O}$ of x such that $F\left(y\right) \geqslant F\left(x\right) - \varepsilon$ for all $y \in \mathcal{O}$.

As we will see later on, the K-, L_{p_0,p_1}- and E-functionals can be obtained as infimal convolutions of two convex functions on $X_0 + X_1$. The definition of the operation of infimal convolution is the following.

Definition 2.1. The *infimal convolution* of two functions F_0 and F_1 from E into $\mathbb{R} \cup \{+\infty\}$ is the function denoted by $F_0 \oplus F_1$ that maps E into $\mathbb{R} \cup \{-\infty, +\infty\}$ and is defined by

$$(F_0 \oplus F_1)(x) = \inf_{x = x_0 + x_1} \{F_0(x_0) + F_1(x_1)\}, \tag{2.1}$$

and it is *exact* at a point $x \in E$ if the infimum is achieved, i.e., $(F_0 \oplus F_1)(x) = \min_{x = x_0 + x_1} \{F_0(x_0) + F_1(x_1)\}$. Suppose that $(F_0 \oplus F_1)(x)$ is *finite and exact*. Then the decomposition $x = x_0 + x_1$, on which the infimum is attained will be called *optimal* and denoted as $x = x_{0,opt} + x_{1,opt}$.

The notion of conjugate function will be important for us:

Definition 2.2. The *conjugate function* of F is the function $F^* : E^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F^*(y) = \sup_{x \in E} \{\langle y, x \rangle - F(x)\}. \tag{2.2}$$

Moreover, we will say that y is dual to x with respect to F if $F^*(y) = \langle y, x \rangle - F(x)$ or, in symmetric form,

$$F(x) + F^*(y) = \langle y, x \rangle .$$

Definition 2.3. A dual element $y \in E^*$ to $x \in E$ is also called a *subgradient* of the convex function F at the point x. The set of all dual elements to x is denoted by $\partial F(x)$ and the function F is called subdifferentiable at the point $x \in E$ if the set $\partial F(x)$ is nonempty.

The next proposition (see [3]) contains examples of functions that will be often used below.

Proposition 2.1. *Consider the following cases:*

a) *Let* $F(x) = \frac{1}{p} \|x\|_E^p$, *where* $1 < p < \infty$. *Then* $F^*(y) = \frac{1}{p'} \|y\|_{E^*}^{p'}$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

b) *Let* $F(x) = \|x\|_E$. *Then*

$$F^*(y) = \begin{cases} 0 & \text{if } \|y\|_{E^*} \leqslant 1 \\ +\infty & \text{if } \|y\|_{E^*} > 1. \end{cases}$$

c) *Let*

$$F(x) = \begin{cases} 0 & \text{if } \|x\|_E \leqslant 1 \\ +\infty & \text{if } \|x\|_E > 1 \end{cases}.$$

Then $F^*(y) = \|y\|_{E^*}$.

For convenience of the reader we will give the proof of a) (the proofs of b) and c) are simpler).

Proof. Let us define the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$ as

$$\varphi(t) = \frac{1}{p} |t|^p .$$

It is clear that this function is convex, lower semicontinuous, proper, and even. The dual φ^* of φ is by definition

$$\varphi^*(t^*) = \sup_{t \in \mathbb{R}} \left(t^* \cdot t - \frac{1}{p} |t|^p \right) .$$

The supremum is attained at $t \in \mathbb{R}$ satisfying $t^* = |t|^{p-1} \operatorname{sgn}(t)$ and we obtain

$$\varphi^*(t^*) = \frac{1}{p'} |t^*|^{p'} .$$

Then the conjugate to the function $F(x) = \frac{1}{p} \|x\|_E^p$ is equal to

$$F^*(y) = \sup_{x \in E} \{ \langle y, x \rangle - \varphi(\|x\|_E) \} .$$

This can be rewritten as follows

$$\begin{aligned} F^*(y) &= \sup_{t \geqslant 0} \sup_{\|x\|_E = t} \{ \langle y, x \rangle - \varphi(t) \} = \sup_{t \geqslant 0} \left\{ t \sup_{\|x\|_E = 1} \langle y, x \rangle - \varphi(t) \right\} \\ &= \sup_{t \geqslant 0} \{ t \|y\|_{E^*} - \varphi(t) \} = \varphi^*(\|y\|_{E^*}) = \frac{1}{p'} \|y\|_{E^*}^{p'} , \end{aligned}$$

where we used the fact that φ is an even function. ■

The following simple observation will also be useful.

Proposition 2.2. *Consider the following cases:*

a) *Let $a \in E$ and $F : E \to \mathbb{R} \cup \{+\infty\}$. Then for the function $F_a(x) = F(x+a)$ we have*

$$\begin{aligned} (F_a)^*(y) &= \sup_{x \in E} \{ \langle y, x \rangle - F(x+a) \} \\ &= \sup_{u \in E} \{ \langle y, u - a \rangle - F(u) \} = F^*(y) - \langle y, a \rangle . \end{aligned}$$

b) *If $\lambda \in \mathbb{R} \setminus \{0\}$ then*

$$(\lambda F)^*(y) = \sup_{x \in E} \{ \langle y, x \rangle - \lambda F(x) \} = \lambda F^* \left(\frac{y}{\lambda} \right) .$$

The following two results show that the operations of addition and infimal convolution of convex functions are dual to each other. The property that the conjugate of infimal convolution of convex functions is equal to the sum of their conjugates holds without any additional requirements. However, the property that the conjugate of the sum is equal to the infimal convolution of the conjugates requires some additional qualification conditions.

Theorem 2.1 (Conjugate of infimal convolution). *Let F_0 and F_1 be convex functions from E into $\mathbb{R} \cup \{+\infty\}$. Then*

$$(F_0 \oplus F_1)^* = F_0^* + F_1^*.$$

The following theorem by H. Attouch and H. Brezis (see [2]) provides a sufficient condition for the conjugate of the sum of two convex, lower semicontinuous and proper functions to be equal to the exact infimal convolution of their conjugates.

Theorem 2.2 (Conjugate of a sum). *Assume $\varphi,\ \psi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ are convex, lower semicontinuous, and proper functions such that*

$$\bigcup_{\lambda \geqslant 0} \lambda \left(dom\varphi - dom\psi \right)$$

is a closed vector subspace of E. Then

$$(\varphi + \psi)^* = \varphi^* \oplus \psi^* \qquad on \quad E^*$$

and, moreover, the infimal convolution $\varphi^* \oplus \psi^*$ is exact.

The following corollary will be very useful. It follows immediately from the Attouch-Brezis theorem and it provides a connection between minimization problem on E and infimal convolution on E^*.

Corollary 2.1. *Let $\varphi,\ \psi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be functions satisfying conditions of the Attouch-Brezis theorem. Then*

$$\inf_{x \in E} \left(\varphi\left(x\right) + \psi\left(x\right) \right) = -\left(\varphi^* \oplus \psi^* \right)(0),$$

where the infimal convolution on the right-hand side is exact.

Proof. From Theorem 2.2 we have that

$$(\varphi + \psi)^* (z) = \left(\varphi^* \oplus \psi^* \right)(z) \qquad \forall z \in E^* \tag{2.3}$$

and $\left(\varphi^* \oplus \psi^* \right)(z)$ is exact. By definitions of convex conjugate and infimal convolution, we can write (2.3) as

$$\sup_{x \in E} \left(\langle z, x \rangle - \varphi(x) - \psi(x) \right) = \inf_{z = z_1 + z_2 \in E^*} \left(\varphi^* \left(z_1 \right) + \psi^* \left(z_2 \right) \right).$$

Since infimal convolution $\left(\varphi^* \oplus \psi^* \right)(z)$ is exact, we can write

$$\sup_{x \in E} \left(\langle z, x \rangle - \varphi(x) - \psi(x) \right) = \min_{z_2 \in E^*} \left(\varphi^* \left(z - z_2 \right) + \psi^* \left(z_2 \right) \right).$$

Let $z = 0$, then we obtain that

$$\sup_{x \in E} \left(-\varphi(x) - \psi(x) \right) = \min_{f \in E^*} \left(\varphi^* \left(-f \right) + \psi^* \left(f \right) \right). \tag{2.4}$$

Note that

$$\sup_{u\in E}\left[-F\left(u\right)\right]=-\inf_{u\in E}F\left(u\right).$$

Then the expression (2.4) can be written as

$$\inf_{x\in E}\left(\varphi(x)+\psi(x)\right)=-\min_{f\in E^{*}}\left(\varphi^{*}\left(-f\right)+\psi^{*}\left(f\right)\right).$$

By definition of infimal convolution, we can rewrite this as

$$\inf_{x\in E}\left(\varphi(x)+\psi(x)\right)=-\left(\varphi^{*}\oplus\psi^{*}\right)(0)$$

where the infimal convolution on the right-hand side is exact. ∎

3. Infimal convolution on couples of Banach spaces

In this section we prove a theorem on subdifferentiability of infimal convolution on couples of Banach spaces. We start with the definition of infimal convolution on couples. Let (X_0, X_1) be a regular Banach couple, i.e. $X_0 \cap X_1$ is dense in both X_0 and X_1, and let $\varphi_0 : X_0 \longrightarrow \mathbb{R}\cup\{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R}\cup\{+\infty\}$ be convex and proper functions. Let us also define the following functions on $X_0 + X_1$:

$$\overline{\varphi}_i\left(u\right)=\begin{cases}\varphi_i\left(u\right) & \text{if } \ u\in X_i\\ +\infty & \text{if } \ u\in\left(X_0+X_1\right)\backslash X_i\end{cases}\qquad i=0,1.$$

Then we can define the infimal convolution $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ on the space $X_0 + X_1$ as follows:

$$(\overline{\varphi}_0\oplus\overline{\varphi}_1)(x)=\inf_{x=x_0+x_1}(\varphi_0(x_0)+\varphi_1(x_1)).$$

Remark 3.1. The functions $\overline{\varphi}_0$, $\overline{\varphi}_1$ may be not lower semicontinuous even when φ_0, φ_1 are lower semicontinuous.

The infimal convolution $(\overline{\varphi}_0\oplus\overline{\varphi}_1)(x)$ can be obtained by solving a minimization problem on the intersection $X_0 \cap X_1$. Indeed, let us fix $x \in (X_0 + X_1)$ such that $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) < +\infty$ and fix a decomposition

$$x=a_0+a_1,\qquad a_0\in X_0,\qquad a_1\in X_1.$$

If $x = x_0 + x_1$, where $x_0 \in X_0$ and $x_1 \in X_1$, then $x_0 + x_1 = a_0 + a_1$. Thus the element $y = a_0 - x_0 = x_1 - a_1 \in (X_0 \cap X_1)$ and we have

$$(\overline{\varphi}_0\oplus\overline{\varphi}_1)(x)=\inf_{x=x_0+x_1}(\varphi_0(x_0)+\varphi_1(x_1))=\inf_{y\in X_0\cap X_1}\left(\varphi_0\left(a_0-y\right)+\varphi_1\left(a_1+y\right)\right).$$

Therefore, if we define the functions

$$S_{a_0}(y)=\varphi_0\left(a_0-y\right),\qquad R_{a_1}(y)=\varphi_1\left(a_1+y\right)\tag{3.1}$$

on the intersection $X_0 \cap X_1$, then we have

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \inf_{y \in X_0 \cap X_1} (S_{a_0}(y) + R_{a_1}(y)) .$$

This representation of infimal convolution is useful because the functions S_{a_0} and R_{a_1} are usually convex, lower semicontinuous, and proper on $X_0 \cap X_1$. The following result will serve as a tool for checking if an infimal convolution defined on a given Banach couple is subdifferentiable.

Theorem 3.1. *Suppose that the functions S_{a_0} and R_{a_1}, defined by (3.1) on $X_0 \cap X_1$, are convex, lower semicontinuous, and proper. Let φ_0^* and φ_1^* be the respective conjugate functions of φ_0 and φ_1, defined on the spaces X_0^* and X_1^*, respectively. Suppose that*

a) *the sets $dom(S_{a_0})$ and $dom(R_{a_1})$ satisfy the equality*

$$\bigcup_{\lambda \geqslant 0} \lambda \left(dom(S_{a_0}) - dom(R_{a_1}) \right) = X_0 \cap X_1;$$

b) *the conjugate function $S_{a_0}^*$ of S_{a_0}, defined on $(X_0 \cap X_1)^* = X_0^* + X_1^*$, is equal to*

$$S_{a_0}^*(z) = \begin{cases} \varphi_0^*(-z) + \langle z, a_0 \rangle & \text{if } z \in X_0^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_0^*; \end{cases}$$

c) *the conjugate function $R_{a_1}^*$ of R_{a_1}, defined on $(X_0 \cap X_1)^* = X_0^* + X_1^*$, is equal to*

$$R_{a_1}^*(z) = \begin{cases} \varphi_1^*(z) + \langle -z, a_1 \rangle & \text{if } z \in X_1^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_1^*. \end{cases}$$

Then the function $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ is subdifferentiable on its domain in $X_0 + X_1$.

Proof. For any given $x \in (X_0 + X_1)$ such that $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) < +\infty$ we consider a decomposition $x = a_0 + a_1$. Then we have

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \inf_{y \in X_0 \cap X_1} (S_{a_0}(y) + R_{a_1}(y)) .$$

From the condition (a) it follows that the functions S_{a_0} and R_{a_1} satisfy the conditions of Corollary 2.1 and therefore by applying this corollary we obtain the equality

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = -\left(S_{a_0}^* \oplus R_{a_1}^* \right)(0)$$

and that the infimal convolution $S_{a_0}^* \oplus R_{a_1}^*$ is exact. Thus there exists an element $y_* \in dom S_{a_0}^* \cap dom R_{a_1}^*$ such that

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = -S_{a_0}^*(-y_*) - R_{a_1}^*(y_*) .$$

By the assumptions (b) and (c) in Theorem 3.1, this is equivalent to

$$\begin{aligned}(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) &= -\varphi_0^*(y_*) + \langle y_*, a_0\rangle - \varphi_1^*(y_*) + \langle y_*, a_1\rangle \\ &= \langle y_*, x\rangle - \varphi_0^*(y_*) - \varphi_1^*(y_*) .\end{aligned}$$

Since the functions $(\overline{\varphi}_i)^*$ $(i = 0, 1)$ on $(X_0 + X_1)^* = X_0^* \cap X_1^*$ coincide with their restrictions φ_i^* $(i = 0, 1)$ on $X_0^* \cap X_1^*$, therefore from Theorem 2.1 we have

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \langle y_*, x\rangle - (\overline{\varphi}_0 \oplus \overline{\varphi}_1)^*(y_*) .$$

Thus the element y_* is dual to the element x with respect to the function $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ and hence the function $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ is *subdifferentiable* on its domain in $X_0 + X_1$. ■

4. Subdifferentiability of *K*-,*L*-, and *E*-functionals

In this section we illustrate Theorem 3.1 by proving subdifferentiability of the K-, L- and E-functionals.

4.1. Some Lemmas

Below we will prove several simple lemmas that show that the conditions of Theorem 3.1 are satisfied for some functions S_{a_0} and R_{a_1} from (3.1). These functions will be used to describe the K-, L_{p_0,p_1}- and E-functionals. Everywhere below the couple (X_0, X_1) is a regular couple.

Lemma 4.1. *Let* $1 \leqslant p_0$, $p_1 < +\infty$ *and let* $a_0 \in X_0$, $a_1 \in X_1$ *be given. Then the functions* $S, R\colon X_0 \cap X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$, *defined by*

$$S(y) = \frac{1}{p_0}\|a_0 - y\|_{X_0}^{p_0}, \qquad R(y) = \frac{t}{p_1}\|a_1 + y\|_{X_1}^{p_1},$$

are convex, proper, and lower semicontinuous.

Proof. We will give the proof only for the function S (the proof for R is similar). It is clear that S is convex and proper. Let us show that it is lower semicontinuous. Suppose that $(u_j)_{j=1}^{+\infty} \in (X_0 \cap X_1)$ converges to y in the norm of $X_0 \cap X_1$:

$$\lim_{j\to+\infty} \|u_j - y\|_{X_0\cap X_1} = 0. \tag{4.1}$$

From the definition of the function S, we can write

$$\begin{aligned}S(y) &= \frac{1}{p_0}\|a_0 - y\|_{X_0}^{p_0} = \frac{1}{p_0}\|a_0 - y + u_j - u_j\|_{X_0}^{p_0} \\ &\leqslant \frac{1}{p_0}\left(\|a_0 - u_j\|_{X_0} + \|u_j - y\|_{X_0}\right)^{p_0} \\ &\leqslant \frac{1}{p_0}\left(\|a_0 - u_j\|_{X_0} + \|u_j - y\|_{X_0\cap X_1}\right)^{p_0} \\ &= \frac{1}{p_0}\left([p_0 S(u_j)]^{1/p_0} + \|u_j - y\|_{X_0\cap X_1}\right)^{p_0} .\end{aligned}$$

Then

$$[p_0 S(y)]^{1/p_0} \leqslant [p_0 S(u_j)]^{1/p_0} + \|u_j - y\|_{X_0 \cap X_1}.$$

Using (4.1) we obtain

$$S(y) \leqslant \liminf_{j \to +\infty} S(u_j)$$

and thus the function S is lower semicontinuous. ■

Lemma 4.2. *Let $\mathcal{B}_{X_1}(\tilde{a}_1; t)$ denote the ball of X_1 of radius $t > 0$ centered at $\tilde{a}_1 \in X_1$ and let $R\colon\ X_0 \cap X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined by*

$$R(u) = \begin{cases} 0 & \text{if } u \in \mathcal{B}_{X_1}(\tilde{a}_1; t) \cap (X_0 \cap X_1) \\ +\infty & \text{otherwise.} \end{cases}$$

Then R is convex, proper, and lower semicontinuous and its conjugate function R^* is equal to

$$R^*(z) = \begin{cases} t\,\|z\|_{X_1^*} + \langle z, \tilde{a}_1 \rangle & \text{if } z \in X_1^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_1^*. \end{cases}$$

Proof. It is clear that R is convex, proper, and lower semicontinuous (as an indicator function of a convex, closed set). Let $z \in X_1^*$. Since

$$R^*(z) = \sup_{\substack{y \in X_0 \cap X_1 \\ \|\tilde{a}_1 - y\|_{X_1} \leqslant t}} \langle z, y \rangle$$

and

$$\sup_{\substack{y \in X_1 \\ \|\tilde{a}_1 - y\|_{X_1} \leqslant t}} \langle z, y \rangle = \sup_{\substack{y \in X_1 \\ \|\tilde{a}_1 - y\|_{X_1} \leqslant t}} \langle z, y - \tilde{a}_1 \rangle + \langle z, \tilde{a}_1 \rangle = t\,\|z\|_{X_1^*} + \langle z, \tilde{a}_1 \rangle$$

therefore to prove the formula for R^* for $z \in X_1^*$ it is sufficient to demonstrate that for any $\overline{y} \in X_1$ such that $\|\tilde{a}_1 - \overline{y}\|_{X_1} \leqslant t$ we have

$$\sup_{\substack{y \in X_0 \cap X_1 \\ \|\tilde{a}_1 - y\|_{X_1} \leqslant t}} \langle z, y \rangle \geqslant \langle z, \overline{y} \rangle.$$

Let $u = \tilde{a}_1 - \overline{y}$. Then

$$\left\| \left(1 - \frac{1}{n}\right) u \right\|_{X_1} = \left(1 - \frac{1}{n}\right) \|u\|_{X_1} \leqslant \left(1 - \frac{1}{n}\right) t.$$

As $X_0 \cap X_1$ is dense in X_1, we can find $u_n \in X_0 \cap X_1$ and $a_{1,n} \in X_0 \cap X_1$ such that

$$\left\| u_n - \left(1 - \frac{1}{n}\right) u \right\|_{X_1} \leqslant \frac{t}{2n} \qquad \text{and} \qquad \|\tilde{a}_1 - a_{1,n}\|_{X_1} \leqslant \frac{t}{2n}.$$

Let us pick $\overline{y_n} = a_{1,n} - u_n \in X_0 \cap X_1$. Then

$$\lim_{n\to+\infty} \|\overline{y} - \overline{y_n}\|_{X_1} = \lim_{n\to+\infty} \|\tilde{a}_1 - u + u_n - a_{1,n}\|_{X_1} = 0$$

and

$$\|\tilde{a}_1 - \overline{y_n}\|_{X_1} = \|\tilde{a}_1 - a_{1,n} + u_n\|_{X_1} \leqslant t.$$

We conclude that

$$\sup_{\substack{y\in X_0\cap X_1 \\ \|\tilde{a}_1 - y\|_{X_1}\leqslant t}} \langle z, y\rangle \geqslant \lim_{n\to+\infty} \langle z, \overline{y_n}\rangle = \langle z, \overline{y}\rangle .$$

Let us now consider the case $z \in (X_0^* + X_1^*) \setminus X_1^*$. We need to show that

$$\sup_{\substack{y\in X_0\cap X_1 \\ \|\tilde{a}_1 - y\|_{X_1}\leqslant t}} \langle z, y\rangle = +\infty.$$

The fact that $z \in (X_0^* + X_1^*) \setminus X_1^*$ implies that z is defined on $X_0 \cap X_1$ and is unbounded on the set $\mathcal{B}_{X_1}(0;1) \cap (X_0 \cap X_1)$. Indeed, if it were bounded on $\mathcal{B}_{X_1}(0;1) \cap (X_0 \cap X_1)$, then, since $X_0 \cap X_1$ is dense in X_1, it would be possible to extend z as a bounded linear functional on X_1. Let $\Omega = \left\{y \in X_0 \cap X_1 : \|\tilde{a}_1 - y\|_{X_1} \leqslant t\right\}$. We need to prove that

$$\sup_{y\in\Omega} \langle z, y\rangle = +\infty. \tag{4.2}$$

Note that $\langle z, y\rangle$ is well-defined because $z \in (X_0^* + X_1^*) = (X_0 \cap X_1)^*$. Let us assume the contrary to (4.2), i.e.

$$\sup_{y\in\Omega} \langle z, y\rangle = C < +\infty.$$

Let us choose $a_{1,t} \in X_0 \cap X_1$ such that $\|\tilde{a}_1 - a_{1,t}\|_{X_1} < \frac{t}{2}$ and consider the set

$$\Omega_{\frac{t}{2}} = \left\{u \in X_0 \cap X_1 : \|u - a_{1,t}\|_{X_1} \leqslant \frac{t}{2}\right\}.$$

Clearly, $\Omega_{\frac{t}{2}} \subset \Omega$. We consider the set $\mathcal{B}_{X_1}(0; \frac{t}{4}) \cap (X_0 \cap X_1)$ and pick an element $v \in \mathcal{B}_{X_1}(0; \frac{t}{4}) \cap (X_0 \cap X_1)$. Then $u = (2v + a_{1,t}) \in \Omega_{\frac{t}{2}}$ and for such v we have

$$\langle z, v\rangle = \left\langle z, -\frac{a_{1,t}}{2}\right\rangle + \frac{1}{2}\langle z, 2v + a_{1,t}\rangle \leqslant \left\langle z, -\frac{a_{1,t}}{2}\right\rangle + \frac{1}{2}C = C_1 < +\infty,$$

i.e.

$$\sup_{v\in\mathcal{B}_{X_1}(0;\frac{t}{4})\cap(X_0\cap X_1)} \langle z, v\rangle \leqslant C_1 < +\infty.$$

This is a contradiction to the fact that z is unbounded on $\mathcal{B}_{X_1}(0;1) \cap (X_0 \cap X_1)$. Therefore

$$\sup_{y\in\Omega} \langle z, y\rangle = +\infty. \qquad \blacksquare$$

Lemma 4.3. *Let $a_0 \in X_0$ and $a_1 \in X_1$ be given. Let also the functions S and R be defined on $X_0 \cap X_1$ by*

$$S(y) = \frac{1}{p_0} \|a_0 - y\|_{X_0}^{p_0} \qquad \text{and} \qquad R(y) = \frac{t}{p_1} \|y + a_1\|_{X_1}^{p_1}$$

and let p_i', $i = 0, 1$ be such that $\frac{1}{p_i} + \frac{1}{p_i'} = 1$. Then

a) *in the case when $1 < p_0, p_1 < +\infty$, we have*

$$S^*(z) = \begin{cases} \frac{1}{p_0'} \|z\|_{X_0^*}^{p_0'} + \langle z, a_0 \rangle & \text{if } z \in X_0^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_0^* \end{cases}$$

and

$$R^*(z) = \begin{cases} \frac{t}{p_1'} \left\| \frac{z}{t} \right\|_{X_1^*}^{p_1'} + \langle -z, a_1 \rangle & \text{if } z \in X_1^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_1^*. \end{cases}$$

b) *in the case when $p_0 = p_1 = 1$, we have*

$$S^*(z) = \begin{cases} \langle z, a_0 \rangle & \text{if } z \in \mathcal{B}_{X_0^*}(0;1) \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus \mathcal{B}_{X_0^*}(0;1) \end{cases}$$

and

$$R^*(z) = \begin{cases} \langle -z, a_1 \rangle & \text{if } z \in \mathcal{B}_{X_1^*}(0;t) \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus \mathcal{B}_{X_1^*}(0;t). \end{cases}$$

Proof. We will only prove the formulas for the function S^* (the proofs for R^* are similar). Let us first consider

The case when $1 < p_0 < +\infty$: Assume that $z \in X_0^*$. From Proposition 2.1 it follows that the conjugate function to the function $T : X_0 \longrightarrow \mathbb{R}_+$, defined by $T(y) = \frac{1}{p_0} \|y\|_{X_0}^{p_0}$, is equal to

$$T^*(z) = \frac{1}{p_0'} \|z\|_{X_0^*}^{p_0'}.$$

Let us calculate S^*. We have

$$S^*(z) = \sup_{y \in X_0 \cap X_1} \left\{ \langle z, y \rangle - \frac{1}{p_0} \|y - a_0\|_{X_0}^{p_0} \right\} = \sup_{y \in X_0 \cap X_1} \{ \langle z, y \rangle - T(y - a_0) \}.$$

Since our couple (X_0, X_1) is regular, i.e. $X_0 \cap X_1$ is dense in both X_0 and X_1 and T is a continuous function with respect to the norm of X_0, then

$$\begin{aligned} S^*(z) &= \sup_{y \in X_0} \{ \langle z, y - a_0 \rangle - T(y - a_0) \} + \langle z, a_0 \rangle \\ &= T^*(z) + \langle z, a_0 \rangle = \frac{1}{p_0'} \|z\|_{X_0^*}^{p_0'} + \langle z, a_0 \rangle. \end{aligned}$$

If $z \in (X_0^* + X_1^*) \setminus X_0^*$, then z is unbounded on $\mathcal{B}_{X_0}(0;1) \cap (X_0 \cap X_1)$. Hence

$$S^*(z) = \sup_{y \in X_0 \cap X_1} \left\{ \langle z, y \rangle - \frac{1}{p_0} \|y - a_0\|_{X_0}^{p_0} \right\} = +\infty.$$

The case when $p_0 = 1$*:* In this case the function T becomes

$$T(y) = \|y\|_{X_0}$$

and its conjugate (see Proposition 2.1) is equal to

$$T^*(z) = \begin{cases} 0 & \text{if } \ z \in \mathcal{B}_{X_0^*}(0;1) \\ +\infty & \text{if } \ z \in (X_0^* + X_1^*) \setminus \mathcal{B}_{X_0^*}(0;1)\,. \end{cases}$$

As before, we have $S^*(z) = T^*(z) + \langle z, a_0 \rangle$. Therefore

$$S^*(z) = \begin{cases} \langle z, a_0 \rangle & \text{if } \ z \in \mathcal{B}_{X_0^*}(0;1) \\ +\infty & \text{if } \ z \in (X_0^* + X_1^*) \setminus \mathcal{B}_{X_0^*}(0;1)\,. \end{cases} \qquad \blacksquare$$

4.2. Subdifferentiability of the E-functional

Given $x \in (X_0 + X_1)$ and $t > 0$, the E-functional is defined by

$$E(t, x; X_0, X_1) = \inf_{\|x_1\|_{X_1} \leqslant t} \|x - x_1\|_{X_0}\,.$$

We can express the E-functional as the following infimal convolution on the sum $X_0 + X_1$

$$E(t, x; X_0, X_1) = (\bar{\varphi}_0 \oplus \bar{\varphi}_1)(x)\,,$$

where $\bar{\varphi}_0$ and $\bar{\varphi}_1$ are functions defined on the sum $X_0 + X_1$ by

$$\bar{\varphi}_0(u) = \begin{cases} \|u\|_{X_0} & \text{if } \ u \in X_0 \\ +\infty & \text{if } \ u \in (X_0 + X_1) \setminus X_0 \end{cases} \tag{4.3}$$

and

$$\bar{\varphi}_1(u) = \begin{cases} 0 & \text{if } \ u \in \mathcal{B}_{X_1}(0;t) \\ +\infty & \text{if } \ u \in (X_0 + X_1) \setminus \mathcal{B}_{X_1}(0;t)\,, \end{cases} \tag{4.4}$$

where $\mathcal{B}_{X_1}(0;t)$ is the ball of X_1 of radius t centered at the origin. In this case the functions $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ are defined by

$$\varphi_0(u) = \|u\|_{X_0} \qquad \text{and} \qquad \varphi_1(u) = \begin{cases} 0 & \text{if } \ u \in \mathcal{B}_{X_1}(0;t) \\ +\infty & \text{if } \ u \in X_1 \setminus \mathcal{B}_{X_1}(0;t)\,. \end{cases} \tag{4.5}$$

Let us fix $a_0 \in X_0$ and $a_1 \in X_1$ such that $x = a_0 + a_1$. Then

$$E(t, x; X_0, X_1) = \inf_{\|a_1 + y\|_{X_1} \leqslant t} \|a_0 - y\|_{X_0}\,, \qquad y \in (X_0 \cap X_1)$$

and, similarly to (3.1), we can rewrite the E-functional as

$$E(t, x; X_0, X_1) = \inf_{y \in X_0 \cap X_1} \{S_{a_0}(y) + R_{a_1}(y)\},$$

where $S_{a_0}(y), R_{a_1}(y) : X_0 \cap X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ are the functions defined by

$$S_{a_0}(y) = \|a_0 - y\|_{X_0}$$

and

$$R_{a_1}(y) = \begin{cases} 0 & \text{if } \ y \in (\mathcal{B}_{X_1}(-a_1; t) \cap (X_0 \cap X_1)) \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 4.1. *The E-functional is subdifferentiable on its whole domain.*

Proof. It follows from Lemma 4.1 that the function S_{a_0} is convex, lower semicontinuous, and proper. The function R_{a_1} is convex and lower semicontinuous as an indicator function of a convex and closed set $\mathcal{B}_{X_1}(-a_1; t) \cap (X_0 \cap X_1)$. Since the space $X_0 \cap X_1$ is dense in X_1, then $\mathcal{B}_{X_1}(-a_1; t) \cap (X_0 \cap X_1) \neq \emptyset$ and therefore the function R_{a_1} is proper. Moreover, since $domS_{a_0} = X_0 \cap X_1$ and $domR_{a_1} = \mathcal{B}_{X_1}(-a_1; t) \cap (X_0 \cap X_1)$ then

$$\bigcup_{\lambda \geqslant 0} \lambda (domS_{a_0} - domR_{a_1}) = X_0 \cap X_1.$$

Thus the condition (a) of the Theorem 3.1 is satisfied. The conjugate function φ_0^* of φ_0 is defined on X_0^* and is given by

$$\varphi_0^*(z) = \sup_{u \in X_0} \left(\langle z, u\rangle - \|u\|_{X_0}\right) = \begin{cases} 0 & \text{if } \ z \in \mathcal{B}_{X_0^*}(0; 1) \\ +\infty & \text{if } \ z \in X_0^* \backslash \mathcal{B}_{X_0^*}(0; 1). \end{cases} \tag{4.6}$$

The conjugate function $S_{a_0}^*$ of S_{a_0} can be obtained from Lemma 4.3:

$$S_{a_0}^*(z) = \begin{cases} \langle z, a_0\rangle & \text{if } \ z \in \mathcal{B}_{X_0^*}(0; 1) \\ +\infty & \text{if } \ z \in (X_0^* + X_1^*) \backslash \mathcal{B}_{X_0^*}(0; 1). \end{cases}$$

Hence

$$S_{a_0}^*(z) = \begin{cases} \varphi_0^*(-z) + \langle z, a_0\rangle & \text{if } \ z \in X_0^* \\ +\infty & \text{if } \ z \in (X_0^* + X_1^*) \backslash X_0^*. \end{cases}$$

The conjugate φ_1^* of φ_1 is defined on X_1^* and is given by

$$\varphi_1^*(z) = \sup_{x \in \mathcal{B}_{X_1}(0; t)} \langle z, x\rangle = t\, \|z\|_{X_1^*}. \tag{4.7}$$

From Lemma 4.2, where we take $\tilde{a}_1 = -a_1$, we obtain

$$R_{a_1}^*(z) = \begin{cases} t\, \|z\|_{X_1^*} - \langle z, a_1\rangle & \text{if } \ z \in X_1^* \\ +\infty & \text{if } \ z \in (X_0^* + X_1^*) \backslash X_1^*, \end{cases}$$

i.e.

$$R_{a_1}^*(z) = \begin{cases} \varphi_1^*(z) - \langle z, a_1 \rangle & \text{if } z \in X_1^* \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_1^*. \end{cases}$$

So all the conditions of Theorem 3.1 are satisfied, therefore the E-functional is subdifferentiable on its domain in $X_0 + X_1$. ■

4.3. Subdifferentiability of the L-functional

Let $x \in (X_0 + X_1)$ and let $t > 0$ be a fixed parameter. We consider the following L-functional

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right), \tag{4.8}$$

where $1 \leqslant p_0,\, p_1 < \infty$. Note that the K-functional corresponds to the particular case when $p_0 = p_1 = 1$.

The L_{p_0,p_1}-functional can be written as the infimal convolution

$$L_{p_0,p_1}(t, x; X_0, X_1) = (\bar{\varphi}_0 \oplus \bar{\varphi}_1)(x),$$

where the functions $\bar{\varphi}_0$ and $\bar{\varphi}_1$ are both defined on the sum $X_0 + X_1$ as follows

$$\bar{\varphi}_0(u) = \begin{cases} \frac{1}{p_0} \|u\|_{X_0}^{p_0} & \text{if } u \in X_0 \\ +\infty & \text{if } u \in (X_0 + X_1) \setminus X_0. \end{cases} \tag{4.9}$$

and

$$\bar{\varphi}_1(u) = \begin{cases} \frac{t}{p_1} \|u\|_{X_1}^{p_1} & \text{if } u \in X_1 \\ +\infty & \text{if } u \in (X_0 + X_1) \setminus X_1. \end{cases} \tag{4.10}$$

In this case, the functions $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ are defined by

$$\varphi_0(u) = \frac{1}{p_0} \|u\|_{X_0}^{p_0} \qquad \text{and} \qquad \varphi_1(u) = \frac{t}{p_1} \|u\|_{X_1}^{p_1}. \tag{4.11}$$

Theorem 4.2. *The L-functional (4.8) is subdifferentiable on $X_0 + X_1$.*

Proof. We will only consider the case when $p_0, p_1 > 1$, as the proofs for the other cases are similar. For given $a_0 \in X_0$ and $a_1 \in X_1$ such that $x = a_0 + a_1$, we can define the L_{p_0,p_1}-functional as

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{y \in X_0 \cap X_1} \{S_{a_0}(y) + R_{a_1}(y)\},$$

where

$$S_{a_0}(y) = \frac{1}{p_0} \|a_0 - y\|_{X_0}^{p_0} \qquad \text{and} \qquad R_{a_1}(y) = \frac{t}{p_1} \|y + a_1\|_{X_1}^{p_1}.$$

Moreover,

$$L_{p_0,p_1}(t, x; X_0, X_1) = (\bar{\varphi}_0 \oplus \bar{\varphi}_1)(x),$$

where the functions $\bar{\varphi}_0$ and $\bar{\varphi}_1$ are defined by (4.9) and (4.10), respectively. From Lemma 4.1 we see that the functions S_{a_0} and R_{a_1} are convex, proper, and lower semicontinuous and since $domS_{a_0} = domR_{a_1} = X_0 \cap X_1$ then

$$\bigcup_{\lambda \geqslant 0} \lambda \left(domS_{a_0} - domR_{a_1} \right) = X_0 \cap X_1.$$

Thus the condition (a) of Theorem 3.1 is satisfied. The respective conjugate functions S^* and R^* of S and R are given in Lemma 4.3:

$$S^*_{a_0}(z) = \begin{cases} \frac{1}{p'_0} \|z\|^{p'_0}_{X^*_0} + \langle z, a_0 \rangle & \text{if } \ z \in X^*_0 \\ +\infty & \text{if } \ z \in (X^*_0 + X^*_1) \backslash X^*_0 \end{cases}$$

and

$$R^*_{a_1}(z) = \begin{cases} \frac{t}{p'_1} \left\| \frac{z}{t} \right\|^{p'_1}_{X^*_1} + \langle -z, a_1 \rangle & \text{if } \ z \in X^*_1 \\ +\infty & \text{if } \ z \in (X^*_0 + X^*_1) \backslash X^*_1. \end{cases}$$

The conjugate functions φ^*_0 of φ_0 and φ^*_1 of φ_1 are defined on X^*_0 and X^*_1, respectively, and are given by (see Propositions 2.1-2.2)

$$\varphi^*_0(z) = \frac{1}{p'_0} \|z\|^{p'_0}_{X^*_0} \qquad \forall z \in X^*_0 \tag{4.12}$$

and

$$\varphi^*_1(z) = \frac{t}{p'_1} \left\| \frac{z}{t} \right\|^{p'_1}_{X^*_1} \qquad \forall z \in X^*_1. \tag{4.13}$$

It is clear that

$$S^*_{a_0}(z) = \begin{cases} \varphi^*_0(-z) + \langle z, a_0 \rangle & \text{if } \ z \in X^*_0 \\ +\infty & \text{if } \ z \in (X^*_0 + X^*_1) \backslash X^*_0 \end{cases}$$

and

$$R^*_{a_1}(z) = \begin{cases} \varphi^*_1(z) + \langle -z, a_1 \rangle & \text{if } \ z \in X^*_1 \\ +\infty & \text{if } \ z \in (X^*_0 + X^*_1) \backslash X^*_1. \end{cases}$$

Thus all the conditions of Theorem 3.1 are satisfied and therefore the L_{p_0,p_1}-functional is subdifferentiable on its domain, which is equal to $X_0 + X_1$. ■

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Addresses: Natan Kruglyak: Division of Mathematics and Applied Mathematics, Department of Mathematics, Linköping University, SE–581 83 Linköping, Sweden;
Japhet Niyobuhungiro: Division of Mathematics and Applied Mathematics, Department of Mathematics, Linköping University, SE–581 83 Linköping, Sweden, and Department of Mathematics, School of Pure and Applied Science, College of Science and Technology, University of Rwanda P.O. Box 3900 Kigali, Rwanda.

E-mail: natan.kruglyak@liu.se, japhet.niyobuhungiro@liu.se, jniyobuhungiro@ur.ac.rw

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