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ALGEBRAIC INDEPENDENCE RESULTS FOR VALUES OF THETA-CONSTANTS

Carsten Elsner

Abstract: Let $\theta(q) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}$ denote the Thetanullwert of the Jacobi Zeta function

$$\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau + 2\pi i\nu z}.$$

For algebraic numbers q with $0 < |q| < 1$ we prove the algebraic independence over $\mathbb{Q}$ of the numbers $\theta(q^n)$ and $\theta(q)$ for $n = 2, 3, \ldots, 12$ and furthermore for all $n \geqslant 16$ which are powers of two. An application for $n = 5$ proves the transcendence of the number

$$\sum_{j=1}^{\infty} (-1)^j \Big(\frac{j}{5}\Big) \frac{jq^j}{1-q^j}.$$

Similar results are obtained for numbers related to modular equations of degree 3, 5, and 7.

Keywords: algebraic independence, theta-constants, Nesterenko's theorem, independence criterion, modular equations.

1. Introduction and statement of results

Let τ with $\Im(\tau) > 0$ denote a complex variable. The series

$$\vartheta_2(\tau) = 2\sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \qquad \vartheta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}, \qquad \vartheta_4(\tau) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q = e^{\pi i\tau}$. Sometimes it is useful to write $\vartheta_2(q), \vartheta_3(q), \vartheta_4(q)$ instead of $\vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau)$, respectively, where q belongs to the unit circle around 0 of the complex plane. The theta-constants are modular forms of weight $1/2$ for the principal congruence subgroup of level 2. In particular, $\theta(q) := \vartheta_3(q)$ is the Thetanullwert of the Jacobi zeta

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function $\theta(z|\tau)=\sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau+2\pi i\nu z}$. Let $n\geqslant 3$ denote an odd positive integer. Set

$$h_j(\tau):=n^2\frac{\vartheta_j^4(n\tau)}{\vartheta_j^4(\tau)}\quad (j=2,3,4),\quad \lambda=\lambda(\tau):=\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)},\quad \psi(n):=n\prod_{p|n}\Big(1+\frac{1}{p}\Big),$$

where p runs through all primes dividing n. Also the function

$$j(\tau):=256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

is a modular function with respect to the group $SL(2,\mathbb{Z})$ (cf. [5, ch.3,18]), for which identities of the form $\Phi_n\big(j(\tau),j(n\tau)\big)$ with polynomials $\Phi_n(X,Y)\in\mathbb{Z}[X,Y]$ are known (cf. [5, ch.5]). Yu.V.Nesterenko [8] proved the existence of integer polynomials $P_n(X,Y)\in\mathbb{Z}[X,Y]$ such that $P_n\big(h_j(\tau),R_j(\lambda(\tau))\big)=0$ holds for $j=2,3,4$, odd integers $n\geqslant 3$, and a suitable rational function R_2,R_3, or R_4, respectively:

Theorem A ([8, Theorem 1.1, Corollary 3]). *For any odd integer $n\geqslant 3$ there exists a polynomial $P_n(X,Y)\in\mathbb{Z}[X,Y]$, $\deg_X P=\psi(n)$, such that*

$$P_n\Big(h_2(\tau),16\frac{\lambda(\tau)-1}{\lambda(\tau)}\Big)=0,$$
$$P_n\big(h_3(\tau),16\lambda(\tau)\big)=0,$$
$$P_n\Big(h_4(\tau),16\frac{\lambda(\tau)}{\lambda(\tau)-1}\Big)=0.$$

The polynomials P_3,P_5,P_7,P_9, and P_{11} are listed in the appendix. P_3 and P_5 are already given in [8], P_7,P_9, and P_{11} are the results of computer-assisted computations of the author.

There are various algebraic relationships between the theta-constants and arithmetic functions like Ramanujan's Eisenstein series $P(q),Q(q),R(q)$ (cf. [6]), the Dedekind eta-function $\eta(\tau)=e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})$, and others. For instance, it follows from Jacobi's triple product identity that $\theta(-q)=\eta^2(\tau)/\eta(2\tau)$ for $\Im(\tau)>0$ and $q=e^{2\pi i\tau}$. Therefore, under suitable circumstances, an algebraic independence result for values of theta-constants can be transformed into an algebraic independence result for functions which are expressed in terms of theta-constants. For example, see [3] and Corollary1.1 below.

In this paper we focus on the problem to decide on the algebraic independence of $\theta(q)$ and $\theta(q^n)$ over $\mathbb{Q}$ for algebraic numbers q and integers $n>1$. We shall use Theorem A in connection with an algebraic independence criterion (Lemma 2.1) to settle the problem for the odd integers $n=3,5,7,9,11$ and for three even numbers $n=6$, $n=10$, and $n=12$. The central point of the algebraic independence criterion is the non-vanishing of a Jacobian determinant, which is hard to decide when the involved polynomials are not given explicitly. Using the double-argument formulae (3.1) for the theta-constants we construct suitable polynomials $P_{2^m}(X,Y)$

(Lemma 3.1). In this case the polynomials $P_{2^m}(X,Y)$ are given recursively such that we can solve the problem of the algebraic independence of $\theta(q)$ and $\theta(q^{2^m})$ for arbitrary integers $m \geqslant 1$. But this method cannot be extended to decide on the algebraic independence of $\theta(q)$ and $\theta(q^n)$ for arbitrary odd integers n by Theorem A. So the main results of this paper are given by the following theorem.

Theorem 1.1. *Let q be an algebraic number with $0 < |q| < 1$. Let $m \geqslant 1$ be an integer. Then, the two numbers $\theta(q)$ and $\theta(q^{2^m})$ are algebraically independent over $\mathbb{Q}$ as well as the two numbers $\theta(q)$ and $\theta(q^n)$ for $n = 3, 5, 6, 7, 9, 10, 11, 12$.*

Let $n \geqslant 3$ be any odd integer. If the polynomial $P_n(X,Y)$ from Theorem A is given explicitly, then by Theorem 4.1 in Section 4 one can decide on the algebraic independence of $\theta(q)$ and $\theta(q^n)$ over $\mathbb{Q}$ for any algebraic number q satisfying the condition of Theorem 1.1.

The following identities are originally due to Ramanujan (cf. [1, §19, Entries 8 and 17]):

$$1 + S_1(q) = 1 + \sum_{j=1}^{\infty} (-1)^j \left(\frac{j}{5}\right) \frac{jq^j}{1-q^j} = \frac{1}{4}\big(5\theta(-q)\theta^3(-q^5) - \theta^3(-q)\theta(-q^5)\big),$$

$$24 + 40S_2(q) = 24 + 40 \sum_{\substack{j=1 \\ j\equiv 1(2)}}^{\infty} \left(\frac{j}{5}\right) \frac{jq^j}{1+q^j} = 25\theta(q)\theta^3(q^5) - \frac{\theta^5(q)}{\theta(q^5)},$$

$$1 + 2S_3(q) = 1 + 2\sum_{j=1}^{\infty} \varepsilon_j \frac{q^j}{1-q^j} = \theta(q)\theta(q^7),$$

where $\big(j/5\big)$ denotes the Legendre symbol, and the cycle of coefficients $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{27})$ of length 28 is given by

$$\big(0,1,-1,-1,1,-1,1,0,1,1,1,1,-1,-1,0,$$
$$1,1,-1,-1,-1,-1,0,-1,1,-1,1,1,-1\big).$$

Corollary 1.1. *Let q be an algebraic number with $0 < |q| < 1$. Then the numbers $S_1(q), S_2(q)$, and $S_3(q)$ are transcendental.*

From Entry 3 and Entry 4 in [1, §19] analogous results can be obtained for modular equations of degree 3.

2. Auxiliary results

A detailed discussion of theta-functions and theta-constants can be found in [4, part2, ch.2] and [9, ch.10]. At first we point out some properties of the functions

$$X_0(\tau) \in \left\{ n^2 \frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_3^2(n\tau)}{\vartheta_3^2(\tau)} \right\} \qquad \text{and} \qquad Y_0(\tau) \in \left\{ 16 \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}, \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)} \right\}.$$

From the theory of modular forms it is well known that in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ the theta-constants $\vartheta_2(\tau), \vartheta_3(\tau)$ and $\vartheta_4(\tau)$ are regular functions for $\tau \in \mathbb{H}$. Moreover, $\vartheta_3(\tau)$ does not vanish in $\mathbb{H}$. Therefore, $X_0(\tau)$ and $Y_0(\tau)$ are regular functions in $\mathbb{H}$.

The most important tool to transfer the algebraic independence of a set of m numbers to another set of m numbers, which all satisfy a system of algebraic identities, is given by the following lemma. We call it an *algebraic independence criterion* (AIC).

Lemma 2.1 ([2, Lemma 3.1]). *Let $x_1, \ldots, x_m \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$ and let $y_1, \ldots, y_m \in \mathbb{C}$ satisfy the system of equations*

$$f_j(x_1, \ldots, x_m, y_1, \ldots, y_m) = 0 \qquad (1 \leqslant j \leqslant m),$$

where $f_j(t_1, \ldots, t_m, u_1, \ldots, u_m) \in \mathbb{Q}[t_1, \ldots, t_m, u_1, \ldots, u_m]$ $(1 \leqslant j \leqslant m)$. Assume that

$$\det\left(\frac{\partial f_j}{\partial t_i}(x_1, \ldots, x_m, y_1, \ldots, y_m)\right) \neq 0.$$

Then the numbers $y_1, \ldots, y_m$ are algebraically independent over $\mathbb{Q}$.

We shall apply the AIC to the sets $\{x_1, x_2\} = \{\vartheta_2(\tau), \vartheta_3(\tau)\}$ and $\{x_1, x_2\} = \{\vartheta_3(\tau), \vartheta_4(\tau)\}$. For this purpose we have to know that these pairs of numbers are algebraically independent.

Lemma 2.2 ([3, Lemma 4]). *Let q be an algebraic number with $q = e^{\pi i \tau}$ and $\Im(\tau) > 0$. Then, the numbers in each of the sets*

$$\{\vartheta_2(\tau), \vartheta_3(\tau)\}, \qquad \{\vartheta_2(\tau), \vartheta_4(\tau)\}, \qquad \{\vartheta_3(\tau), \vartheta_4(\tau)\}$$

are algebraically independent over $\mathbb{Q}$.

This result can be derived from Yu.V.Nesterenko's theorem [7] on the algebraic independence of the values $P(q)$, $Q(q)$, $R(q)$ of the Ramanujan functions P, Q, R at a nonvanishing algebraic point q.

3. Preparation of the proof of Theorem 1.1

In this section, we prove the following lemmas which are required to prove Theorem 1.1 when n is a power of two.

Lemma 3.1. *For every integer $m \geqslant 1$ let $n = 2^m$. There exists a polynomial $P_n(X, Y) \in \mathbb{Z}[X, Y]$ such that*

$$P_n\Big(\frac{\vartheta_3^2(n\tau)}{\vartheta_3^2(\tau)}, \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}\Big) = 0$$

with $\deg_X P_2(X, Y) = 1$, *and* $\deg_X P_n(X, Y) = 2^{m-2}$ *for* $m \geqslant 2$.

Proof. For simplicity we introduce the notation $\vartheta_3 := \vartheta_3(\tau)$ and $\vartheta_4 := \vartheta_4(\tau)$. Then

$$\left.\begin{aligned} 2\vartheta_2^2(2\tau) &= \vartheta_3^2 - \vartheta_4^2, \\ 2\vartheta_3^2(2\tau) &= \vartheta_3^2 + \vartheta_4^2, \\ \vartheta_4^2(2\tau) &= \vartheta_3\vartheta_4. \end{aligned}\right\} \tag{3.1}$$

For every integer $m \geqslant 1$ let

$$z_1 := \vartheta_3^2(n\tau),$$
$$z_2 := (\vartheta_3 + \vartheta_4)^2,$$
$$z_3 := \vartheta_3\vartheta_4.$$

Note that z_1 depends on $n = 2^m$, while z_2 and z_3 do not depend on n. First, we compute the polynomials $P_n(X, Y)$ for $n = 2, 4, 8$.

$n = 2$: From (3.1) we have

$$2\vartheta_3^2(2\tau) - (\vartheta_3 + \vartheta_4)^2 + 2\vartheta_3\vartheta_4 = 2z_1 - z_2 + 2z_3 = 0. \tag{3.2}$$

Dividing by ϑ_3^2, it follows that

$$2\Big(\frac{\vartheta_3(2\tau)}{\vartheta_3}\Big)^2 - \Big(1 + \frac{\vartheta_4}{\vartheta_3}\Big)^2 + 2\frac{\vartheta_4}{\vartheta_3} = 0.$$

Hence, $P_2(X, Y) = 2X - (1 + Y)^2 + 2Y$.

$n = 4$: In the second identity of (3.1) we replace τ by 2τ and then express $\vartheta_3^2(2\tau)$ and $\vartheta_4^2(2\tau)$ on the right-hand side again by (3.1) in terms of ϑ_3 and ϑ_4:

$$4\vartheta_3^2(4\tau) - (\vartheta_3 + \vartheta_4)^2 = 4z_1 - z_2 = 0. \tag{3.3}$$

Dividing by ϑ_3^2, it follows that

$$4\Big(\frac{\vartheta_3(4\tau)}{\vartheta_3}\Big)^2 - \Big(1 + \frac{\vartheta_4}{\vartheta_3}\Big)^2 = 0.$$

Hence, $P_4(X, Y) = 4X - (1 + Y)^2$.

$n = 8$: In (3.3) we replace τ by 2τ. In order to express $\vartheta_3(2\tau)$ and $\vartheta_4(2\tau)$ in terms of ϑ_3 and ϑ_4, it becomes necessary to solve the identity for $\vartheta_3\vartheta_4$ and square the equation. After some straightforward computations it turns out that

$$\begin{aligned} 0 &= \big(8\vartheta_3^2(8\tau) - (\vartheta_3 + \vartheta_4)^2\big)^2 - 8\big((\vartheta_3 + \vartheta_4)^2 - 2\vartheta_3\vartheta_4\big)\vartheta_3\vartheta_4 \\ &= (8z_1 - z_2)^2 - 8(z_2 - 2z_3)z_3. \end{aligned} \tag{3.4}$$

Dividing by ϑ_3^4, we find that

$$P_8(X, Y) = \big(8X - (1 + Y)^2\big)^2 - 8\big((1 + Y)^2 - 2Y\big)Y.$$

The polynomials in terms of z_1, z_2, z_3 in (3.2 - 3.4) are homogeneous of degrees 1,1, and 2 respectively. Therefore, we try to prove the following statement by induction with respect to m:

For every $m \geqslant 1$ there is a homogeneous polynomial $T_n(t_1,t_2,t_3) \in \mathbb{Z}[t_1,t_2,t_3]$ of total degree λ such that $T_n(z_1,z_2,z_3) = 0$ with $\lambda = \deg_{t_1} T_n(t_1,t_2,t_3) = 2^{m-2}$ for $m \geqslant 2$ and $\lambda = 1$ when $m = 1$.

We have already shown the existence of T_2, T_4, and T_8. For T_8 we have $\lambda = 2$ by (3.4). So, let us assume that for some $m \geqslant 3$ such a homogeneous polynomial T_{2^m} with $\lambda = 2^{m-2}$ do exist. Then,

$$T_{2^m}\Big(\vartheta_3^2(2^m\tau), (\vartheta_3+\vartheta_4)^2, \vartheta_3\vartheta_4\Big) = 0, \tag{3.5}$$

where

$$T_{2^m}(t_1,t_2,t_3) = \sum_\nu a_\nu t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3}, \tag{3.6}$$

say, with $\nu = (\nu_1,\nu_2,\nu_3) \in \mathbb{N}^3$, $a_\nu \in \mathbb{Z}$, and $\nu_1+\nu_2+\nu_3 = \lambda = \deg_{t_1} T_{2^m}(t_1,t_2,t_3)$. Here, $\mathbb{N}$ denotes the set of nonnegative integers. The leading term with respect to t_1 occurs once only for $\nu = (\lambda,0,0)$. Next, in (3.5) we replace τ by 2τ:

$$T_{2^m}\Big(\vartheta_3^2(2^{m+1}\tau), \big(\vartheta_3(2\tau)+\vartheta_4(2\tau)\big)^2, \vartheta_3(2\tau)\vartheta_4(2\tau)\Big) = 0. \tag{3.7}$$

Setting $w := \vartheta_3(2\tau)\vartheta_4(2\tau)$, we have, using (3.1),

$$\big(\vartheta_3(2\tau)+\vartheta_4(2\tau)\big)^2 = \frac{z_2}{2} + 2w.$$

For $m+1$ we set $z_1 := \vartheta_3^2(2^{m+1}\tau) = \vartheta_3^2\big((n+1)\tau\big)$. Then, (3.7) and (3.6) can be expressed in terms of z_1, z_2, and w:

$$\begin{aligned} 0 &= T_{2^m}\Big(z_1, \frac{z_2}{2}+2w, w\Big) \\ &= \sum_\nu a_\nu z_1^{\nu_1}\Big(\frac{z_2}{2}+2w\Big)^{\nu_2} w^{\nu_3} \\ &= \sum_\mu b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3} \end{aligned}$$

with $\mu = (\mu_1,\mu_2,\mu_3) \in \mathbb{N}^3$, $b_\mu \in \mathbb{Q}$, and $\mu_1+\mu_2+\mu_3 = \lambda$. We separate the sum on $\mu = (\mu_1,\mu_2,\mu_3)$ into two parts according to the parity of μ_3:

$$0 = \sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 0 \pmod 2}} b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3} + \sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 1 \pmod 2}} b_\mu z_1^{\mu_1} z_2^{\mu_2} w^{\mu_3},$$

where the leading term with respect to z_1 is $b_{(\lambda,0,0)}z_1^\lambda \not\equiv 0$ occurring in the left-hand sum. It follows that

$$\Big(\sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 0 \pmod 2}} b_\mu z_1^{\mu_1}z_2^{\mu_2}w^{\mu_3}\Big)^2 - w^2\Big(\sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 1 \pmod 2}} b_\mu z_1^{\mu_1}z_2^{\mu_2}w^{\mu_3-1}\Big)^2 = 0. \qquad (3.8)$$

Using (3.1) we express w^2 in terms of z_2 and z_3:

$$w^2 = \frac{1}{2}(z_2 - 2z_3)z_3.$$

Substituting this expression into (3.8), we obtain

$$\begin{aligned} 0 &= \Big(\sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 0 \pmod 2}} b_\mu z_1^{\mu_1}z_2^{\mu_2}2^{-\mu_3/2}(z_2-2z_3)^{\mu_3/2}z_3^{\mu_3/2}\Big)^2 \\ &\quad - \frac{1}{2}(z_2-2z_3)z_3\Big(\sum_{\substack{\mu=(\mu_1,\mu_2,\mu_3)\\ \mu_3\equiv 1 \pmod 2}} b_\mu z_1^{\mu_1}z_2^{\mu_2}2^{-(\mu_3-1)/2}(z_2-2z_3)^{(\mu_3-1)/2}z_3^{(\mu_3-1)/2}\Big)^2 \\ &= \sum_\kappa c_\kappa z_1^{\kappa_1}z_2^{\kappa_2}z_3^{\kappa_3}, \end{aligned}$$

where $\kappa = (\kappa_1,\kappa_2,\kappa_3) \in \mathbb{N}^3$, $c_\kappa \in \mathbb{Q}$, and $\kappa_1+\kappa_2+\kappa_3 = 2\lambda$. The leading term with respect to z_1 is $c_{(2\lambda,0,0)}z_1^{2\lambda} \not\equiv 0$. The homogeneous polynomial $T_{2^{m+1}} \in \mathbb{Z}[t_1,t_2,t_3]\setminus\{0\}$ can be chosen by

$$T_{2^{m+1}}(t_1,t_2,t_3) := 2^{2\lambda}\sum_\kappa c_\kappa t_1^{\kappa_1}t_2^{\kappa_2}t_3^{\kappa_3}.$$

For this polynomial we have $2\lambda = 2^{m-1}$. This completes the proof of the existence of the homogeneous polynomials $T_n(t_1,t_2,t_3)$ for every integer $m \geqslant 1$ with $n = 2^m$ satisfying $T_n(z_1,z_2,z_3) = 0$. Let us consider a monomial of such a homogeneous polynomial T_n of degree λ given by (3.6). Then we have $\nu_1+\nu_2+\nu_3 = \lambda$. After dividing T_n by $\vartheta_3^{2\lambda}$, the monomial takes the form

$$\begin{aligned} \frac{a_\nu}{\vartheta_3^{2\lambda}}\cdot z_1^{\nu_1}z_2^{\nu_2}z_3^{\nu_3} &= \frac{a_\nu}{\vartheta_3^{2\lambda}}\cdot\big(\vartheta_3(2^m\tau)\big)^{2\nu_1}(\vartheta_3+\vartheta_4)^{2\nu_2}(\vartheta_3\vartheta_4)^{\nu_3} \\ &= a_\nu\Big(\frac{\vartheta_3(2^m\tau)}{\vartheta_3}\Big)^{2\nu_1}\Big(1+\frac{\vartheta_4}{\vartheta_3}\Big)^{2\nu_2}\Big(\frac{\vartheta_4}{\vartheta_3}\Big)^{\nu_3} \\ &= a_\nu X^{\nu_1}(1+Y)^{2\nu_2}Y^{\nu_3} \end{aligned}$$

with

$$X := \frac{\vartheta_3^2(2^m\tau)}{\vartheta_3^2} \qquad\text{and}\qquad Y := \frac{\vartheta_4}{\vartheta_3}.$$

Introducing the polynomial

$$P_n(X,Y) := \sum_\nu a_\nu X^{\nu_1}(1+Y)^{2\nu_2}Y^{\nu_3} = T_n\big(X,(1+Y)^2,Y\big),$$

we finish the proof of Lemma 3.1. ∎

The polynomials P_2, P_4, P_8, P_{16}, and P_{32} are listed in the appendix. The proof of the algebraic independence of $\vartheta_3(q^{2^m})$ and $\vartheta_3(q)$ over $\mathbb{Q}$ will require some more information on the polynomials $T_n(t_1,t_2,t_3)$ introduced in the proof of the preceding lemma.

Lemma 3.2. *For every integer $m \geqslant 3$ let $n = 2^m$. Then there is a polynomial $U_n(t_1,t_2,t_3) \in \mathbb{Q}[t_1,t_2,t_3]$ such that the polynomial $T_n(t_1,t_2,t_3)$ from (3.5) can be written as*

$$T_n(t_1,t_2,t_3) = \bigl(nt_1 - t_2\bigr)^{2^{m-2}} + t_3 U_n(t_1,t_2,t_3) \tag{3.9}$$

with

$$U_n\Bigl(\frac{1}{n},1,0\Bigr) = -2^{2^{m-1}-1}. \tag{3.10}$$

Proof. Lemma 3.2 is true for $m=3$ and $m=4$. We have

$$\begin{aligned}
T_8(t_1,t_2,t_3) &= (8t_1-t_2)^2 - 8(t_2-2t_3)t_3,\\
U_8(t_1,t_2,t_3) &= -8(t_2-2t_3);\\
T_{16}(t_1,t_2,t_3) &= (16t_1-t_2)^4 + t_3\Bigl(16(t_2-2t_3)(16t_1-t_2)^2 + 64(t_2-2t_3)^2 t_3\\
&\qquad - 128(t_2-2t_3)\Bigl(8t_1+\frac{t_2}{2}\Bigr)^2\Bigr),\\
U_{16}(t_1,t_2,t_3) &= 16(t_2-2t_3)(16t_1-t_2)^2 + 64(t_2-2t_3)^2 t_3\\
&\qquad - 128(t_2-2t_3)\Bigl(8t_1+\frac{t_2}{2}\Bigr)^2.
\end{aligned} \tag{3.11}$$

For $m \geqslant 4$ we prove a more precise result on the particular shape of the polynomials T_{2^m}. We shall show the following. For every integer $m \geqslant 4$ we have

$$\begin{aligned}
T_{2^m}(t_1,t_2,t_3) &= (2^m t_1 - t_2)^{2^{m-2}}\\
&\quad + t_3 \sum_{\substack{\nu_1,\dots,\nu_5\\ \nu_1\geqslant 2 \vee \nu_5 \geqslant 1\\ \nu_2\geqslant 1}} a_\nu (2^m t_1 - t_2)^{\nu_1}(t_2-2t_3)^{\nu_2} t_1^{\nu_3} t_2^{\nu_4} t_3^{\nu_5}\\
&\quad - 2^{2^{m-1}-1} t_2^{2^{m-3}-2}(t_2-2t_3)\Bigl(2^{m-1}t_1 + \frac{t_2}{2}\Bigr)^{2^{m-3}} t_3.
\end{aligned} \tag{3.12}$$

Here, $\nu = (\nu_1,\dots,\nu_5) \in \mathbb{N}^5$, and the numbers a_ν are rationals. Only finitely many a_ν do not vanish. One can show that $T_{2^m}(t_1,t_2,t_3)$ is a polynomial with integer coefficients, but we do not need this fact. We point out that the conditions on the summation variables $\nu_1,\dots,\nu_5$ read as follows: it is either $\nu_1 \geqslant 2$ or $\nu_5 \geqslant 1$ (or both), and we always have $\nu_2 \geqslant 1$. The second and third term on the right-hand side of (3.12) form $t_3 U_{2^m}(t_1,t_2,t_3)$, which implies (3.9). In particular, for $t_1 = 1/2^m$, $t_2 = 1$, and $t_3 = 0$, we have

$$2^m t_1 - t_2 = 0, \qquad t_2 - 2t_3 = 1, \qquad 2^{m-1}t_1 + \frac{t_2}{2} = 1,$$

such that

$$U_{2^m}\left(\frac{1}{2^m},1,0\right) = -2^{2^{m-1}-1}$$

proves (3.10) in Lemma 3.2.

Proof of (3.12). We proceed by induction on m. For $m = 4$ see (3.11). Next let us assume that (3.12) holds for some integer $m \geqslant 4$. Following the lines in the proof of Lemma 3.1, we construct step by step the new polynomial $T_{2^{m+1}}(t_1,t_2,t_3)$ from (3.12).

Step 1: After substituting the new expressions

$$t_1 \to t_1, \qquad t_2 \to \frac{t_2}{2} + 2w, \qquad t_3 \to w$$

into (3.12), we see that the resulting term equals to zero. Hence,

$$\begin{aligned} 0 &= \left(2^m t_1 - \frac{t_2}{2} - 2w\right)^{2^{m-2}} \\ &+ w \sum_{\substack{\nu_1,\dots,\nu_5 \\ \nu_1\geqslant 2 \vee \nu_5 \geqslant 1 \\ \nu_2 \geqslant 1}} a_\nu \left(2^m t_1 - \frac{t_2}{2} - 2w\right)^{\nu_1} \left(\frac{t_2}{2}\right)^{\nu_2} t_1^{\nu_3} \left(\frac{t_2}{2} + 2w\right)^{\nu_4} w^{\nu_5} \\ &- 2^{2^{m-1}-1} \left(\frac{t_2}{2} + 2w\right)^{2^{m-3}-2} \frac{t_2}{2} \left(2^{m-1} t_1 + \frac{t_2}{4} + w\right)^{2^{m-3}} w. \end{aligned}$$

Using four times the binomial theorem, the above expression becomes

$$\begin{aligned} 0 &= \left(2^m t_1 - \frac{t_2}{2}\right)^{2^{m-2}} + \sum_{\mu_1=1}^{2^{m-2}} \binom{2^{m-2}}{\mu_1} (-1)^{\mu_1} \left(2^m t_1 - \frac{t_2}{2}\right)^{2^{m-2}-\mu_1} (2w)^{\mu_1} \\ &+ w \sum_{\substack{\nu_1,\dots,\nu_5 \\ \nu_1\geqslant 2 \vee \nu_5 \geqslant 1 \\ \nu_2 \geqslant 1}} a_\nu \left(\sum_{\mu_2=0}^{\nu_1} \binom{\nu_1}{\mu_2} (-1)^{\mu_2} \left(2^m t_1 - \frac{t_2}{2}\right)^{\nu_1-\mu_2} (2w)^{\mu_2} \right) \left(\frac{t_2}{2}\right)^{\nu_2} t_1^{\nu_3} \\ &\times \left(\sum_{\mu_3=0}^{\nu_4} \binom{\nu_4}{\mu_3} \left(\frac{t_2}{2}\right)^{\nu_4-\mu_3} (2w)^{\mu_3} \right) w^{\nu_5} \\ &- 2^{2^{m-1}-1} \left(\frac{t_2}{2}\right)^{2^{m-3}-2} \frac{t_2}{2} \left(2^{m-1} t_1 + \frac{t_2}{4} + w\right)^{2^{m-3}} w \\ &- 2^{2^{m-1}-1} \left(\sum_{\mu_4=1}^{2^{m-3}-2} \binom{2^{m-3}-2}{\mu_4} \left(\frac{t_2}{2}\right)^{2^{m-3}-2-\mu_4} (2w)^{\mu_4} \right) \frac{t_2}{2} \\ &\times \left(2^{m-1} t_1 + \frac{t_2}{4} + w\right)^{2^{m-3}} w. \end{aligned} \tag{3.13}$$

The last but one term on the right-hand side of (3.13) can be expanded by

$$\begin{aligned}
&2^{2^{m-1}-1}\Big(\frac{t_2}{2}\Big)^{2^{m-3}-2}\frac{t_2}{2}\cdot\frac{1}{2^{2^{m-3}}}\Big(2^m t_1+\frac{t_2}{2}+2w\Big)^{2^{m-3}}w\\
&=2^{2^{m-1}-2^{m-3}-2^{m-3}-1-1+2}t_2^{2^{m-3}-1}\Big(2^m t_1+\frac{t_2}{2}+2w\Big)^{2^{m-3}}w\\
&=2^{2^{m-2}}t_2^{2^{m-3}-1}\Big(2^m t_1+\frac{t_2}{2}\Big)^{2^{m-3}}w\\
&\quad+2^{2^{m-2}}t_2^{2^{m-3}-1}\left(\sum_{\mu_5=1}^{2^{m-3}}\binom{2^{m-3}}{\mu_5}\Big(2^m t_1+\frac{t_2}{2}\Big)^{2^{m-3}-\mu_5}(2w)^{\mu_5}\right)w.
\end{aligned}\tag{3.14}$$

Substituting (3.14) for the last but one term into (3.13), we summarize the terms as follows.

$$\begin{aligned}
0=&\Big(2^m t_1-\frac{t_2}{2}\Big)^{2^{m-2}}+\sum_{\substack{\mu_6,\dots,\mu_9\\ \mu_6\geqslant 2\vee\mu_9\geqslant 2\\ \mu_9\geqslant 1}}b_\mu(2^{m+1}t_1-t_2)^{\mu_6}t_1^{\mu_7}t_2^{\mu_8}w^{\mu_9}\\
&+2^{2^{m-2}}t_2^{2^{m-3}-1}\Big(2^m t_1+\frac{t_2}{2}\Big)^{2^{m-3}}w.
\end{aligned}\tag{3.15}$$

Here, we abbreviate by $\mu=(\mu_6,\dots,\mu_9)$, and the coefficients b_μ are again rational numbers.

Step 2: In (3.15) we separate the terms with an even power of w from those with an odd power of w. This gives

$$\begin{aligned}
&\Big(2^m t_1-\frac{t_2}{2}\Big)^{2^{m-2}}+\sum_{\substack{\mu_6,\dots,\mu_9\\ \mu_9\geqslant 2\\ \mu_9\equiv 0\pmod 2}}b_\mu(2^{m+1}t_1-t_2)^{\mu_6}t_1^{\mu_7}t_2^{\mu_8}w^{\mu_9}\\
&=-\sum_{\substack{\mu_6,\dots,\mu_9\\ \mu_6\geqslant 2\vee\mu_9\geqslant 2\\ \mu_9\equiv 1\pmod 2\\ \mu_9\geqslant 1}}b_\mu(2^{m+1}t_1-t_2)^{\mu_6}t_1^{\mu_7}t_2^{\mu_8}w^{\mu_9}\\
&\quad-2^{2^{m-2}}t_2^{2^{m-3}-1}\Big(2^m t_1+\frac{t_2}{2}\Big)^{2^{m-3}}w.
\end{aligned}\tag{3.16}$$

Step 3: Squaring (3.16), we obtain

$$
\begin{aligned}
&\Big(2^m t_1 - \frac{t_2}{2}\Big)^{2^{m-1}} + \Big(\sum_{\substack{\mu_6,\dots,\mu_9 \\ \mu_9 \geqslant 2 \\ \mu_9 \equiv 0 \pmod 2}} b_\mu (2^{m+1}t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9} \Big)^2 \\
&+ 2\Big(2^m t_1 - \frac{t_2}{2}\Big)^{2^{m-2}} \sum_{\substack{\mu_6,\dots,\mu_9 \\ \mu_9 \geqslant 2 \\ \mu_9 \equiv 0 \pmod 2}} b_\mu (2^{m+1}t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9} \\
= &\Big(\sum_{\substack{\mu_6,\dots,\mu_9 \\ \mu_6 \geqslant 2 \vee \mu_9 \geqslant 2 \\ \mu_9 \equiv 1 \pmod 2 \\ \mu_9 \geqslant 1}} b_\mu (2^{m+1}t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{\mu_9} \Big)^2 \\
&+ 2^{2^{m-2}+1} t_2^{2^{m-3}-1} \Big(2^m t_1 + \frac{t_2}{2}\Big)^{2^{m-3}} \sum_{\substack{\mu_6,\dots,\mu_9 \\ \mu_6 \geqslant 2 \vee \mu_9 \geqslant 2 \\ \mu_9 \equiv 1 \pmod 2 \\ \mu_9 \geqslant 1}} b_\mu (2^{m+1}t_1 - t_2)^{\mu_6} t_1^{\mu_7} t_2^{\mu_8} w^{1+\mu_9} \\
&+ 2^{2^{m-1}} t_2^{2^{m-2}-2} \Big(2^m t_1 + \frac{t_2}{2}\Big)^{2^{m-2}} w^2.
\end{aligned}
$$

This identity can be summarized as follows.

$$
\begin{aligned}
0 = &\Big(2^m t_1 - \frac{t_2}{2}\Big)^{2^{m-1}} + \sum_{\substack{\nu_6,\dots,\nu_9 \\ \nu_6 \geqslant 2 \vee \nu_7 \geqslant 2 \\ \nu_7 \geqslant 1}} c_\nu \Big(2^m t_1 - \frac{t_2}{2}\Big)^{\nu_6} w^{2\nu_7} t_1^{\nu_8} t_2^{\nu_9} \\
&- 2^{2^{m-1}} t_2^{2^{m-2}-2} \Big(2^m t_1 + \frac{t_2}{2}\Big)^{2^{m-2}} w^2,
\end{aligned}
\tag{3.17}
$$

where $\nu = (\nu_6, \dots, \nu_9)$ and $c_\nu \in \mathbb{Q}$.

Step 4: Multiplying (3.17) by $2^{2^{m-1}}$ and replacing w^2 by $\frac{1}{2}(t_2 - 2t_3)t_3$, the right-hand side of (3.17) becomes the polynomial $T_{2^{m+1}}(t_1, t_2, t_3)$. Thus we obtain

$$\begin{aligned}
T_{2^{m+1}}(t_1,t_2,t_3) &= \Big(2^{m+1}t_1 - t_2\Big)^{2^{m-1}} \\
&\quad + \sum_{\substack{\nu_6,\dots,\nu_9 \\ \nu_6\geqslant 2\vee\nu_7\geqslant 2 \\ \nu_7\geqslant 1}} 2^{2^{m-1}} c_\nu \Big(2^m t_1 - \frac{t_2}{2}\Big)^{\nu_6} \frac{1}{2^{\nu_7}}(t_2-2t_3)^{\nu_7} t_3^{\nu_7} t_1^{\nu_8} t_2^{\nu_9} \\
&\quad - 2^{2^{m-1}+2^{m-1}} t_2^{2^{m-2}-2} \Big(2^m t_1 + \frac{t_2}{2}\Big)^{2^{m-2}} \frac{1}{2}(t_2-2t_3)t_3 \\
&= \Big(2^{m+1}t_1 - t_2\Big)^{2^{m-1}} \\
&\quad + t_3 \sum_{\substack{\nu_6,\dots,\nu_{10} \\ \nu_6\geqslant 2\vee\nu_{10}\geqslant 1 \\ \nu_7\geqslant 1}} d_\nu \big(2^{m+1}t_1 - t_2\big)^{\nu_6} (t_2-2t_3)^{\nu_7} t_1^{\nu_8} t_2^{\nu_9} t_3^{\nu_{10}} \\
&\quad - 2^{2^m-1} t_2^{2^{m-2}-2} (t_2-2t_3) \Big(2^m t_1 + \frac{t_2}{2}\Big)^{2^{m-2}} t_3,
\end{aligned} \tag{3.18}$$

where $\nu = (\nu_6,\dots,\nu_{10})$, and $d_\nu \in \mathbb{Q}$. Hence, (3.18) corresponds to (3.12) with m replaced by $m+1$. This proves (3.12). ∎

4. Proof of Theorem 1.1

By $\mathrm{Res}_t\big(f(t),g(t)\big)$ we denote the resultant of two polynomials $f(t),g(t)$ with respect to the variable t. It is consistent with the notation of theta-constants to write $\vartheta_3(q)$ and $\vartheta_3(\tau)$ instead of $\theta(q)$ and $\theta(\tau)$, respectively.
We divide the proof of Theorem 1.1 into several steps. The first step is an interim result given by the following theorem.

Theorem 4.1. *Let n be either an odd integer $\geqslant 3$ or $n = 2^m$ with $m \geqslant 1$. Let q be an algebraic number with $q = e^{\pi i\tau}$ and $\Im(\tau) > 0$. If the polynomial*

$$Res_X\Big(P_n(X,Y), \frac{\partial}{\partial Y}P_n(X,Y)\Big)$$

does not vanish identically, then the numbers $\vartheta_3(n\tau)$ and $\vartheta_3(\tau)$ are algebraically independent over $\mathbb{Q}$.

Proof. For any given odd integer $n \geqslant 3$ let

$$\begin{aligned}
X_0 &:= n^2\frac{\vartheta_3^4(n\tau)}{\vartheta_3^4(\tau)}, & Y_0 &:= 16\frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}; \\
x_1 &:= \vartheta_2(\tau), & x_2 &:= \vartheta_3(\tau), \\
y_1 &:= \vartheta_3(n\tau), & y_2 &:= x_2 = \vartheta_3(\tau).
\end{aligned}$$

We know by Theorem A that $P_n(X_0,Y_0) = 0$, and by Lemma 2.2 and the conditions of Theorem 4.1 that x_1 and x_2 are algebraically independent over $\mathbb{Q}$. Let

$$P_n(X,Y) = \sum_{\nu=0}^{N}\sum_{\mu=0}^{M} a_{\nu,\mu}X^\nu Y^\mu, \tag{4.1}$$

where $a_{\nu,\mu}$ are the integer coefficients of the polynomial P_n. Consider the polynomials

$$\begin{aligned} f_1 := f_1(t_1,t_2,u_1,u_2) &:= t_2^{4M}u_2^{4N}P_n\Big(\frac{n^2u_1^4}{u_2^4},\frac{16t_1^4}{t_2^4}\Big) \\ &= \sum_{\nu=0}^{N}\sum_{\mu=0}^{M} a_{\nu,\mu}t_2^{4M}u_2^{4N}\Big(\frac{n^2u_1^4}{u_2^4}\Big)^{\nu}\Big(\frac{16t_1^4}{t_2^4}\Big)^{\mu} \\ &= \sum_{\nu=0}^{N}\sum_{\mu=0}^{M} 16^{\mu}n^{2\nu}a_{\nu,\mu}t_1^{4\mu}t_2^{4(M-\mu)}u_1^{4\nu}u_2^{4(N-\nu)}, \\ f_2 := f_2(t_1,t_2,u_1,u_2) &:= u_2 - t_2. \end{aligned}$$

Note that $f_j(x_1,x_2,y_1,y_2)=0$ for $j=1,2$. To prove the algebraic independence of y_1 and y_2 using the AIC (Lemma 2.1) we have to show that the determinant

$$\Delta := \det\begin{pmatrix} \dfrac{\partial f_1}{\partial t_1} & \dfrac{\partial f_1}{\partial t_2} \\ \dfrac{\partial f_2}{\partial t_1} & \dfrac{\partial f_2}{\partial t_2} \end{pmatrix}$$

does not vanish at (x_1,x_2,y_1,y_2). Since

$$\frac{\partial f_2}{\partial t_1} = 0 \qquad \text{and} \qquad \frac{\partial f_2}{\partial t_2} = -1,$$

the condition $\Delta \neq 0$ is equivalent with the nonvanishing of the number

$$\frac{\partial f_1}{\partial t_1}(x_1,x_2,y_1,y_2) := \frac{\partial f_1(t_1,t_2,u_1,u_2)}{\partial t_1}\Big|_{(t_1=x_1,t_2=x_2,u_1=y_1,u_2=y_2)}.$$

We have

$$\begin{aligned} \frac{\partial f_1}{\partial t_1}(x_1,x_2,y_1,y_2) &= \sum_{\nu=0}^{N}\sum_{\mu=1}^{M} 16^{\mu}n^{2\nu}a_{\nu,\mu}4\mu x_1^{4\mu-1}x_2^{4(M-\mu)}y_1^{4\nu}y_2^{4(N-\nu)} \\ &= x_2^{4M}y_2^{4N}\sum_{\nu=0}^{N}\sum_{\mu=1}^{M} a_{\nu,\mu}\left(\frac{n^2y_1^4}{y_2^4}\right)^{\nu}\mu\left(\frac{16x_1^4}{x_2^4}\right)^{\mu-1}\left(64\frac{x_1^3}{x_2^4}\right) \\ &= 64x_1^3x_2^{4M-4}y_2^{4N}\frac{\partial P_n}{\partial Y}\left(\frac{n^2y_1^4}{y_2^4},\frac{16x_1^4}{x_2^4}\right). \end{aligned}$$

Since both, x_1 and $x_2(=y_2)$ do not vanish, it is clear that

$$\Delta \neq 0 \quad\Longleftrightarrow\quad \frac{\partial f_1}{\partial t_1}(x_1,x_2,y_1,y_2) \neq 0 \quad\Longleftrightarrow\quad \frac{\partial P_n}{\partial Y}\left(\frac{n^2y_1^4}{y_2^4},\frac{16x_1^4}{x_2^4}\right) \neq 0. \quad (4.2)$$

By the hypothesis of Theorem 4.1 the polynomial

$$R = \mathrm{Res}_X\Big(P_n(X,Y), \frac{\partial}{\partial Y}P_n(X,Y)\Big) \in \mathbb{Z}[Y]$$

does not vanish identically. For $Y = Y_0 = 16x_1^4/x_2^4$ we have $R \in \mathbb{Q}(x_1,x_2)$, so that the algebraic independence of x_1, x_2 proves $R \neq 0$. In particular, $P_n(X,Y)$ and $\frac{\partial}{\partial Y}P_n(X,Y)$ as polynomials in X have no common root for fixed $Y = Y_0 = 16x_1^4/x_2^4$. Since $P_n(X,Y)$ vanishes at $(X_0,Y_0) = \big(n^2y_1^4/y_2^4, 16x_1^4/x_2^4\big)$, it follows that

$$\frac{\partial P_n}{\partial Y}\left(\frac{n^2y_1^4}{y_2^4}, \frac{16x_1^4}{x_2^4}\right) \neq 0.$$

Thus, Theorem 4.1 for odd integers $n \geqslant 3$ follows from (4.2) and the AIC (Lemma 2.1).

In the case $n = 2^m$ $(m \geqslant 1)$ we introduce the quantities

$$\begin{aligned} X_0 &:= \frac{\vartheta_3^2(n\tau)}{\vartheta_3^2(\tau)}, & Y_0 &:= \frac{\vartheta_4(\tau)}{\vartheta_3(\tau)}; \\ x_1 &:= \vartheta_4(\tau), & x_2 &:= \vartheta_3(\tau), \\ y_1 &:= \vartheta_3(n\tau), & y_2 &:= x_2 = \vartheta_3(\tau). \end{aligned}$$

Here, we have $P_n(X_0,Y_0) = 0$ by Lemma 3.1, and

$$\begin{aligned} f_1(t_1,t_2,u_1,u_2) &:= t_2^M u_2^{2N} P_n\Big(\frac{u_1^2}{u_2^2}, \frac{t_1}{t_2}\Big) = \sum_{\nu=0}^{N}\sum_{\mu=0}^{M} a_{\nu,\mu} t_1^{\mu} t_2^{M-\mu} u_1^{2\nu} u_2^{2(N-\nu)}, \\ \frac{\partial f_1}{\partial t_1}(x_1,x_2,y_1,y_2) &= x_2^{M-1} y_2^{2N} \frac{\partial P_n}{\partial Y}\Big(\frac{y_1^2}{y_2^2}, \frac{x_1}{x_2}\Big), \\ f_2(t_1,t_2,u_1,u_2) &:= u_2 - t_2. \end{aligned}$$

Then,

$$\Delta \neq 0 \iff \frac{\partial P_n}{\partial Y}\Big(\frac{y_1^2}{y_2^2}, \frac{x_1}{x_2}\Big) \neq 0.$$

Using similar arguments as above by considering the particular point $(X_0,Y_0) = \big(y_1^2/y_2^2, x_1/x_2\big)$, the algebraic independence of $\vartheta_3(q^n)$ and $\vartheta_3(q)$ for $n = 2^m$ can be derived from the AIC. ■

First, using Theorem 4.1 we prove the algebraic independence of $\vartheta_3(q)$ and $\vartheta_3(q^n)$ for $n = 2,3,4,5,7,8,9,11$ by computing the resultant of the polynomials $P_n(X,Y)$ and $\partial P_n(X,Y)/\partial Y$. We have to show that these resultants do not vanish identically. So, it suffices to compute the values of the resultants at the point $Y = 0$ for $n = 3,4,5,7,8,11$ and at the point $Y = 2$ for $n = 2,9$. Note that

$Res_X\big(P_n(X,Y), \frac{\partial P_n(X,Y)}{\partial Y}\big)$ vanishes at $Y=0$ for $n=2,9$.

$$\begin{aligned}
Res_X\Big(P_2(X,2), \frac{\partial P_2}{\partial Y}(X,2)\Big) &= -2^2,\\
Res_X\Big(P_3(X,0), \frac{\partial P_3}{\partial Y}(X,0)\Big) &= 2^{16}\cdot 3^2,\\
Res_X\Big(P_4(X,0), \frac{\partial P_4}{\partial Y}(X,0)\Big) &= -2,\\
Res_X\Big(P_5(X,0), \frac{\partial P_5}{\partial Y}(X,0)\Big) &= 2^{60}\cdot 3^{10}\cdot 5^2,\\
Res_X\Big(P_7(X,0), \frac{\partial P_7}{\partial Y}(X,0)\Big) &= 2^{142}\cdot 3^{14}\cdot 7^2,\\
Res_X\Big(P_8(X,0), \frac{\partial P_8}{\partial Y}(X,0)\Big) &= 2^{12},\\
Res_X\Big(P_9(X,2), \frac{\partial P_9}{\partial Y}(X,2)\Big) &= 2^{132}\cdot 3^{96}\cdot 7^2\cdot 37^2\cdot 193^2\cdot 5387^2\\
&\qquad\times 3683832193^2\cdot 94686353323^2,\\
Res_X\Big(P_{11}(X,0), \frac{\partial P_{11}}{\partial Y}(X,0)\Big) &= 2^{336}\cdot 3^{22}\cdot 5^{22}\cdot 11^2.
\end{aligned}$$

Next we prove the algebraic independence of the numbers in each of the sets

$$\big\{\vartheta_3(6\tau), \vartheta_3(\tau)\big\} \qquad \text{and} \qquad \big\{\vartheta_3(10\tau), \vartheta_3(\tau)\big\}.$$

We shall not treat these two problems by the method shown before, but again the AIC will play an important role. We first consider the numbers $\vartheta_3(6\tau)$ and $\vartheta_3(\tau)$. Given any odd integer $n \geqslant 3$ one can deduce the algebraic independence of $\vartheta_3(2n\tau)$ and $\vartheta_3(\tau)$ as follows. First we replace τ by 2τ in Theorem A. Then,

$$P_n(X,Y) = 0 \tag{4.3}$$

holds for

$$X_0 := n^2\frac{\vartheta_3^4(2n\tau)}{\vartheta_3^4(2\tau)} \qquad \text{and} \qquad Y_0 := 16\frac{\vartheta_2^4(2\tau)}{\vartheta_3^4(2\tau)}.$$

Next we express $\vartheta_2^4(2\tau)$ and $\vartheta_3^4(2\tau)$ in terms of $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$:

$$\begin{aligned}
\vartheta_2^4(2\tau) &= \frac{1}{4}\big(\vartheta_3^2(\tau) - \vartheta_4^2(\tau)\big)^2,\\
\vartheta_3^4(2\tau) &= \frac{1}{4}\big(\vartheta_3^2(\tau) + \vartheta_4^2(\tau)\big)^2.
\end{aligned}$$

Hence (4.3) holds for

$$X_0 = \frac{4n^2\vartheta_3^4(2n\tau)}{\big(\vartheta_3^2(\tau)+\vartheta_4^2(\tau)\big)^2} \qquad \text{and} \qquad Y_0 = \frac{16\big(\vartheta_3^2(\tau)-\vartheta_4^2(\tau)\big)^2}{\big(\vartheta_3^2(\tau)+\vartheta_4^2(\tau)\big)^2}.$$

Setting

$$x_1 := \vartheta_3(\tau), \qquad x_2 := \vartheta_4(\tau),$$
$$y_1 := \vartheta_3(2n\tau), \qquad y_2 := x_1 = \vartheta_3(\tau),$$

we know that (4.3) holds for

$$X_0 = \frac{4n^2y_1^4}{(y_2^2+x_2^2)^2} \qquad \text{and} \qquad Y_0 = \frac{16(x_1^2-x_2^2)^2}{(x_1^2+x_2^2)^2}. \tag{4.4}$$

Beside (4.3) we have the identity $y_2 - x_1 = 0$, and the numbers x_1, x_2 are known to be algebraically independent over $\mathbb{Q}$ for any algebraic number $q = e^{\pi i\tau}$ with $\Im(\tau) > 0$ by Lemma 2.2. Using

$$P_n(X,Y) = \sum_{\nu=1}^{N}\sum_{\mu=1}^{M} a_{\nu,\mu}X^\nu Y^\mu,$$

we now introduce the polynomials

$$f_1(t_1,t_2,u_1,u_2) := (t_2^2+u_2^2)^{2N}(t_1^2+t_2^2)^{2M}P_n\left(\frac{4n^2u_1^4}{(t_2^2+u_2^2)^2}, \frac{16(t_1^2-t_2^2)^2}{(t_1^2+t_2^2)^2}\right), \tag{4.5}$$
$$f_2(t_1,t_2,u_1,u_2) := u_2 - t_1.$$

Using the AIC we have to show the nonvanishing of

$$\Delta := \det\begin{pmatrix} \dfrac{\partial f_1}{\partial t_1} & \dfrac{\partial f_1}{\partial t_2} \\ \dfrac{\partial f_2}{\partial t_1} & \dfrac{\partial f_2}{\partial t_2}\end{pmatrix} = \frac{\partial f_1}{\partial t_2}$$

at (x_1,x_2,y_1,y_2). From now on we restrict the investigation to particular cases. First, let $n = 3$. For $P_3(X,Y)$ we have $N = 4$ and $M = 2$ (cf. Appendix). We now compute $\Delta = \frac{\partial f_1}{\partial t_2}(x_1,x_2,y_1,y_2)$, where f_1 is as in (4.5). Setting $y_2 = x_1$, we get

$$\begin{aligned}\Delta = 72x_2(x_2^2+x_1^2)^3\big(&3440x_2^2y_1^4x_1^{10} + 7536y_1^4x_2^{10}x_1^2 - 19936x_1^6y_1^4x_2^6 \\ &+ 34560x_1^6y_1^8x_2^2 - 10344x_1^8y_1^4x_2^4 - 186624x_1^2y_1^{12}x_2^2 + 920x_1^4y_1^4x_2^8 \\ &+ 51840x_1^4y_1^8x_2^4 + 34560x_1^2y_1^8x_2^6 + 8640y_1^8x_2^8 - 93312x_1^4y_1^{12} + 210x_1^8x_2^8 \\ &+ 168x_1^6x_2^{10} + 84x_1^4x_2^{12} + 168x_1^{10}x_2^6 + 84x_1^{12}x_2^4 + 24x_1^{14}x_2^2 + 744x_1^{12}y_1^4 \\ &+ 24x_1^2x_2^{14} - 93312y_1^{12}x_2^4 + 3x_2^{16} + 186624y_1^{16} + 8640x_1^8y_1^8 + 3x_1^{16} - 280y_1^4x_2^{12}\big).\end{aligned}$$

To prove that $\Delta \neq 0$ it suffices to consider the polynomial within the second parentheses, since $72x_2(x_2^2+x_1^2)^3$ does not vanish by the algebraic independence of x_1 and x_2:

$$\begin{aligned} h_1(x_1,x_2,y_1) := {} & 3440x_2^2y_1^4x_1^{10} + 7536y_1^4x_2^{10}x_1^2 - 19936x_1^6y_1^4x_2^6 \\ & + 34560x_1^6y_1^8x_2^2 - 10344x_1^8y_1^4x_2^4 - 186624x_1^2y_1^{12}x_2^2 + 920x_1^4y_1^4x_2^8 \\ & + 51840x_1^4y_1^8x_2^4 + 34560x_1^2y_1^8x_2^6 + 8640y_1^8x_2^8 - 93312x_1^4y_1^{12} \\ & + 210x_1^8x_2^8 + 168x_1^6x_2^{10} + 84x_1^4x_2^{12} + 168x_1^{10}x_2^6 + 84x_1^{12}x_2^4 \\ & + 24x_1^{14}x_2^2 + 744x_1^{12}y_1^4 + 24x_1^2x_2^{14} - 93312y_1^{12}x_2^4 + 3x_2^{16} \\ & + 186624y_1^{16} + 8640x_1^8y_1^8 + 3x_1^{16} - 280y_1^4x_2^{12}. \end{aligned}$$

Let us assume that $\Delta = 0$, hence $h_1(x_1,x_2,y_1) = 0$. From (4.5) we have $(x_1^2+x_2^2)^{2(N+M)}P_3(X_0,Y_0) = 0$. Using $y_2 = x_1$, it follows that

$$\begin{aligned} 0 = {} & 9(x_2^2+x_1^2)^4(-x_2^8 + 80y_1^2x_1^2x_2^4 - 72y_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_1^8 \\ & - 6x_2^4x_1^4 - 144y_1^4x_1^2x_2^2 + 80x_1^4y_1^2x_2^2 - 4x_2^6x_1^2 - 72y_1^4x_2^4 - 16y_1^2x_2^6 \\ & - 16x_1^6y_1^2)(-x_2^8 - 80y_1^2x_1^2x_2^4 - 72y_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_1^8 - 6x_2^4x_1^4 \\ & - 144y_1^4x_1^2x_2^2 - 80x_1^4y_1^2x_2^2 - 4x_2^6x_1^2 - 72y_1^4x_2^4 + 16y_1^2x_2^6 + 16x_1^6y_1^2). \end{aligned}$$

The algebraic independence of x_1, x_2 over $\mathbb{Q}$ shows that $9(x_2^2+x_1^2)^4 \neq 0$, hence the number

$$\begin{aligned} h_2(x_1,x_2,y_1) := {} & (-x_2^8 + 80y_1^2x_1^2x_2^4 - 72y_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_1^8 - 6x_2^4x_1^4 \\ & - 144y_1^4x_1^2x_2^2 + 80x_1^4y_1^2x_2^2 - 4x_2^6x_1^2 - 72y_1^4x_2^4 - 16y_1^2x_2^6 - 16x_1^6y_1^2) \\ & \times (-x_2^8 - 80y_1^2x_1^2x_2^4 - 72y_1^4x_1^4 - 4x_2^2x_1^6 + 432y_1^8 - x_1^8 - 6x_2^4x_1^4 \\ & - 144y_1^4x_1^2x_2^2 - 80x_1^4y_1^2x_2^2 - 4x_2^6x_1^2 - 72y_1^4x_2^4 + 16y_1^2x_2^6 + 16x_1^6y_1^2) \end{aligned}$$

vanishes. By the assumption $h_1 = 0$ it follows that $\mathrm{Res}_{y_1}\big(h_1(x_1,x_2,y_1), h_2(x_1,x_2,y_1)\big) = 0$. We obtain

$$\begin{aligned} 0 = {} & 2^{240}3^{72}x_1^{16}x_2^8(8x_1^4 + 29x_2^2x_1^2 + 27x_2^4)^4(x_2-x_1)^{12}(x_1+x_2)^{12} \\ & \times (x_2^2 - 2x_1x_2 - x_1^2)^{16}(x_2^2 + 2x_1x_2 - x_1^2)^{16}(x_2^2+x_1^2)^{64}, \end{aligned}$$

a contradiction to the algebraic independence of x_1, x_2 over $\mathbb{Q}$. Thus the AIC proves the algebraic independence of $\vartheta_3(6\tau)$ and $\vartheta_3(\tau)$ over $\mathbb{Q}$.

Next, let $n = 5$. With $N = 6$, $M = 4$, and the polynomial $P_5(X,Y)$ listed in the appendix, an analogous computation finally gives the identity

$$\begin{aligned} 0 = {} & 2^{592}5^{200}x_1^{32}x_2^8(128x_1^{12} - 816x_1^8x_2^4 + 603x_1^6x_2^6 + 5775x_1^4x_2^8 + 7569x_1^2x_2^{10} \\ & + 3125x_2^{12})^4(243x_2^{24} - 3580x_1^2x_2^{22} - 315034x_1^4x_2^{20} + 1780x_1^6x_2^{18} + 1040093x_1^8x_2^{16} \\ & + 774920x_1^{10}x_2^{14} - 2001516x_1^{12}x_2^{12} + 774920x_1^{14}x_2^{10} + 1040093x_1^{16}x_2^8 \\ & + 1780x_1^{18}x_2^6 - 315034x_1^{20}x_2^4 - 3580x_1^{22}x_2^2 + 243x_1^{24})^8(x_1^2 - 2x_1x_2 - x_2^2)^{16} \\ & \times (x_1^2 + 2x_1x_2 - x_2^2)^{16}(x_1-x_2)^{20}(x_1+x_2)^{20}(x_1^2+x_2^2)^{96}. \end{aligned}$$

The contradiction proves the algebraic independence of $\vartheta_3(10\tau)$ and $\vartheta_3(\tau)$ over $\mathbb{Q}$. For the proof of the algebraic independence of $\vartheta_3(12\tau)$ and $\vartheta_3(\tau)$ over $\mathbb{Q}$ we have to modify the above formulae. From the double-argument formulae (3.1) we obtain

$$\vartheta_2^4(4\tau) = \frac{1}{16}\big(\vartheta_3 - \vartheta_4\big)^4,$$
$$\vartheta_3^4(4\tau) = \frac{1}{16}\big(\vartheta_3 + \vartheta_4\big)^4.$$

In Theorem A we replace τ by 4τ such that (4.3) holds with

$$X_0 = \frac{n^2\vartheta_3^4(4n\tau)}{\vartheta_3^4(4\tau)} = \frac{16n^2y_1^4}{(y_2+x_2)^4},$$
$$Y_0 = \frac{16\vartheta_2^4(4\tau)}{\vartheta_3^4(4\tau)} = \frac{16(x_1-x_2)^4}{(x_1+x_2)^4},$$

where $y_1 = \vartheta_3(4n\tau)$. Finally, we replace (4.5) by

$$f_1(t_1,t_2,u_1,u_2) = \big(t_2+u_2\big)^{4N}\big(t_1+t_2\big)^{4M}P_n\Big(\frac{16n^2u_1^4}{(t_2+u_2)^4}, \frac{16(t_1-t_2)^4}{(t_1+t_2)^4}\Big).$$

Setting $n=3$, $N=4$, $M=2$, and following the above lines of computations, we deduce the following identity:

$$0 = 2^{376}3^{72}x_1^8x_2^8(x_1^4 - 12x_1^3x_2 - 12x_1x_2^3 + x_2^4 + 6x_1^2x_2^2)^{16}$$
$$\times\,(3x_1^4 + 16x_1^3x_2 + 30x_1^2x_2^2 + 32x_1x_2^3 + 27x_2^4)^4(x_1-x_2)^{32}(x_1+x_2)^{128}.$$

Here the contradiction proves the algebraic independence of $\vartheta_3(12\tau)$ and $\vartheta_3(\tau)$ over $\mathbb{Q}$.

Finally, for Theorem 1.1 it remains to prove the algebraic independence of $\vartheta_3(q^{2^m})$ and $\vartheta_3(q)$ over $\mathbb{Q}$ for any $m \geqslant 3$. Let $n = 2^m$. By Theorem 4.1 it suffices to show that the polynomial

$$\mathrm{Res}_X\Big(P_n(X,Y), \frac{\partial}{\partial Y}P_n(X,Y)\Big) \in \mathbb{Z}[Y]$$

does not vanish identically. We know from (3.9) in Lemma 3.2 that

$$P_n(X,Y) = T_n\big(X,(1+Y)^2,Y\big) = \big(nX - (1+Y)^2\big)^{2^{m-2}} + YU_n\big(X,(1+Y)^2,Y\big).$$

Hence we obtain

$$P_n(X,0) = T_n(X,1,0) = \big(2^mX - 1\big)^{2^{m-2}}, \tag{4.6}$$
$$\frac{\partial P_n}{\partial Y}(X,0) = -2^{m-1}\big(2^mX-1\big)^{2^{m-2}-1} + U_n(X,1,0). \tag{4.7}$$

On the one hand the polynomial $P_n(X,0)$ in (4.6) has a 2^{m-2}-fold root X_0 at $X_0 = 1/2^m$. On the other hand we know by (4.7) and (3.10) in Lemma 3.2 that

$$\frac{\partial P_n}{\partial Y}(X_0,0) = U_n\Big(\frac{1}{n},1,0\Big) = -2^{2^{m-1}-1} \neq 0.$$

This shows that for $Y = 0$ the polynomials $P_n(X,Y)$ and $\partial P_n(X,Y)/\partial Y$ have no common root. Therefore, the resultant of both polynomials with respect to X does not vanish identically. This completes the proof of Theorem 1.1. ■

5. Appendix

The polynomials P_3, P_5, P_7, P_9, and P_{11} listed below were derived from the proof of Theorem 1.1 in [8].

$$\begin{aligned}
P_3 &= 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4,\\
P_5 &= 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X + (255 + 1920Y - 120Y^2)X^2\\
&\quad + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6,\\
P_7 &= 49 - (344 - 17568Y + 20554Y^2 - 6528Y^3 + 844Y^4 - 48Y^5 + Y^6)X\\
&\quad + (1036 + 156800Y + 88760Y^2 - 12320Y^3 + 385Y^4)X^2\\
&\quad - (1736 - 185024Y + 18732Y^2 - 896Y^3 + 28Y^4)X^3\\
&\quad + (1750 + 31360Y - 1960Y^2)X^4 - (1064 - 2464Y + 154Y^2)X^5\\
&\quad + 364X^6 - 56X^7 + X^8,\\
P_9 &= 6561 - (60588 - 18652032Y + 56033208Y^2 - 40036032Y^3 + 11743542Y^4\\
&\quad - 1715904Y^5 + 132516Y^6 - 5184Y^7 + 81Y^8)X\\
&\quad + (250146 + 427613184Y + 2083563072Y^2 + 86274432Y^3 - 57982860Y^4\\
&\quad + 4249728Y^5 - 99288Y^6 + 576Y^7 - 9Y^8)X^2\\
&\quad - (607420 - 1418904064Y + 2511615520Y^2 - 353755456Y^3 + 19071754Y^4\\
&\quad - 612736Y^5 + 13960Y^6 - 64Y^7 + Y^8)X^3\\
&\quad + (959535 + 856286208Y + 8468928Y^2 - 2145024Y^3 - 808488Y^4\\
&\quad + 65664Y^5 - 1368Y^6)X^4\\
&\quad - (1028952 + 22899456Y + 1430352Y^2 - 505152Y^3 + 38826Y^4\\
&\quad - 1728Y^5 + 36Y^6)X^5\\
&\quad + (757596 - 13138944Y + 4160448Y^2 - 417408Y^3 + 13044Y^4)X^6\\
&\quad - (378072 + 1138176Y + 16416Y^2 - 10944Y^3 + 342Y^4)X^7\\
&\quad + (122895 + 64512Y - 4032Y^2)X^8 - (24060 - 11136Y + 696Y^2)X^9\\
&\quad + 2466X^{10} - 108X^{11} + X^{12},
\end{aligned}$$

$$\begin{aligned}
P_{11} = 121 &- (1332 - 2214576Y + 15234219Y^2 - 21424896Y^3 + 11848792Y^4 \\
&- 3309152Y^5 + 522914Y^6 - 48896Y^7 + 2684Y^8 - 80Y^9 + Y^{10})X \\
&+ (6666 + 111458688Y + 2532888424Y^2 + 2367855776Y^3 - 327773413Y^4 \\
&- 9982720Y^5 + 3230480Y^6 - 161920Y^7 + 2530Y^8)X^2 \\
&- (20020 - 864654912Y + 12880909668Y^2 - 5289254784Y^3 + 744094076Y^4 \\
&- 43914992Y^5 + 967461Y^6 - 2816Y^7 + 44Y^8)X^3 \\
&+ (40095 + 1748954240Y - 175142088Y^2 + 372281536Y^3 - 68516998Y^4 \\
&+ 4266240Y^5 - 88880Y^6)X^4 \\
&- (56232 - 1061669664Y + 132688050Y^2 - 10724736Y^3 + 715308Y^4 \\
&- 28512Y^5 + 594Y^6)X^5 \\
&+ (56364 + 211953280Y - 7454568Y^2 - 724064Y^3 + 22627Y^4)X^6 \\
&- (40392 - 24140864Y + 2162116Y^2 - 81664Y^3 + 2552Y^4)X^7 \\
&+ (20295 + 1448832Y - 90552Y^2)X^8 - (6820 - 36784Y + 2299Y^2)X^9 \\
&+ 1386X^{10} - 132X^{11} + X^{12}.
\end{aligned}$$

The polynomials P_2, P_4, P_8, P_{16}, and P_{32} listed below were derived from the proof of Lemma 3.1:

$$\begin{aligned}
P_2 &= 2X - Y^2 - 1, \\
P_4 &= 4X - (1+Y)^2, \\
P_8 &= 64X^2 - 16(1+Y)^2X + (1-Y)^4, \\
P_{16} &= 65536X^4 - 16384(1+Y)^2X^3 + 512(3Y^4 + 4Y^3 + 18Y^2 + 4Y + 3)X^2 \\
&\quad - 64(1+Y)^2(Y^4 + 28Y^3 + 6Y^2 + 28Y + 1)X + (1-Y)^8, \\
P_{32} &= 2^{40}X^8 - 2^{38}(1+Y)^2X^7 + 2^{32}(7Y^4 + 20Y^3 + 42Y^2 + 20Y + 7)X^6 \\
&\quad - 2^{28}(1+Y)^2(7Y^4 + 164Y^3 + 42Y^2 + 164Y + 7)X^5 \\
&\quad + 2^{21}(35Y^8 + 552Y^7 + 2260Y^6 + 3864Y^5 + 5010Y^4 \\
&\quad + 3864Y^3 + 2260Y^2 + 552Y + 35)X^4 \\
&\quad - 2^{18}(1+Y)^2(7Y^8 + 424Y^7 + 7492Y^6 + 2968Y^5 + 15082Y^4 \\
&\quad + 2968Y^3 + 7492Y^2 + 424Y + 7)X^3 \\
&\quad + 2^{12}(7Y^{12} - 5924Y^{11} + 4174Y^{10} + 33900Y^9 + 33161Y^8 + 36536Y^7 \\
&\quad + 58436Y^6 + 36536Y^5 + 33161Y^4 + 33900Y^3 + 4174Y^2 - 5924Y + 7)X^2 \\
&\quad - 2^8(1+Y)^2(Y^{12} + 660Y^{11} + 15170Y^{10} + 68420Y^9 + 121327Y^8 \\
&\quad + 212520Y^7 + 212380Y^6 + 212520Y^5 + 121327Y^4 + 68420Y^3 \\
&\quad + 15170Y^2 + 660Y + 1)X + (1-Y)^{16}.
\end{aligned}$$

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Address: Carsten Elsner: Fachhochschule für die Wirtschaft, University of Applied Sciences, Freundallee 15, D-30173 Hannover, Germany.

E-mail: carsten.elsner@fhdw.de

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