

ON THE PROXIMITY OF ADDITIVE AND MULTIPLICATIVE FUNCTIONS

JEAN-MARIE DE KONINCK, NICOLAS DOYON, PATRICK LETENDRE

Abstract: Given an additive function f and a multiplicative function g , let $E(f, g; x) = \#\{n \leq x : f(n) = g(n)\}$. We study the size of $E(f, g; x)$ for functions f such that $f(n) \neq 0$ for at least one integer $n > 1$. In particular, we show that for those additive functions f whose values $f(n)$ are concentrated around their mean value $\lambda(n)$, one can find a multiplicative function g such that, given any $\varepsilon > 0$, then $E(f, g; x) \gg x/\lambda(x)^{1+\varepsilon}$. We also show that given any additive function satisfying certain regularity conditions, no multiplicative function can coincide with it on a set of positive density. It follows that if $\omega(n)$ stands for the number of distinct prime factors of n , then, given any $\varepsilon > 0$, there exists a multiplicative function g such that $E(\omega, g; x) \gg x/(\log \log x)^{1+\varepsilon}$, while for all multiplicative functions g , we have $E(\omega, g; x) = o(x)$ as $x \rightarrow \infty$.

Keywords: additive functions, multiplicative functions, number of distinct prime factors of an integer.

1. Introduction

We investigate the deceptively simple question of whether the behavior of a multiplicative function can resemble that of an additive function. More precisely, letting \mathbb{N} stand for the set of all positive integers, we ask the following question: can one find an additive function f and a multiplicative function g such that $f(n) = g(n)$ for all $n \in R$, where R is a large subset of \mathbb{N} ?

For convenience, let us write \mathcal{A} for the set of all additive functions f such that $f(1) = 0$ and \mathcal{M} for the set of all multiplicative functions g such that $g(1) = 1$. From here on, we only consider real arithmetic functions. Given $f \in \mathcal{A}$ and $g \in \mathcal{M}$, we define

$$E(f, g; x) := \#\{n \leq x : f(n) = g(n)\}.$$

A *strongly additive function* (resp. *strongly multiplicative function*) h is a function in \mathcal{A} (resp. in \mathcal{M}) such that $h(p^a) = h(p)$ for all integers $a \geq 1$ and all primes p . We shall write \mathcal{A}^* (resp. \mathcal{M}^*) for the set of strongly additive functions (resp. strongly multiplicative functions). Given an integer $n \geq 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n , and set $\omega(1) = 0$.

Of course, we are not interested in the trivial case $f(n) = 0$ for all $n \geq 1$ and $g(n) = 0$ for all $n \geq 2$, since in this case we simply get $E(f, g; x) = \lfloor x \rfloor - 1$. We are instead interested in studying the size of $E(f, g; x)$ when the functions $f \in \mathcal{A}$ and $g \in \mathcal{M}$ are such that $f(n) \neq 0$ for at least one integer $n > 1$, in which case one can prove that they differ infinitely often.

Here we show that for those additive functions f whose values $f(n)$ are concentrated around their mean value $\lambda(n)$ in the sense that, given any $\varepsilon > 0$,

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{n < x : |f(n) - \lambda(x)| > \lambda(x)^{1-\varepsilon}\} = 0,$$

then, one can find a multiplicative function g such that

$$E(f, g; x) \gg \frac{x}{\lambda(x)^{1+\varepsilon}}.$$

We also show that given an additive function f satisfying certain regularity conditions, no function $g \in \mathcal{M}$ can coincide with f on a set of positive density.

In the case where the additive function f is chosen to be ω , it follows that, given any $\varepsilon > 0$, there exists a multiplicative function g such that

$$E(\omega, g; x) \gg \frac{x}{(\log \log x)^{1+\varepsilon}},$$

while for all $g \in \mathcal{M}$ we have

$$E(\omega, g; x) = o(x) \quad (x \rightarrow \infty).$$

2. Notation and preliminary results

Before stating our results, we introduce additional notation and state preliminary results.

We shall write \wp for the set of all prime numbers, while the letter p will always stand for a prime number. The letter c , with or without subscript, always denotes a constant, but not necessarily the same at each occurrence. By $\log_2 x$, we mean $\max(1, \log \log x)$; by $\log_3 x$, we mean $\max(1, \log \log \log x)$; and so on. We will use $P^+(n)$ for the largest prime factor of $n \geq 2$ and $P^-(n)$ for the smallest prime factor of $n \geq 2$. For convenience, we set $P^+(1) = P^-(1) = 1$. We also define as is customary, for $2 \leq y \leq x$,

$$\Psi(x, y) := \#\{n \leq x : P^+(n) \leq y\} \quad \text{and} \quad \Phi(x, y) := \#\{n \leq x : P^-(n) > y\}.$$

Lemma 1. *Let $D \geq 3$ be a fixed integer and let $0 < a_1 < a_2 < 1$ with $a_2 > 2a_1$. Then,*

$$S_D(x) := \sum_{\substack{a_1x \leq n \leq a_2x \\ P^-(n) > D}} \frac{1}{n} \ll \frac{\log(a_2/a_1)}{\log D}. \tag{2.1}$$

Proof. Using the Stieltjes integral representation of the sum in (2.1), we obtain that

$$S_D(x) = \int_{a_1x}^{a_2x} \frac{1}{t} d\Phi(t, D) = \frac{\Phi(t, D)}{t} \Big|_{a_1x}^{a_2x} + \int_{a_1x}^{a_2x} \frac{\Phi(t, D)}{t^2} dt. \tag{2.2}$$

Setting $u = \log t / \log D$ and using the estimate

$$\Phi(t, D) = \frac{t\omega_0(u)}{\log D} - \frac{D}{\log D} + O\left(\frac{t}{\log^2 D}\right),$$

where $\omega_0(u)$ stands for the Buchstab function (see Theorem 6.4 in the book of Tenenbaum [7]) and recalling that $\frac{1}{2} \leq \omega_0(u) \leq 1$ for all $u \geq 1$, it follows from (2.2) that

$$\begin{aligned} S_D(x) &= \frac{\omega_0(u)}{\log D} \Big|_{a_1x}^{a_2x} + O\left(\frac{1}{\log^2 D}\right) + \int_{a_1x}^{a_2x} \left(\frac{\omega_0(u)}{t \log D} + O\left(\frac{1}{t \log^2 D}\right)\right) dt \\ &= O\left(\frac{1}{\log D}\right) + O\left(\frac{1}{\log D} \int_{a_1x}^{a_2x} \frac{dt}{t}\right) \\ &\ll \frac{\log(a_2/a_1)}{\log D}, \end{aligned}$$

thus establishing our claim. ■

Lemma 2. *There exists a positive constant c_3 such that*

$$\Psi(x, y) \leq c_3x \exp\left\{-\frac{1}{2} \frac{\log x}{\log y}\right\} \quad (2 \leq y \leq x).$$

Proof. For a proof, see Theorem 5.1 in the book of Tenenbaum [7]. ■

Lemma 3. *Uniformly for $x \geq 3$ and $0 \leq \xi(x) \leq \sqrt{\log \log x}$,*

$$\#\{n \leq x : |\omega(n) - \log \log x| > \xi(x)\sqrt{\log \log x}\} \ll xe^{-\xi(x)^2/3}.$$

Proof. This result follows immediately from Theorem 3.8 in the book of Tenenbaum [7]. ■

Lemma 4. *For each positive integer k , let $\pi_k(x) := \#\{n \leq x : \omega(n) = k\}$. Then the maximum value of $\pi_k(x)$ is attained when $k = k_0 = \log \log x + O(1)$, in which case we have $\pi_{k_0}(x) = (1 + o(1))x/\sqrt{\log \log x}$ as $x \rightarrow \infty$.*

Proof. This follows from a result of Balazard [1]. ■

Lemma 5. *Let f be a complex-valued additive function and set*

$$A(x) = A_f(x) := \sum_{p^\alpha \leq x} \frac{f(p^\alpha)}{p^\alpha} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad B(x)^2 = B_f(x)^2 := \sum_{p^\alpha \leq x} \frac{|f(p^\alpha)|^2}{p^\alpha}. \tag{2.3}$$

Then there exists an absolute constant $C > 0$ such that for all $x \geq 2$, we have

$$\frac{1}{x} \sum_{n \leq x} |f(n) - A(x)|^2 \leq CB(x)^2,$$

where the constant C can be replaced by $\frac{3}{2} + O\left(\frac{1}{\sqrt{\log x}}\right)$.

Proof. See Kubilius [5] and Hildebrand [4]. ■

To each real number $D > 1$ we associate the multiplicative function

$$G_D(n) = \prod_{\substack{p^\alpha \parallel n \\ p \leq D}} p^\alpha.$$

Lemma 6. *Let $\varepsilon > 0$ be a small number and let D be a large number satisfying*

$$D > e^{12/\varepsilon}. \tag{2.4}$$

Then, there exist two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$, with $c_1 < 1 < c_2$, and a real number $x_0 = x_0(\varepsilon)$ such that

$$\#\{n \leq x : D^{c_1} < G_D(n) < D^{c_2}\} \geq (1 - \varepsilon)x \quad (x \geq x_0). \tag{2.5}$$

Proof. In order to prove (2.5), we will prove that for some large $x_1 = x_1(\varepsilon)$,

$$\#\{n \leq x : G_D(n) \leq D^{c_1}\} \leq \frac{\varepsilon}{2}x \quad (x \geq x_1) \tag{2.6}$$

and that

$$\#\{n \leq x : G_D(n) \geq D^{c_2}\} \leq \frac{\varepsilon}{2}x \quad (x \geq 1). \tag{2.7}$$

The result will then follow by choosing $x_0 = x_1$.

First recall that, for some positive constant c_4 ,

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c_4 + \frac{\theta_x}{\log^2 x} \quad (x \geq 286), \tag{2.8}$$

where $|\theta_x| \leq 1/2$, an estimate due to Rosser and Schoenfeld [6].

With $c_1 < 1$, it follows from (2.8) that there exists $x_1 = x_1(\varepsilon) \geq 286$ such that for all $x \geq x_1$,

$$\begin{aligned}
 \#\{n \leq x : G_D(n) \leq D^{c_1}\} &\leq \#\{n \leq x : \text{there exists no prime } p \mid n \text{ with } D^{c_1} < p \leq D\} \\
 &\leq \frac{3}{2}x \prod_{D^{c_1} < p \leq D} \left(1 - \frac{1}{p}\right) = \frac{3}{2}x \exp \sum_{D^{c_1} < p \leq D} \log \left(1 - \frac{1}{p}\right) \\
 &< \frac{3}{2}x \exp \left(- \sum_{D^{c_1} < p \leq D} \frac{1}{p} + \frac{1}{2D^{c_1}}\right) \\
 &\leq \frac{3}{2}x \exp \left(\log \log D^{c_1} - \log \log D + \frac{1}{2D^{c_1}} + \frac{1}{2 \log^2 D^{c_1}}\right) \\
 &= \frac{3}{2}x c_1 \exp \left(\frac{1}{2D^{c_1}} + \frac{1}{\log^2 D^{c_1}}\right) \\
 &< \frac{3}{2}x c_1 2 = 3c_1 x, \tag{2.9}
 \end{aligned}$$

where we used (2.4). Choosing $c_1 = \varepsilon/6$, then (2.9) yields (2.6).

On the other hand, in order to prove (2.7), first apply Lemma 5 to the additive function $f(n) = \log G_D(n)$ for which the corresponding sums $A(x)$ and $B(x)$ satisfy

$$A(x) = \log D + O(1) \quad \text{and} \quad B^2(x) = \frac{\log^2 D}{2} + O(1),$$

from which it follows that, for all $x \geq 1$,

$$\begin{aligned}
 \#\{n \leq x : G_D(n) \geq D^{c_2}\} &= \#\{n \leq x : f(n) \geq c_2 \log D\} \\
 &\leq \#\{n \leq x : |f(n) - \log D|^2 \geq (c_2 - 1)^2 \log^2 D\} \\
 &\leq \sum_{n \leq x} \frac{|f(n) - \log D|^2}{(c_2 - 1)^2 \log^2 D} \\
 &\leq \frac{1}{(c_2 - 1)^2 \log^2 D} Cx \frac{\log^2 D}{2} \\
 &< \frac{\varepsilon}{2} x,
 \end{aligned}$$

provided we choose $c_2 > \sqrt{\frac{C}{\varepsilon}} + 1$, thus establishing (2.7). ■

3. Lower bounds

It is easy to show that for any $f \in \mathcal{A}$, there exists $g \in \mathcal{M}$ for which $E(f, g; x) > x/\log x$ for all $x \geq 11$. Indeed, since any additive or multiplicative function is entirely determined by its values at prime powers, we only need to choose $g(p^\alpha) =$

$f(p^\alpha)$ for all $p \in \wp$ and $\alpha \in \mathbb{N}$. Then, letting R stand for the set of all prime powers, it is clear that $g(n) = f(n)$ for all $n \in R$, whose counting function $R(x)$ satisfies $R(x) > x/\log x$ for all $x \geq 11$.

We now turn to a general question: Given $f \in \mathcal{A}$, can one find $g \in \mathcal{M}$ such that $E(f, g; x)$ is maximal? The answer is of course highly dependant on the particular function f considered and we will show that the size of $E(x)$ can range from $E(x) = 0$ (simply choose $f \in \mathcal{A}^*$ defined by $f(p) = -1$ for all $p \in \wp$ and $g \in \mathcal{M}^*$ defined by $g(p) = 1$ for all $p \in \wp$) to $E(x) \geq cx$ for any positive constant $c < 1$.

To show that for any given positive constant $c < 1$, there exist a function $f \in \mathcal{A}$ and a corresponding function $g \in \mathcal{M}$ which yields $E(f, g; x) \geq cx$, as $x \rightarrow \infty$, we first let $\mathcal{S} \subset \wp$ be a set satisfying

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty \tag{3.1}$$

and define $f \in \mathcal{A}^*$ by

$$f(p) = \begin{cases} 1 & \text{if } p \in \mathcal{S}, \\ 0 & \text{if } p \in \wp \setminus \mathcal{S}. \end{cases}$$

It is easy to see that $f(n)$ will be zero on a subset of \mathbb{N} of positive density $c = \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right)$. It follows that the strongly multiplicative function g defined by $f(p) = g(p)$ on all primes p will give $E(f, g; x) \geq cx$. It is also straightforward, given the construction of \mathcal{S} , that the constant c can be chosen arbitrarily close to 1.

More generally, if f is a strongly additive function such that $f(p) \neq 0$ on a set of primes \mathcal{S} satisfying (3.1), then one can construct a strongly multiplicative function g such that $E(f, g; x) \geq c_1x$ for some positive constant c_1 depending on f . One way to construct this corresponding function g is to choose

$$g(p) = \begin{cases} f(p) & \text{if } p \in \mathcal{S}, \\ 1 & \text{if } p \in \wp \setminus \mathcal{S}. \end{cases}$$

This will give $g(n) = f(n) = f(p)$ for a certain $p \in \mathcal{S}$, that is provided that

$$\prod_{\substack{q|n \\ q \in \mathcal{S}}} q = p,$$

and thus

$$E(f, g; x) \gtrsim \sum_{p \in \mathcal{S}} \sum_{a \geq 1} \frac{x}{p^a} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right) = c \cdot x \sum_{p \in \mathcal{S}} \sum_{a \geq 1} \frac{1}{p^a} =: c_1x.$$

In some instances, it is easy to show that this strategy is optimal. Indeed, consider the additive function f defined by $f(2) = b$, where b is any fixed real number, and $f(p) = 0$ for all primes $p \geq 3$, and take the strongly multiplicative function g as $g(2) = b$ and $g(p) = 1$ for all primes $p \geq 3$. Then, since $f(n)$ and $g(n)$ agree at even integers, we easily see that $E(f, g; x) = \lfloor x/2 \rfloor$. But we cannot hope to do any better. Indeed, for an arbitrary multiplicative function g , let κ be the proportion of those odd integers m for which $g(m) = f(m) = 0$. Then, since $g(2^a m) \neq f(2^a m)$ for all $a \in \mathbb{N}$, the proportion of even integers for which g agrees with f is at most $1 - \kappa$. It follows from this observation that

$$E(f, g; x) \lesssim x \left(\frac{1}{2}\kappa + \frac{1}{2}(1 - \kappa) \right) = \frac{x}{2}.$$

If one considers a more common additive function, say $f = \omega$, then determining the maximal size of $E(\omega, g; x)$ for an optimal g is much more difficult. A natural way to proceed is the following. Fix an arbitrary integer $k \geq 2$. One can prove that for $f = \omega \in \mathcal{A}$, there exists $g \in \mathcal{M}$ such that, for some positive constant $C > 0$,

$$E(\omega, g; x) > C \frac{x}{\log x} \cdot \frac{(\log \log x)^{k-1}}{(k-1)!} \quad (\text{for } x \text{ sufficiently large}). \tag{3.2}$$

Indeed, consider the strongly multiplicative function g defined on primes p by $g(p) = k^{1/k}$. Then, clearly, for those integers n such that $\omega(n) = k$, we have $g(n) = (k^{1/k})^{\omega(n)} = k$.

It remains to count the number of integers $n \leq x$ such that $\omega(n) = k$. But this is well known. Indeed, letting $\pi_r(x)$ stand for the number of positive integers $n \leq x$ such that $\omega(n) = r$, it is known since Landau (see Theorem 10.3 in De Koninck and Luca [3]) that, for each fixed $r \in \mathbb{N}$,

$$\pi_r(x) = (1 + o(1)) \frac{x}{\log x} \cdot \frac{(\log \log x)^{r-1}}{(r-1)!} \quad (x \rightarrow \infty).$$

It is also known that $\pi(x) > x/\log x$ for all $x \geq 11$ (see for instance Rosser and Schoenfeld [6]). Using this, one can easily prove by induction that for each integer $r \geq 2$, there exists a positive constant c_r and a number x_r such that

$$\pi_r(x) > c_r \frac{x}{\log x} \cdot \frac{(\log \log x)^{r-1}}{(r-1)!} \quad (\text{for all } x > x_r). \tag{3.3}$$

Hence, choosing $r = k$ in (3.3), estimate (3.2) follows immediately.

Does this provide an optimal result? Can we construct a multiplicative function g which agrees with ω on a larger set? Let us first consider a multiplicative function g which has “natural” properties. For example, since the mean and normal orders of $\omega(n)$ are $\log \log n$, it would seem natural to look for a multiplicative function $g(n)$ which also has mean and normal orders of $\log \log n$. But this is impossible. Indeed, Birch [2] has proven that the only unbounded multiplicative

functions $g(n)$ with a non-decreasing normal order are the powers of n . Let us therefore consider a seemingly unnatural multiplicative function g that agrees with ω on a large set of values. One key idea to construct such a tricky function is to choose g such that $g(p) \neq 1$ only if $p \in \mathcal{S}$, where \mathcal{S} is an infinite set of primes satisfying

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty.$$

Given an integer $n \geq 2$, we shall write

$$n = \prod_{\substack{p^a \parallel n \\ p \in \mathcal{S}}} p^a \cdot \prod_{\substack{p^a \parallel n \\ p \notin \mathcal{S}}} p^a = s \cdot m,$$

say. This means that the value of $g(n)$ will be determined by s while the value of $\omega(n)$ will essentially be determined by $\omega(m)$. Thus, in order for $g(n)$ to be equal to $\omega(n)$ for many $n \leq x$, we will arrange for $g(s)$ to be close to $\log \log x$ as often as possible. This idea is expressed in the following theorem, where we consider a more general strongly additive function f .

Theorem 1. *Let $\mathcal{S} := \{s_1, s_2, \dots\}$ be an infinite set of primes such that $s_1 < s_2 < \dots$ and satisfying $\sum_{j=1}^{\infty} \frac{1}{s_j} < \infty$. Assume that f is a non negative integer valued strongly additive function and $\lambda(x)$ a function which tends to infinity with x such that*

$$\#\left\{n \leq \frac{x}{s^{\lceil \lambda(x) \rceil}} : f(n) > \lambda(x)\right\} = o\left(\frac{x}{s^{\lceil \lambda(x) \rceil}}\right) \quad (x \rightarrow \infty). \tag{3.4}$$

Then, letting g be the strongly multiplicative function defined on the primes p by

$$g(p) = \begin{cases} j + f(s_j) & \text{if } p = s_j \text{ for some } s_j \in \mathcal{S}, \\ 1 & \text{if } p \notin \mathcal{S}, \end{cases}$$

we have

$$E(f, g; x) \gg \frac{x}{s^{\lceil \lambda(x) \rceil}}. \tag{3.5}$$

Remark 1. Clearly Theorem 1 implies in particular that, given any $\varepsilon > 0$,

$$E(f, g; x) \gg \frac{x}{\lambda(x)^{1+\varepsilon}}.$$

Proof of Theorem 1. In order to find a lower bound for $E(f, g; x) = \#\{n \leq x : f(n) = g(n)\}$, we will only count those positive integers $n \leq x$ which can be written as $n = m \cdot s_j$ with an m such that $s_k \nmid m$ for all integers $k \geq 1$. Hence,

$$E(f, g; x) \geq \sum_{j \geq 1} \#\{n = m \cdot s_j \leq x : s_k \nmid m \ \forall k, f(n) = g(n)\}. \tag{3.6}$$

Given that $g(n) = g(m \cdot s_j) = g(s_j) = j + f(s_j)$ and that $f(n) = f(m \cdot s_j) = f(m) + f(s_j)$, the condition $f(n) = g(n)$ is equivalent to $f(m) = j$. Inequality (3.6) can therefore be replaced by

$$E(f, g; x) \geq \sum_{j \geq 1} \#\{n = m \cdot s_j \leq x : s_k \nmid m \ \forall k, f(m) = j\},$$

which is equivalent to

$$E(f, g; x) \geq \sum_{j \geq 1} \#\left\{m \leq \frac{x}{s_j} : s_k \nmid m \ \forall k, f(m) = j\right\}. \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned} E(f, g; x) &\geq \sum_{j \leq \lambda(x)} \#\left\{m \leq \frac{x}{s_j} : s_k \nmid m \ \forall k, f(m) = j\right\} \\ &\geq \sum_{j \leq \lambda(x)} \#\left\{m \leq \frac{x}{s_{\lceil \lambda(x) \rceil}} : s_k \nmid m \ \forall k, f(m) = j\right\} \\ &\geq \#\left\{m \leq \frac{x}{s_{\lceil \lambda(x) \rceil}} : s_k \nmid m \ \forall k\right\} \\ &\quad + O\left(\#\left\{m \leq \frac{x}{s_{\lceil \lambda(x) \rceil}} : f(m) > \lambda(x)\right\}\right). \end{aligned} \tag{3.8}$$

Setting $C(\mathcal{S}) := \prod_{j=1}^{\infty} \left(1 - \frac{1}{s_j}\right)$, which is positive because $\sum_{j=1}^{\infty} \frac{1}{s_j} < \infty$, we obtain, in light of inequality (3.8) and hypothesis (3.4),

$$E(f, g; x) \gg C(\mathcal{S}) \frac{x}{s_{\lceil \lambda(x) \rceil}} + o\left(\frac{x}{s_{\lceil \lambda(x) \rceil}}\right),$$

from which conclusion (3.5) follows immediately. ■

A direct application of this theorem is the following.

Corollary 1. *Given an arbitrary integer $k \geq 1$ and any real number $\delta > 0$, there exists a multiplicative function g such that*

$$E(\omega, g; x) \gg \frac{x}{\left(\prod_{r=1}^k \log_{r+1} x\right) (\log_{k+2} x)^{1+\delta}}. \tag{3.9}$$

In particular, given any $\varepsilon > 0$, there exists a multiplicative function g such that

$$E(\omega, g; x) \gg \frac{x}{(\log \log x)^{1+\varepsilon}}. \tag{3.10}$$

Proof. In Theorem 1, take $f = \omega$. Then, for any positive integer k and fixed $\delta > 0$, let $\mathcal{S} := \{s_1, s_2, \dots\}$ be an infinite set of primes defined as follows: first set $s_1 = 2$ and, for each $j \geq 2$, let s_j be the smallest prime number larger than

$$\max \left(s_{j-1}, j \left(\prod_{r=1}^k \log_r j \right) (\log_{k+1} j)^{1+\delta} \right).$$

Then, choosing $\lambda(x) = \log \log x + \xi(x)\sqrt{\log \log x}$, where $\xi(x)$ is any function tending to infinity with x , we immediately obtain (3.9).

Finally, (3.10) is an easy consequence of (3.9). ■

Can we improve the lower bound obtained in (3.10)? While an optimal answer to this question is difficult to provide, the next theorem and the remark that follows provide a partial answer.

Theorem 2. *Given any small number $\varepsilon > 0$, there exists a multiplicative function g and an infinite sequence of integers r such that*

$$E(\omega, g; r) \gg \frac{r}{(\log \log r)^{1/2+\varepsilon}}.$$

Proof. Let $\delta > 0$ be a fixed small number. Let $s_1 = 2$, and for each integer $j \geq 2$, let s_j be the smallest prime number larger than $\max(s_{j-1}, j^{1+\delta})$ and set $\mathcal{S} := \{s_1, s_2, \dots\}$. Define the sequence of positive real numbers r_1, r_2, \dots by $r_j = e^{e^{2^j}}$. Further consider the sequence of positive integers (z_j) , where each z_j is the integer maximizing the quantity

$$\# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1 \right\}.$$

Consider the strongly multiplicative function g defined on the primes p by

$$g(p) = \begin{cases} z_j & \text{if } p = s_j \text{ for some } s_j \in \mathcal{S}, \\ 1 & \text{if } p \notin \mathcal{S}. \end{cases}$$

Setting $I_j := [\log \log r_j - (\log \log r_j)^{1/2+\varepsilon}, \log \log r_j + (\log \log r_j)^{1/2+\varepsilon}]$, we have

$$\# \left\{ m \leq \frac{r_j}{s_j} : \omega(m) \notin I_j \right\} = o \left(\frac{r_j}{s_j} \right) \quad (j \rightarrow \infty), \tag{3.11}$$

which allows us to write that, as $j \rightarrow \infty$,

$$\begin{aligned} & \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S} \right\} \\ &= \sum_{\nu \in I_j} \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = \nu \right\} + o \left(\frac{r_j}{s_j} \right). \end{aligned} \tag{3.12}$$

Since by the very nature of the sequence (z_j) , we have that

$$\sum_{\nu \in I_j} \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = \nu \right\} \leq 2(\log \log r_j)^{1/2+\varepsilon} \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m, \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1 \right\},$$

it follows from (3.12), using (3.11), that

$$\begin{aligned} \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1 \right\} \\ \geq \frac{\#\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}\} + o(\frac{r_j}{s_j})}{2(\log \log r_j)^{1/2+\varepsilon}}. \end{aligned} \tag{3.13}$$

On the other hand, letting $C(\delta) := \prod_{j=1}^{\infty} \left(1 - \frac{1}{s_j}\right)$, it is easy to see that

$$\# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S} \right\} = (1 + o(1))C(\delta)\frac{r_j}{s_j} \quad (j \rightarrow \infty). \tag{3.14}$$

We thus obtain, combining (3.13) and (3.14), that as $j \rightarrow \infty$,

$$\begin{aligned} \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1 \right\} \\ \geq \left(\frac{1}{2} + o(1)\right) C(\delta) \frac{r_j}{s_j (\log \log r_j)^{1/2+\varepsilon}}. \end{aligned} \tag{3.15}$$

Now, it is clear that, writing each integer $n \leq r_j$ as $n = s_j \cdot m$, we have

$$\begin{aligned} E(\omega, g; r_j) &\geq \#\{n \leq r_j : g(n) = z_j, \omega(n) = z_j\} \\ &\geq \#\{n \leq r_j : s_j | n, s_k \nmid n \text{ for } k \neq j, \omega(n) = z_j\} \\ &\geq \# \left\{ m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1 \right\}, \end{aligned}$$

which implies, in light of (3.15), that as $j \rightarrow \infty$,

$$E(\omega, g; r_j) \geq \left(\frac{1}{2} + o(1)\right) C(\delta) \frac{r_j}{s_j (\log \log r_j)^{1/2+\varepsilon}}. \tag{3.16}$$

Now, using the fact that, for j sufficiently large, we have

$$s_j < j^{1+2\delta} < (2^j)^\varepsilon = (\log \log r_j)^\varepsilon,$$

it follows from (3.16) that

$$E(\omega, g; r_j) \gg \frac{r_j}{(\log \log r_j)^{1/2+2\varepsilon}},$$

thus completing the proof of Theorem 2. ■

Remark 2. The lower bound of Theorem 2 is better than the one in Corollary 1 but only over a thin subset of the integers, while Theorem 1 provides a stronger uniform bound.

The next theorem states the limitation of the strategy used to construct the “optimal” function g of Theorem 1.

Theorem 3. *Let g be a multiplicative function for which the corresponding set $\mathcal{S}_g := \{p \in \wp : g(p) \neq 1\}$ is such that*

$$\sum_{p \in \mathcal{S}_g} \frac{1}{p} < \infty. \tag{3.17}$$

Then, for any $\varepsilon > 0$, there exists a sequence of real number x_k tending to infinity such that

$$E(\omega, g; x_k) \leq \frac{x_k}{(\log \log x_k)^{1-\varepsilon}} \quad (k = 1, 2, \dots).$$

Proof. First, we introduce the sets

$$\mathfrak{A} := \{n \in \mathbb{N} : p|n \Rightarrow p \in \mathcal{S}_g\} \quad \text{and} \quad \mathfrak{B} := \{n \in \mathbb{N} : p|n \Rightarrow p \notin \mathcal{S}_g\}.$$

Observe that the condition (3.17) implies that there exists a real number C_g such that

$$C_g = \sum_{a \in \mathfrak{A}} \frac{1}{a}.$$

For each positive integer j , define the numbers c_j implicitly by the relation

$$\sum_{\substack{a \in \mathfrak{A} \\ g(a)=j}} \frac{1}{a} = c_j C_g, \tag{3.18}$$

so that in particular we have

$$\sum_{j=1}^{\infty} c_j \leq 1. \tag{3.19}$$

Let us now introduce the additive function ω^* defined by

$$\omega^*(n) = \sum_{\substack{p|n \\ p \in \mathcal{S}_g}} 1.$$

If we apply Lemma 5 to the function ω^* , we obtain that the corresponding sums $A(x)$ and $B(x)$ defined in (2.3) satisfy $A(x) \ll 1$ and $B(x) \ll 1$, thus yielding

$$\#\{n \leq x : \omega^*(n) > (\log \log x)^{1-\delta}\} \ll \frac{x}{(\log \log x)^{2-2\delta}}$$

for any $\delta > 0$. Thus from now on we can assume that $\omega^*(n) \leq (\log \log x)^{1-\delta}$.

Now fix an arbitrary $\varepsilon \in (0, 1)$. For each integer $k \geq 1$, let $y_k = k^{2+6\varepsilon}$ and consider the intervals

$$J_k := \left(y_k - y_k^{1/2+\varepsilon}, y_k + y_k^{1/2+\varepsilon} \right] \quad (k = 1, 2, 3, \dots).$$

Observe that these intervals do not overlap for k large enough (say $k > k_0$) since

$$\begin{aligned} y_{k+1} - y_{k+1}^{1/2+\varepsilon} &= (k+1)^{2+6\varepsilon} - (k+1)^{(2+6\varepsilon)(1/2+\varepsilon)} \\ &> k^{2+6\varepsilon} + (2+6\varepsilon)k^{1+6\varepsilon} - (k+1)^{1+5\varepsilon+6\varepsilon^2} \\ &= y_k + (2+6\varepsilon)k^{1+6\varepsilon} - k^{\frac{\log(k+1)}{\log k}(1+5\varepsilon+6\varepsilon^2)} > y_k + k^{1+6\varepsilon} \\ &> y_k + k^{(2+6\varepsilon)(1/2+\varepsilon)} = y_k + y_k^{1/2+\varepsilon}. \end{aligned}$$

Finally define the sequence (D_k) by

$$D_k := \sum_{j \in J_k} c_j \quad (k > k_0). \tag{3.20}$$

The fact that $\sum_{k > k_0} D_k \leq 1$ (because of (3.19)) implies that $D_k \leq \frac{1}{k}$ for infinitely many integers k . For each such integer k , let $x_k = e^{e^{y_k}}$. Then, for each such k ,

$$\begin{aligned} E(\omega, g; x_k) &= \sum_{j \in J_k} \#\{n \leq x_k : \omega(n) = j, g(n) = j\} \\ &\quad + O\left(x_k \exp(-(\log \log x_k)^{2\varepsilon}/3)\right), \end{aligned}$$

where for the error term we used Lemma 3 to account for those integers $n \leq x$ counted by $E(\omega, g; x_k)$ for which the corresponding $\omega(n)$ value lies outside J_k .

On the other hand, for each $j \in J_k$, we have

$$\#\{n \leq x_k : \omega(n) = j, g(n) = j\} \leq \sum_{\substack{a \in \mathfrak{A} \\ g(a)=j}} \#\left\{b \leq \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\}. \tag{3.21}$$

In the above sum, first consider those $a \in \mathfrak{A}$ such that $a < \frac{x_k}{e^{\sqrt{\log x_k}}}$. Then, in light of Lemma 4, there exists a positive constant c such that

$$\begin{aligned} \#\left\{b \leq \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\} &\leq c \frac{x_k}{a \sqrt{\log \log (x_k/a)}} \\ &\leq \frac{2c}{a \sqrt{\log \log x_k}}. \end{aligned} \tag{3.22}$$

Secondly, for those $a \in \mathfrak{A}$ with $a \geq \frac{x_k}{e^{\sqrt{\log x_k}}}$, we have that $b \leq \frac{x_k}{a} \leq e^{\sqrt{\log x_k}}$, which implies that $\log \log b \leq \frac{1}{2} \log \log x_k$ and that $j - \omega(a) \geq \frac{2}{3} \log \log x_k$. Applying one

more time Lemma 5 yields

$$\begin{aligned} \#\left\{b \leq \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\} &\lesssim \frac{2x_k}{a} \frac{\log \log(x_k/a)}{((1/6) \log \log x_k)^2} \\ &\lesssim \frac{6x_k}{a \log \log x_k}. \end{aligned} \tag{3.23}$$

Using (3.22) and (3.23) in (3.21), we get that

$$\#\{n \leq x_k : \omega(n) = j, g(n) = j\} \leq c \sum_{\substack{a \in \mathfrak{A} \\ g(a)=j}} \frac{x_k}{a \sqrt{\log \log x_k}}.$$

We thus obtain, recalling (3.18) and (3.20),

$$\begin{aligned} E(\omega, g; x_k) &\leq c C_g \sum_{j \in J_k} c_j \frac{x_k}{\sqrt{\log \log x_k}} + O(x_k \exp(-(\log \log x_k)^{2\varepsilon}/3)) \\ &= c C_g D_k \frac{x_k}{\sqrt{\log \log x_k}} + O(x_k \exp(-(\log \log x_k)^{2\varepsilon}/3)) \\ &\leq \frac{c C_g x_k}{k \sqrt{\log \log x_k}} + O(x_k \exp(-(\log \log x_k)^{2\varepsilon}/3)). \end{aligned} \tag{3.24}$$

Since $k = y_k^{1/(2+6\varepsilon)} \geq y_k^{1/2-2\varepsilon} = (\log \log x_k)^{1/2-2\varepsilon}$, we obtain from (3.24) that

$$E(\omega, g; x_k) \leq \frac{c C_g x_k}{(\log \log x_k)^{1-2\varepsilon}} + O(x_k \exp(-(\log \log x_k)^{2\varepsilon}/3)),$$

thus completing the proof of Theorem 3. ■

This last theorem essentially confirms that if one hopes to obtain significant improvements of the results in this section, it can only be possible using a different approach.

4. An upper bound

In this section, we show that, given an additive function f satisfying certain basic conditions, no function $g \in \mathcal{M}$ can agree with f on a set of positive density. More precisely, our goal in this section is to prove the following result.

Theorem 4. *Let f be an integer valued additive function for which the corresponding sums $A(x)$ and $B(x)$ defined in (2.3) satisfy the conditions*

- (i) $\varphi(x) = \varphi_f(x) := \frac{B(x)}{A(x)} \rightarrow 0$ as $x \rightarrow \infty$,
- (ii) $\max_{z \in \mathbb{N}} \#\{n \leq x : f(n) = z\} = O\left(\frac{x}{H(x)}\right)$, where $H(x) = H_f(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Then, given any multiplicative function g , we have $E(f, g; x) = o(x)$ as $x \rightarrow \infty$.

Remark 3. Observe that in the case $f = \omega$, we have $A(x) = (1 + o(1)) \log \log x$ and $B(x) = (1 + o(1)) \sqrt{\log \log x}$ as $x \rightarrow \infty$, so that $\varphi(x) = (1 + o(1)) / \sqrt{\log \log x}$ as $x \rightarrow \infty$, while in light of Lemma 4 we have $H(x) = \sqrt{\log \log x}$.

Before we start the proof of Theorem 4, observe that it is clear that we can discard those integers $n \leq x$ for which

$$|f(n) - A(x)| > \varphi(x)^{-1/3} B(x)$$

since, in light of Lemma 5, their contribution to $E(f, g; x)$ is $o(x)$. This means that throughout the rest of the proof we only need to consider those $n \leq x$ for which

$$|f(n) - A(x)| \leq \varphi(x)^{-1/3} B(x) = \varphi(x)^{2/3} A(x). \tag{4.1}$$

We now introduce a key set of integers. Let D and k be positive integers. Later, D and k will be chosen large. Define the set $\mathcal{T}_{D,k}$ by

$$\mathcal{T}_{D,k} := \{n \in \mathbb{N} : P^+(n) \leq D \text{ and } p^{k+1} \nmid n \text{ for any prime } p\}.$$

Observe that for any given D and k , the set $\mathcal{T}_{D,k}$ is finite, which implies in particular that if $g(n) \neq 1$ for some positive integer n , then, for D large enough and depending only on g , the quantity

$$V_{D,k} := \min_{\substack{m, n \in \mathcal{T}_{D,k} \\ |g(n)| > |g(m)|}} \frac{|g(n)|}{|g(m)|}$$

is well defined, in which case $V_{D,k} > 1$. Now choose x large enough so that

$$1 + \varphi(x)^{1/3} < V_{D,k}. \tag{4.2}$$

Let us further introduce the set \mathcal{U}_D defined by

$$\mathcal{U}_D := \{n \in \mathbb{N} : P^-(n) > D\}.$$

Note that the counting function of the set \mathcal{U}_D is the function $\Phi(x, D)$. Finally, for each positive integer k , let \mathcal{S}_k stand for the set of those positive integers n such that $p^{k+1} | n$ for some prime number p , and let $\mathcal{S}_k(x)$ be the counting function of \mathcal{S}_k . Observe that

$$\mathcal{S}_k(x) \leq \sum_{p \leq x} \left\lfloor \frac{x}{p^{k+1}} \right\rfloor \leq x \int_2^x \frac{dt}{t^{k+1}} < \frac{x}{2^k}. \tag{4.3}$$

Observe also that, given any fixed positive integers D and k , the prime powers dividing an arbitrary integer $n \geq 2$ either belong to $\mathcal{T}_{D,k}$ or to \mathcal{U}_D or to \mathcal{S}_k .

Before we move on to the proof of Theorem 4, we need an additional result which essentially says that if we gather the prime powers p^a dividing n into two parts, namely the part $u(n)$ made up of the product of those p^a which are not in $\mathcal{T}_{D,k}$ and the part $t(n)$ made up of the product of those p^a which belong to $\mathcal{T}_{D,k}$, and if two integers $n \leq x$ such that $g(n) = f(n)$ happen to have an identical $u(n)$ part, then their $t(n)$ parts have the same image under f . This is formally expressed in the following lemma.

Lemma 7. *Let g be a multiplicative function. Under the assumption (4.1), write each integer $n \in [2, x]$ as*

$$n = \prod_{\substack{p^a \parallel n \\ p \notin \mathcal{T}_{D,k}}} p^a \cdot \prod_{\substack{p^a \parallel n \\ p \in \mathcal{T}_{D,k}}} p^a = u(n) \cdot t(n). \tag{4.4}$$

Let n_1 and n_2 be two integers such that $g(n_1) = f(n_1)$ and $g(n_2) = f(n_2)$. Write $u_j = u(n_j)$ and $t_j = t(n_j)$ for $j = 1, 2$. If $u_1 = u_2$, then $f(t_1) = f(t_2)$.

Proof. We will first show that $g(t_1) = g(t_2)$. Indeed, it is clear that

$$\frac{f(n_1)}{f(n_2)} = \frac{g(n_1)}{g(n_2)} = \frac{g(u_1 t_1)}{g(u_2 t_2)} = \frac{g(t_1)}{g(t_2)}. \tag{4.5}$$

Hence, assuming that $g(t_1) \neq g(t_2)$, we have by the definition of $V_{D,k}$ and in light of (4.2), that

$$\max \left(\frac{g(t_1)}{g(t_2)}, \frac{g(t_2)}{g(t_1)} \right) \geq V_{D,k} > 1 + \varphi(x)^{1/3}. \tag{4.6}$$

Taking into account (4.5), (4.6) and assumption (4.1), we have, assuming say that $f(n_2) \geq f(n_1)$ and that x is sufficiently large,

$$\begin{aligned} f(n_2) &= f(n_1) \cdot \frac{f(n_2)}{f(n_1)} > \left(A(x) - \varphi(x)^{-1/3} B(x) \right) \left(1 + \varphi(x)^{1/3} \right) \\ &> A(x) + \frac{1}{2} \varphi(x)^{1/3} A(x) \end{aligned}$$

while

$$\begin{aligned} f(n_1) &= f(n_2) \cdot \frac{f(n_1)}{f(n_2)} < \left(A(x) + \varphi(x)^{-1/3} B(x) \right) \left(1 + \varphi(x)^{1/3} \right)^{-1} \\ &< A(x) - \frac{3}{4} \varphi(x)^{1/3} A(x), \end{aligned}$$

from which it follows that $f(n_2) - f(n_1) > \frac{5}{4} \varphi(x)^{1/3} A(x)$, thus contradicting our assumption (4.1). It follows that we must have $g(t_1) = g(t_2)$. Hence, by (4.5), we have that $f(n_1) = f(n_2)$ and therefore that $f(t_1) = f(t_2)$, as requested. ■

We are now ready to begin the proof of Theorem 4.

Proof. As we did in the statement of Lemma 7, write those numbers $n \leq x$ for which $g(n) = f(n)$ as in (4.4). Then, Lemma 7 guarantees that, for each fixed positive integer $u \leq x$, we only need to count the number of positive integers $t \leq x/u$, with $t \in \mathcal{T}_{D,k}$, knowing that these t 's have the same value under f , say the value z .

This allows us to write that

$$E(f, g; x) \leq \sum_{\substack{u \leq x \\ u \in \mathcal{U}_D}} \# \left\{ t \leq \frac{x}{u} : t \in \mathcal{T}_{D,k}, g(u \cdot t) = f(u \cdot t) \right\} + O \left(\frac{x}{2^k} \right) + o(x), \tag{4.7}$$

where the first error term is a consequence of (4.3) while the second one comes from the fact that we chose to discard those $n \leq x$ for which $|f(n) - A(x)| > \varphi(x)^{-1/3} B(x)$.

Suppose that for a fixed value of $u \leq x$, the numbers t_1, t_2, \dots, t_N are the integers counted by the counting function

$$\# \left\{ t \leq \frac{x}{u} : t \in \mathcal{T}_{D,k}, g(u \cdot t) = f(u \cdot t) \right\},$$

then by Lemma 7, there exists an integer z such that

$$f(t_1) = f(t_2) = \dots = f(t_N) = z.$$

We thus obtain from (4.7) that

$$\begin{aligned} E(f, g; x) &\leq \sum_{\substack{u \leq x \\ u \in \mathcal{U}_D}} \max_{z \geq 1} \# \left\{ t \leq \frac{x}{u} : t \in \mathcal{T}_{D,k}, f(t) = z \right\} + O\left(\frac{x}{2^k}\right) + o(x) \\ &= R(x) + O\left(\frac{x}{2^k}\right) + o(x), \end{aligned} \tag{4.8}$$

say.

Lemma 6 allows us to discard those $n \leq x$ for which $t(n) \leq D^{c_1}$ or $t(n) \geq D^{c_2}$, that is those $n \leq x$ for which $u(n) \geq n/D^{c_1}$ or $u(n) \leq n/D^{c_2}$, and therefore it follows that

$$R(x) = \sum_{\substack{\frac{x}{D^{c_2}} < u < \frac{x}{D^{c_1}} \\ P^-(u) > D}} \max_{z \geq 1} \# \left\{ t \leq \frac{x}{u} : t \in \mathcal{T}_{D,k}, f(t) = z \right\} + o(x) = R_0(x) + o(x), \tag{4.9}$$

say. Using condition (ii) and Lemma 1 with $a_1 = 1/D^{c_2}$ and $a_2 = 1/D^{c_1}$, we obtain that

$$\begin{aligned} R_0(x) &\leq \sum_{\substack{\frac{x}{D^{c_2}} < u < \frac{x}{D^{c_1}} \\ P^-(u) > D}} \max_{z \geq 1} \# \left\{ t \leq \frac{x}{u} : f(t) = z \right\} \\ &\leq \frac{x}{H(D^{c_1})} \sum_{\substack{\frac{x}{D^{c_2}} < u < \frac{x}{D^{c_1}} \\ P^-(u) > D}} \frac{1}{u} \\ &\ll \frac{x}{H(D^{c_1})} \frac{(c_2 - c_1)}{\log D} \log D \ll \frac{x}{H(D^{c_1})}. \end{aligned} \tag{4.10}$$

Substituting (4.9) and (4.10) in (4.8), it follows that

$$E(f, g; x) \leq \frac{x}{H(D^{c_1})} + O\left(\frac{x}{2^k}\right) + o(x).$$

Since both D and k can be chosen to be arbitrarily large, this completes the proof of Theorem 4. ■

Acknowledgements. The authors wish to thank Professor Régis de la Bretèche and Professor Andrzej Schinzel for some very useful comments. The authors would also like to thank the referee for some useful remarks. The work of the first author was supported by a grant from NSERC.

References

- [1] M. Balazard, *Unimodalité de la distribution du nombre de diviseurs premiers d'un entier*, Ann. Inst. Fourier **40**(2) (1990), 255–270.
- [2] B.J. Birch, *Multiplicative functions with non-decreasing normal order*, J. London Math. Soc. **42** (1967), 149–151.
- [3] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.
- [4] A. Hildebrand, *An asymptotic formula for the variance of an additive function*, Math. Z. **183** (1983), no. 2, 145–170.
- [5] J. Kubilius, *Estimate of the second central moment for any additive arithmetic functions*, Lith. Math. J. **23** (1983), no. 2, 183–189.
- [6] J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
- [7] G. Tenenbaum, *Introduction à la théorie analytique des nombres*, Collection Échelles, Belin, 2008.

Address: Jean-Marie De Koninck, Nicolas Doyon, and Patrick Letendre: Dép. de mathématiques, Université Laval, Québec, Québec G1V 0A6, Canada.

E-mail: jmdk@mat.ulaval.ca, nicolas.doyon@mat.ulaval.ca, patrick.letendre@mat.ulaval.ca

Received: 2 September 2014; **revised:** 21 November 2014