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TAME KERNELS OF NON-ABELIAN GALOIS EXTENSIONS OF NUMBER FIELDS OF DEGREE q^3

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Abstract: Let E/F be a non-abelian Galois extension of number fields of degree q^3. We give some expressions for the order of the Sylow p-subgroup of tame kernel of E and some of its subfields containing F, where p is a prime, q is an odd prime, $p \neq q$. As applications, we give some results about the orders of the Sylow p-subgroups of tame kernels when $E/\mathbb{Q}(\zeta_3)$ is a Galois extension of number fields with non-abelian Galois group of order 27.

Keywords: Tame kernels, non-abelian extensions of number fields.

1. Introduction

Let F be a number field, $\mathcal{O}_F$ the ring of integers in F, $K_2(F)$ the Milnor K-group of F. The tame kernel of F is the kernel of the following map

$$\tau = \oplus \tau_{\mathfrak{p}} : K_2(F) \to \bigoplus_{\mathfrak{p}-\text{finite}} k_{\mathfrak{p}}^*,$$

where for every finite prime ideal $\mathfrak{p}$, $k_{\mathfrak{p}}$ is the residue field modulo $\mathfrak{p}$ and $\tau_{\mathfrak{p}} : K_2(F) \to k_{\mathfrak{p}}^*$ defined by

$$\tau_{\mathfrak{p}}\{a, b\} \equiv (-1)^{v_{\mathfrak{p}}(a)v_{\mathfrak{p}}(b)} \frac{a^{v_{\mathfrak{p}}(b)}}{b^{v_{\mathfrak{p}}(a)}} \pmod{\mathfrak{p}}.$$

It is well-known that $K_2(\mathcal{O}_F)$ called the tame kernel of F, is a finite abelian group. The 2-primary part of the tame kernel $K_2(\mathcal{O}_F)$ for number field F has been intensively studied (See [3], [10]-[12]). There are also some results concerning the p-primary part of the tame kernel when p is odd (See [1], [2], [15]-[17]).

There are various conjectures about the order of $K_2(\mathcal{O}_F)$. Birch-Tate Conjecture states that if F is a totally real number field, then

$$|K_2(\mathcal{O}_F)| = \omega_2(F)|\zeta_F(-1)|, \tag{1.1}$$

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where $\omega_2(F)$ is the maximal order of the root of unity belonging to the compositum of all quadratic extensions of F, and $\zeta_F(s)$ denotes the Dedekind zeta function of F. By work on the main conjecture of Iwasawa theory (See [8]), the Birch-Tate Conjecture was confirmed up to 2-torsion, and was confirmed for abelian extensions F over $\mathbb{Q}$ (See [7], [14]).

Let E/F be a Galois extension of number fields with Galois group G. For every cyclic subgroup H of G denote

$$c_G(H) = \frac{1}{(G:H)} \sum_{H^* \text{ cyclic}, \ H \subseteq H^* \subseteq G} \mu((H^*:H)), \tag{1.2}$$

where μ is the *Möbius* function. R. Brauer and S. Kuroda have independently given the following multiplicative relations (See [4]):

$$\zeta_F(s) = \prod_{H \text{ cyclic}, \ H \subseteq G} \zeta_{E^H}^{c_G(H)}(s). \tag{1.3}$$

Throughout the paper we use the following notation:

- p is a prime, q is an odd prime.
- C_q is a cyclic group of order q.
- $A(p)$ denotes the Sylow p-subgroup of a finite group A.
- $|A|$ denotes the order of a finite group A.
- $x =_p y$ means $v_p(x) = v_p(y)$, where $x, y \in \mathbb{Z}$.
- $\langle a \rangle$ denotes the cyclic group generated by a.
- $G_1 = G_1(q) = \langle g_1, g_2, g_3 | g_1^q = g_2^q = g_3^q = 1, g_2g_1 = g_1g_2g_3, g_1g_3 = g_3g_1, g_2g_3 = g_3g_2 \rangle$.
- $G_2 = G_2(q) = \langle g_1, g_2 | g_1^{q^2} = 1, g_2^q = 1, g_2g_1 = g_1^{1+q}g_2 \rangle$.

Let $E/\mathbb{Q}$ be a Galois extension of number fields with Galois group $C_q \times C_q \times \cdots \times C_q$. In [15], Wu proved that $(K_2(\mathcal{O}_E))_p = \bigoplus (K_2(\mathcal{O}_k))_p$, where k runs over all cyclic subfields of E, q is the degree of k over $\mathbb{Q}$, and $p \neq q$ is an odd prime. Let E/F be a Galois extension of number fields with Galois group $C_q \times C_q$. Denote by $K_2(E/F)$ the kernel of the map $\mathrm{tr}_{E/F} : K_2(\mathcal{O}_E) \to K_2(\mathcal{O}_F)$. In Section 2, by the same approach as in [16], we prove that for every prime p $(p \neq q)$,

$$K_2(E/F)(p) \cong K_2(k_0/F)(p) \times K_2(k_1/F)(p) \times \cdots \times K_2(k_q/F)(p),$$

where k_i/F $(i = 0, 1, \cdots, q)$ are all cyclic subextensions of E/F. This generalizes Wu's results when $F \neq \mathbb{Q}$ and the Galois group $Gal(E/F)$ is $C_q \times C_q$.

From [5], we know that up to isomorphism the two non-abelian groups of order q^3 are G_1 and G_2, where q is an odd prime. In 1937, A. Scholz and H. Reichardt proved the following results: For an odd prime q, every finite $q-$group occurs as a Galois group over $\mathbb{Q}$. In [9], Ivo M. Michailov and Nikola P. Ziapkov surveyed the realizability of $q-$groups as Galois groups over arbitrary fields containing the primitive q-th roots of unity. Furthermore, C. Jensen, A. Ledet and N. Yui examined the realizability of G_1 as Galois group over arbitrary field in [6]. Let E/F

be a Galois extension of number fields with Galois group G_1 or G_2. In Section 3, we prove some relations between the order of the Sylow p-subgroup of tame kernel of E and of some of its subfields. In particular, let $E/\mathbb{Q}$ be a Galois extension of number fields with Galois group G_1 or G_2. Then E is a totally real number field. Assuming the Birch-Tate Conjecture (1.1) and applying the Brauer-Kuroda relations (1.3), we give some expressions for the order of tame kernel of E and of some of its subfields. As applications, in section 4, we give some results about the orders of the Sylow p-subgroups of tame kernels when $E/\mathbb{Q}(\zeta_3)$ is a Galois extension of number fields with Galois group $G_1(3)$ or $G_2(3)$.

2. The tame kernels of bi-cyclic extensions of number fields

We begin with some preliminary results.

Let E/F be a finite extension of number fields. There exists a group homomorphism, called the transfer map and denoted by $\mathrm{tr}_{E/F}$, mapping $K_2(E)$ into $K_2(F)$. An explicit description of this map is hard, but we list here some well-known facts which will be the basis in this paper (See [16]).

(1) Let $j : K_2F \to K_2E$ denote the canonical map, which is induced by $F \subset E$, then
$$\mathrm{tr}_{E/F}(j(\alpha)) = \alpha^{[E:F]}, \text{ for all } \alpha \in K_2(F).$$
(2) If L is an intermediate field of E/F, then $\mathrm{tr}_{E/F} = \mathrm{tr}_{L/F} \cdot \mathrm{tr}_{E/L}$.

(3) If E/F is a Galois extension with Galois group G, then
$$j(\mathrm{tr}_{E/F}(\alpha)) = N_{E/F}(\alpha) = \alpha^{\sum_{\sigma\in G}\sigma}, \text{ for all } \alpha \in K_2(E).$$
(4) If $j : K_2F \to K_2E$ and $\mathrm{tr}_{E/F} : K_2E \to K_2F$ are restricted to the groups $K_2(\mathcal{O}_E)$, $K_2(\mathcal{O}_F)$, then the analogues of (1), (2) and (3) hold for these groups as well.

Obviously, the Sylow p-subgroup $K_2(E/F)(p)$ of $K_2(E/F)$ is the kernel of the map $\mathrm{tr}_{E/F} : K_2(\mathcal{O}_E)(p) \to K_2(\mathcal{O}_F)(p)$.

Lemma 1 ([16]). *Let E/F be a Galois extension of number fields, then for every prime $p \nmid (E : F)$, $j : K_2(\mathcal{O}_F)(p) \to K_2(\mathcal{O}_E)(p)$ is injective, the transfer $\mathrm{tr}_{E/F} : K_2(\mathcal{O}_E)(p) \to K_2(\mathcal{O}_F)(p)$ is surjective, and $K_2(\mathcal{O}_E)(p) \cong K_2(E/F)(p) \times K_2(\mathcal{O}_F)(p)$.*

Theorem 1. *Let E/F be a Galois extension of number fields with Galois group $C_q \times C_q = \langle a\rangle \times \langle b\rangle$. Its non-trivial subgroups are: $\langle a\rangle$, $\langle ab\rangle$, $\langle a^2b\rangle$, $\cdots$, $\langle a^{q-1}b\rangle$, $\langle b\rangle$, and the corresponding fixed subfields are k_0, k_1, k_2, $\cdots$, k_{q-1}, k_q. Then for every prime p, $p \neq q$,*
$$K_2(E/F)(p) \cong K_2(k_0/F)(p) \times K_2(k_1/F)(p) \times \cdots \times K_2(k_q/F)(p), \tag{2.1}$$
and
$$|K_2(O_E)||K_2(O_F)|^q =_p |K_2(O_{k_0})||K_2(O_{k_1})| \cdots |K_2(O_{k_q})|. \tag{2.2}$$

Proof. For every prime p, $p \neq q$, raising to the power q is an automorphism of the $p-$part of any finite abelian group. So for every $c \in K_2(E/F)(p)$, there is an unique element $d \in K_2(E/F)(p)$, such that $c = d^q$.

From $\mathrm{tr}_{E/F} = \mathrm{tr}_{k_i/F} \cdot \mathrm{tr}_{E/k_i}$, $i = 0, 1, \cdots, q$, it follows that $\mathrm{tr}_{E/k_i}(\ker(\mathrm{tr}_{E/F})) \subseteq \ker(\mathrm{tr}_{k_i/F})$, hence $\mathrm{tr}_{E/k_i}(K_2(E/F)(p)) \subseteq K_2(k_i/F)(p)$.

From $d \in K_2(E/F)(p)$, we get

$$1 = \mathrm{tr}_{E/F}(d) = d^{1+a+a^2+\cdots+a^{q-1}+ab+a^2b^2+\cdots+a^{q-1}b^{q-1}+\cdots+b^{q-1}},$$

and

$$\mathrm{tr}_{E/k_0}(d) = d^{1+a+a^2+\cdots+a^{q-1}} \in K_2(k_0/F)(p),$$
$$\mathrm{tr}_{E/k_i}(d) = d^{1+a^ib+(a^ib)^2+\cdots+(a^ib)^{q-1}} \in K_2(k_i/F)(p), \qquad i = 1, 2, \cdots, q.$$

We define

$$\varphi : K_2(E/F)(p) \to K_2(k_0/F)(p) \times K_2(k_1/F)(p) \times \cdots \times K_2(k_q/F)(p)$$

by

$$\varphi(c) = (d^{1+a+a^2+\cdots+a^{q-1}}, d^{1+ab+a^2b^2+\cdots+a^{q-1}b^{q-1}}, \cdots, d^{1+b+b^2+\cdots+b^{q-1}}).$$

Obviously, φ is a homomorphism.

If $\varphi(c) = 1$, then

$$d^{1+a+a^2+\cdots+a^{q-1}} = d^{1+a^ib+(a^ib)^2+\cdots+(a^ib)^{q-1}} = 1, \qquad i = 1, 2, \cdots, q.$$

Hence

$$\begin{aligned} c &= d^q = d^q \cdot \mathrm{tr}_{E/F}(d) \\ &= d^{1+a+a^2+\cdots+a^{q-1}} \cdot d^{1+ab+a^2b^2+\cdots+a^{q-1}b^{q-1}} \cdots d^{1+b+b^2+\cdots+b^{q-1}} = 1, \end{aligned}$$

so φ is injective.

For every $b_i \in K_2(k_i/F)(p)$, $i = 0, 1, \cdots, q$, there exists $d_i \in K_2(k_i/F)(p)$ such that $b_i = d_i^q$. Since d_0 is fixed by a, d_i is fixed by a^ib, $i = 1, 2, \cdots, q$, we get

$$d_0^{1+a+a^2+\cdots+a^{q-1}} = d_0^q = b_0,$$
$$d_i^{1+a^ib+(a^ib)^2+\cdots+(a^ib)^{q-1}} = d_i^q = b_i, \qquad i = 1, 2, \cdots, q.$$

Hence taking $d := d_0 d_1 \cdots d_q$ and $c := d^q$, we have

$$\begin{aligned} \varphi(c) &= \left(d^{1+a+a^2+\cdots+a^{q-1}}, d^{1+ab+a^2b^2+\cdots+a^{q-1}b^{q-1}}, \cdots, d^{1+b+b^2+\cdots+b^{q-1}}\right) \\ &= \left(d_0^{1+a+a^2+\cdots+a^{q-1}}, d_1^{1+ab+a^2b^2+\cdots+a^{q-1}b^{q-1}}, \cdots, d_q^{1+b+b^2+\cdots+b^{q-1}}\right) \\ &= \left(b_0, b_1, \cdots, b_q\right), \end{aligned}$$

so φ is surjective.

Therefore we have proved (2.1). By (2.1), we have

$$|K_2(E/F)| =_p |K_2(k_0/F)||K_2(k_1/F)| \cdots |K_2(k_q/F)| \tag{2.3}$$

By Lemma 1, we have $|K_2(O_E)| =_p |K_2(E/F)||K_2(O_F)|$, and

$$|K_2(O_{k_i})| =_p |K_2(k_i/F)||K_2(O_F)|, \qquad i = 0, 1, \cdots, q.$$

Substituting this in (2.3) proves (2.2). ■

Remark. When $F = \mathbb{Q}$ in Theorem 1, E is a totally real abelian number field of degree q^2. We assume that

$$\omega_2(E) = \omega_2(\mathbb{Q}) = \omega_2(k_i) = 24, \qquad i = 0, 1, \cdots, q.$$

By the Birch-Tate Conjecture (1.1) and the Brauer-Kuroda relations (1.3), one has

$$|K_2(O_E)| = \frac{|K_2(O_{k_0})||K_2(O_{k_1})| \cdots |K_2(O_{k_q})|}{2^q}.$$

3. The tame kernels of non-abelian extensions of number fields of degree q^3

Let E/F be a Galois extension of number fields with Galois group G_1 or G_2. In this section, we give some expressions for the order of the Sylow p-subgroup of tame kernel of E and of some of its subfields containing F.

Applying the standard methods of group theory (See [5]), we firstly get the following basis information about G_1.

G_1 has $q^2 + q + 1$ subgroups of order q which belong to $q + 2$ conjugate classes, and every conjugate class has q subgroups except the third class:

- $\langle g_1 \rangle$, $\langle g_1 g_3 \rangle$, $\cdots$, $\langle g_1 g_3^{q-1} \rangle$;
- $\langle g_2 \rangle$, $\langle g_2 g_3 \rangle$, $\cdots$, $\langle g_2 g_3^{q-1} \rangle$;
- $\langle g_3 \rangle$ is a normal subgroup;
- $\langle g_1 g_2^i \rangle, \langle g_1^2 g_2^{2i} \rangle, \cdots, \langle g_1^{q-1} g_2^{(q-1)i} \rangle, \langle g_1 g_2^i g_3 \rangle$, $i = 1, 2, \cdots, q-1$.

G_1 has $q + 1$ subgroups of order q^2: $\langle g_1 \rangle \times \langle g_3 \rangle$, $\langle g_2 \rangle \times \langle g_3 \rangle$, $\langle g_1 g_2^i \rangle \times \langle g_3 \rangle$, $i = 1, 2, \cdots, q-1$, and all of them are isomorphic to $C_q \times C_q$.

We denote by E_H the subfield of E fixed by the subgroup H.

Theorem 2. *Let E/F be a Galois extension of number fields with Galois group G_1, its subgroups as stated above. Then for every prime p, $p \neq q$,*

$$|K_2(\mathcal{O}_E)|^{q+1}|K_2(\mathcal{O}_F)|^{q^2} =_p \left(\prod_{j=1,2} |K_2(\mathcal{O}_{E_{\langle g_j \rangle}})|^q \right) |K_2(\mathcal{O}_{E_{\langle g_3 \rangle}})| \times \prod_{i=1,2,\cdots,q-1} |K_2(\mathcal{O}_{E_{\langle g_1 g_2^i \rangle}})|^q. \tag{3.1}$$

Proof. Since E/F is a Galois extension, by Galois theory, $E/E_{\langle g_1\rangle\times\langle g_3\rangle}$, $E/E_{\langle g_2\rangle\times\langle g_3\rangle}$, $E/E_{\langle g_1 g_2^i\rangle\times\langle g_3\rangle}$ and $E_{\langle g_3\rangle}/F$ are Galois extensions with Galois group $C_q\times C_q$, where $i=1,2,\cdots,q-1$.

$\langle g_1\rangle$, $\langle g_1 g_3\rangle$, $\cdots$, $\langle g_1 g_3^{q-1}\rangle$ are conjugate subgroups, so they have isomorphic fixed fields. From Theorem 1, it is clear that

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{E_{\langle g_1\rangle\times\langle g_3\rangle}})|^q =_p |K_2(\mathcal{O}_{E_{\langle g_1\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})|. \tag{3.2}$$

Similarly,

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{E_{\langle g_2\rangle\times\langle g_3\rangle}})|^q =_p |K_2(\mathcal{O}_{E_{\langle g_2\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})|, \tag{3.3}$$

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{E_{\langle g_1 g_2^i\rangle\times\langle g_3\rangle}})|^q =_p |K_2(\mathcal{O}_{E_{\langle g_1 g_2^i\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})|, \quad i=1,2,\cdots,q-1, \tag{3.4}$$

$$|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})||K_2(\mathcal{O}_F)|^q =_p \left(\prod_{j=1,2}|K_2(\mathcal{O}_{E_{\langle g_j\rangle\times\langle g_3\rangle}})|\right) \times \prod_{i=1,2,\cdots,q-1}|K_2(\mathcal{O}_{E_{\langle g_1 g_2^i\rangle\times\langle g_3\rangle}})|. \tag{3.5}$$

We can get (3.1) easily by raising both sides of (3.5) to the power q and comparing it with (3.2), (3.3), (3.4). This completes the proof. ■

Next, we consider the other non-abelian group of order q^3, and give the following basis information.

G_2 has $q+1$ subgroups of order q which belong to 2 conjugate classes:

(1) $\langle g_1^q\rangle$ is a normal subgroup;

(2) $\langle g_2\rangle, \langle g_1^q g_2\rangle, \langle g_1^{2q} g_2\rangle, \cdots, \langle g_1^{(q-1)q} g_2\rangle$ are conjugate subgroups.

G_2 has $q+1$ subgroups of order q^2: $\langle g_1\rangle, \langle g_1 g_2\rangle, \langle g_1^2 g_2\rangle, \cdots, \langle g_1^{q-1} g_2\rangle$ and $\langle g_1^q\rangle\times\langle g_2\rangle$, where $\langle g_1^q\rangle\times\langle g_2\rangle$ is isomorphic to $C_q\times C_q$.

Theorem 3. *Let E/F be a Galois extension with Galois group G_2, its subgroups as stated above. Then for every prime p, $p\neq q$,*

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_F)|^{q^2}|K_2(\mathcal{O}_{E_{\langle g_1^q\rangle}})|^{q-1} =_p \left(\prod_{j=1,2}|K_2(\mathcal{O}_{E_{\langle g_j\rangle}})|^q\right) \times \prod_{i=1,2,\cdots,q-1}|K_2(\mathcal{O}_{E_{\langle g_1^i g_2\rangle}})|^q. \tag{3.6}$$

Proof. Since E/F is a Galois extension, by Galois theory, $E/E_{\langle g_1^q\rangle\times\langle g_2\rangle}$ and $E_{\langle g_1^q\rangle}/F$ are Galois extensions with Galois group $C_q\times C_q$.

From Theorem 1, it is clear that

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{E_{\langle g_1^q\rangle\times\langle g_2\rangle}})|^q =_p |K_2(\mathcal{O}_{E_{\langle g_2\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_1^q\rangle}})|, \tag{3.7}$$

$$\begin{aligned}|K_2(\mathcal{O}_{E_{\langle g_1^q\rangle}})||K_2(\mathcal{O}_F)|^q =_p\ & |K_2(\mathcal{O}_{E_{\langle g_1\rangle}})||K_2(\mathcal{O}_{E_{\langle g_1^q\rangle\times\langle g_2\rangle}})| \\ & \times \prod_{i=1,\cdots,q-1} |K_2(\mathcal{O}_{E_{\langle g_1^i g_2\rangle}})|.\end{aligned} \tag{3.8}$$

We can get (3.6) easily by raising both sides of (3.8) to the power q and comparing it with (3.7). This completes the proof. ■

Remark. When $F = \mathbb{Q}$ in Theorem 2 and Theorem 3, E is a totally real non-abelian number field of degree q^3. One has the Brauer-Kuroda relations

$$\zeta_E^{q+1}(s)\zeta_{\mathbb{Q}}^{q^2}(s) = \zeta_{E_{\langle g_1\rangle}}^q(s)\zeta_{E_{\langle g_2\rangle}}^q(s)\zeta_{E_{\langle g_3\rangle}}(s)\prod_{i=1,\cdots,q-1}\zeta_{E_{\langle g_1 g_2^i\rangle}}^q(s), \tag{3.9}$$

where $Gal(E/\mathbb{Q}) = G_1$, and

$$\zeta_E(s)\zeta_{\mathbb{Q}}^{q^2}(s)\zeta_{E_{\langle g_1^q\rangle}}^{q-1}(s) = \zeta_{E_{\langle g_1\rangle}}^q(s)\zeta_{E_{\langle g_2\rangle}}^q(s)\prod_{i=1,\cdots,q-1}\zeta_{E_{\langle g_1^i g_2\rangle}}^q(s), \tag{3.10}$$

where $Gal(E/\mathbb{Q}) = G_2$. We assume that $\omega_2(\bullet)$ is equal to 24, where $\bullet$ runs over all fields in (3.9) and (3.10). Applying the Birch-Tate Conjecture (1.1), by (3.9) and (3.10), one has, for every prime $p \neq 2$,

$$\begin{aligned}|K_2(\mathcal{O}_E)|^{q+1} =_p\ & |K_2(\mathcal{O}_{E_{\langle g_1\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_2\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})| \\ & \times \prod_{i=1,2,\cdots,q-1} |K_2(\mathcal{O}_{E_{\langle g_1 g_2^i\rangle}})|^q,\end{aligned} \tag{3.11}$$

where $Gal(E/\mathbb{Q}) = G_1$. And

$$\begin{aligned}|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{E_{\langle g_1^q\rangle}})|^{q-1} =_p\ & |K_2(\mathcal{O}_{E_{\langle g_1\rangle}})|^q|K_2(\mathcal{O}_{E_{\langle g_2\rangle}})|^q \\ & \times \prod_{i=1,2,\cdots,q-1} |K_2(\mathcal{O}_{E_{\langle g_1^i g_2\rangle}})|^q,\end{aligned} \tag{3.12}$$

where $Gal(E/\mathbb{Q}) = G_2$.

Combining (3.11) (3.12) with (3.1) (3.6), we have

$$\begin{aligned}|K_2(\mathcal{O}_E)|^{q+1}|K_2(\mathcal{O}_{\mathbb{Q}})|^{q^2} = & \left(\prod_{j=1,2}|K_2(\mathcal{O}_{E_{\langle g_j\rangle}})|^q\right)|K_2(\mathcal{O}_{E_{\langle g_3\rangle}})| \\ & \times \prod_{i=1,2,\cdots,q-1}|K_2(\mathcal{O}_{E_{\langle g_1 g_2^i\rangle}})|^q,\end{aligned}$$

where $Gal(E/\mathbb{Q}) = G_1$. And

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_\mathbb{Q})|^{q^2}|K_2(\mathcal{O}_{E_{\langle g_1^q\rangle}})|^{q-1} = \left(\prod_{j=1,2}|K_2(\mathcal{O}_{E_{\langle g_j\rangle}})|^q\right) \times \prod_{i=1,2,\cdots,q-1}|K_2(\mathcal{O}_{E_{\langle g_1^i g_2\rangle}})|^q,$$

where $Gal(E/\mathbb{Q}) = G_2$.

4. Applications

Let σ and τ be generators for $C_q \times C_q$. Then G_1 (or G_2) maps onto $C_q \times C_q$ by $\pi : g_1 \mapsto \sigma, g_2 \mapsto \tau$, and we can consider G_1-extension (or G_2-extension) by looking at embeddings along π.

Let M/F be a Galois extension of number fields with Galois group $C_q \times C_q$. If the primitive q-th roots of unity μ_q are contained in F^*, then by Kummer Theory, we have

$$M = F(\sqrt[q]{a}, \sqrt[q]{b}),$$

where $a, b \in F^*$ are q-independent, i.e., the classes of a and b are linearly independent in $F^*/(F^*)^q$. We pick a primitive q-th root of unity ζ, and define σ and τ in $C_q \times C_q = Gal(M/F)$ by

$$\sigma : \sqrt[q]{a} \mapsto \zeta\sqrt[q]{a}, \quad \sqrt[q]{b} \mapsto \sqrt[q]{b},$$
$$\tau : \sqrt[q]{a} \mapsto \sqrt[q]{a}, \quad \sqrt[q]{b} \mapsto \zeta\sqrt[q]{b}.$$

Lemma 2 ([6]). *Let M/F be a $C_q \times C_q$-extension as above. Then*

(1) *M/F can be embedded into a G_1-extension along π if and only if b is a norm in $F(\sqrt[q]{a})/F$. Furthermore, if $b = N_{F(\sqrt[q]{a})/F}(z)$ for some $z \in F(\sqrt[q]{a})$, the embeddings along π are $M/F \subseteq F(\sqrt[q]{r\omega}, \sqrt[q]{b})/F$ for $r \in F^*$, where $\omega = z^{q-1}\sigma z^{q-2}\cdots\sigma^{q-2}z$;*

(2) *M/F can be embedded into a G_2-extension along π if and only if $b\zeta$ is a norm in $F(\sqrt[q]{a})/F$. Furthermore, if $b\zeta = N_{F(\sqrt[q]{a})/F}(z)$ for some $z \in F(\sqrt[q]{a})$, the embeddings along π are $M/F \subseteq F(\sqrt[q]{a}, \sqrt[q]{b}, \sqrt[q]{r\sqrt[q]{a}^{-1}\omega})/F$ for $r \in F^*$, where $\omega = z^{q-1}\sigma z^{q-2}\cdots\sigma^{q-2}z$.*

Example 1. Let $F = \mathbb{Q}(\zeta_3)$. Take $a = \zeta_3$, then $\mathbb{Q}(\zeta_3, \sqrt[3]{a})$ is the ninth cyclotomic field $\mathbb{Q}(\zeta_9)$. Next we take $z = \zeta_9 + 2 \in \mathbb{Q}(\zeta_9)$, and get

$$b = z \cdot \sigma z \cdot \sigma^2 z = \zeta_3 + 8.$$

From Lemma 2 (1), we know $Gal(\mathbb{Q}(\zeta_9, \sqrt[3]{\zeta_3+8}, \sqrt[3]{\omega})/\mathbb{Q}(\zeta_3)) = G_1(3)$, where $w = z^2\sigma z = \zeta_9^6 + 4\zeta_9^5 + 4\zeta_9^4 + 2\zeta_9^2 + 8\zeta_9 + 8$, and

$$g_1 : \zeta_9 \mapsto \zeta_9^4, \quad \sqrt[3]{\zeta_3+8} \mapsto \sqrt[3]{\zeta_3+8}, \quad \sqrt[3]{\omega} \mapsto \frac{\sqrt[3]{\zeta_3+8}}{\zeta_9+2}\sqrt[3]{\omega},$$
$$g_2 : \zeta_9 \mapsto \zeta_9, \quad \sqrt[3]{\zeta_3+8} \mapsto \zeta_3\sqrt[3]{\zeta_3+8}, \quad \sqrt[3]{\omega} \mapsto \sqrt[3]{\omega},$$
$$g_3 : \zeta_9 \mapsto \zeta_9, \quad \sqrt[3]{\zeta_3+8} \mapsto \sqrt[3]{\zeta_3+8}, \quad \sqrt[3]{\omega} \mapsto \zeta_3\sqrt[3]{\omega}.$$

Let $E=\mathbb{Q}(\zeta_9, \sqrt[3]{\zeta_3+8}, \sqrt[3]{\omega})$, we have

$$\begin{aligned}
E_{\langle g_1\rangle\times\langle g_3\rangle} &= F(\sqrt[3]{b}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3+8}) := F_1,\\
E_{\langle g_2\rangle\times\langle g_3\rangle} &= F(\sqrt[3]{a}) = \mathbb{Q}(\zeta_9) := F_2,\\
E_{\langle g_1g_2\rangle\times\langle g_3\rangle} &= F(\sqrt[3]{a^2b}) = \mathbb{Q}(\zeta_3, \sqrt[3]{1+8\zeta_3^2}) := F_3,\\
E_{\langle g_1g_2^2\rangle\times\langle g_3\rangle} &= F(\sqrt[3]{ab}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3^2+8\zeta_3}) := F_4,\\
E_{\langle g_3\rangle} &= F(\sqrt[3]{a}, \sqrt[3]{b}) = \mathbb{Q}(\zeta_9, \sqrt[3]{\zeta_3+8}) := F_5.
\end{aligned}$$

$\{1, g_2, g_2^2\}$, $\{1, g_2g_3, g_2^2g_3^2\}$ and $\{1, g_2g_3^2, g_2^2g_3\}$ are conjugate subgroups, so they have isomorphic fixed fields

$$E_{\langle g_2\rangle} \cong E_{\langle g_2g_3\rangle} \cong E_{\langle g_2g_3^2\rangle} = F(\sqrt[3]{a}, \sqrt[3]{w}) = \mathbb{Q}(\zeta_9, \sqrt[3]{\omega}) := F_6.$$

From lemma 2 (1), $F(\sqrt[3]{a}, \sqrt[3]{b})/F$ can be embedded into an $G_1(3)$-extension E/F, we can get a is a norm in $F(\sqrt[3]{b})/F$, i.e., there is an element $z_1 \in F(\sqrt[3]{b})$ such that $a = N_{F(\sqrt[3]{b})/F}(z_1)$. Take $\omega_1 = z_1^2\tau z_1$, then

$$E_{\langle g_1\rangle} \cong E_{\langle g_1g_3\rangle} \cong E_{\langle g_1g_3^2\rangle} = F(\sqrt[3]{b}, \sqrt[3]{w_1}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3+8}, \sqrt[3]{\omega_1}) := F_7.$$

Similarly, $a = N_{F(\sqrt[3]{a^2b})/F}(z_2)(z_2 \in F(\sqrt[3]{a^2b}))$, $\omega_2 = z_2^2\tau z_2$,

$$E_{\langle g_1g_2\rangle} \cong E_{\langle g_1g_2g_3\rangle} \cong E_{\langle g_1g_2g_3^2\rangle} = F(\sqrt[3]{a^2b}, \sqrt[3]{w_2}) = \mathbb{Q}(\zeta_3, \sqrt[3]{1+8\zeta_3^2}, \sqrt[3]{\omega_2}) := F_8.$$

And $a = N_{F(\sqrt[3]{ab})/F}(z_3)(z_3 \in F(\sqrt[3]{ab}))$, $\omega_3 = z_3^2\tau z_3$,

$$E_{\langle g_1g_2^2\rangle} \cong E_{\langle g_1g_2^2g_3\rangle} \cong E_{\langle g_1g_2^2g_3^2\rangle} = F(\sqrt[3]{ab}, \sqrt[3]{w_3}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3^2+8\zeta_3}, \sqrt[3]{\omega_3}) := F_9.$$

It is well-known that $K_2(\mathcal{O}_{\mathbb{Q}(\zeta_3)})$ is trivial ([3] and [13]). From Theorem 2, we get, for every prime $p \neq 3$,

$$\begin{aligned}
|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_1})|^3 &=_p |K_2(\mathcal{O}_{F_7})|^3|K_2(\mathcal{O}_{F_5})|,\\
|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_2})|^3 &=_p |K_2(\mathcal{O}_{F_6})|^3|K_2(\mathcal{O}_{F_5})|,\\
|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_3})|^3 &=_p |K_2(\mathcal{O}_{F_8})|^3|K_2(\mathcal{O}_{F_5})|,\\
|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_4})|^3 &=_p |K_2(\mathcal{O}_{F_9})|^3|K_2(\mathcal{O}_{F_5})|,\\
|K_2(\mathcal{O}_{F_5})| &=_p |K_2(\mathcal{O}_{F_1})||K_2(\mathcal{O}_{F_2})||K_2(\mathcal{O}_{F_3})||K_2(\mathcal{O}_{F_4})|,
\end{aligned}$$

and

$$|K_2(\mathcal{O}_E)|^4 =_p |K_2(\mathcal{O}_{F_7})|^3|K_2(\mathcal{O}_{F_6})|^3|K_2(\mathcal{O}_{F_5})||K_2(\mathcal{O}_{F_8})|^3|K_2(\mathcal{O}_{F_9})|^3.$$

Example 2. Let $F = \mathbb{Q}(\zeta_3)$. Take $a = \zeta_3$, then $\mathbb{Q}(\zeta_3, \sqrt[3]{a}) = \mathbb{Q}(\zeta_9)$. Next we take $z = \zeta_9 + 2 \in \mathbb{Q}(\zeta_9)$, and get

$$b = \zeta_3^{-1} z\sigma z\sigma^2 z = 1 + 8\zeta_3^2.$$

From Lemma 2 (2), we know $Gal(\mathbb{Q}(\zeta_9, \sqrt[3]{1+8\zeta_3^2}, \sqrt[3]{\zeta_9^{-1}\omega})/\mathbb{Q}(\zeta_3)) = G_2(3)$, where $w = z^2\sigma z = \zeta_9^6 + 4\zeta_9^5 + 4\zeta_9^4 + 2\zeta_9^2 + 8\zeta_9 + 8$, and

$$\begin{aligned} g_1 &: \zeta_9 \mapsto \zeta_9^4, \quad \sqrt[3]{1+8\zeta_3^2} \mapsto \sqrt[3]{1+8\zeta_3^2}, \quad \sqrt[3]{\zeta_9^{-1}\omega} \mapsto \frac{\sqrt[3]{1+8\zeta_3^2}}{\zeta_9+2}\sqrt[3]{\zeta_9^{-1}\omega}, \\ g_2 &: \zeta_9 \mapsto \zeta_9, \quad \sqrt[3]{1+8\zeta_3^2} \mapsto \zeta_3\sqrt[3]{1+8\zeta_3^2}, \quad \sqrt[3]{\zeta_9^{-1}\omega} \mapsto \sqrt[3]{\zeta_9^{-1}\omega}. \end{aligned}$$

Let $E = \mathbb{Q}(\zeta_9, \sqrt[3]{1+8\zeta_3^2}, \sqrt[3]{\zeta_9^{-1}\omega})$, we have

$$\begin{aligned} E_{\langle g_1\rangle} &= F(\sqrt[3]{b}) = \mathbb{Q}(\zeta_3, \sqrt[3]{1+8\zeta_3^2}) := F_1, \\ E_{\langle g_1^3\rangle\times\langle g_2\rangle} &= F(\sqrt[3]{a}) = \mathbb{Q}(\zeta_9) := F_2, \\ E_{\langle g_1 g_2\rangle} &= F(\sqrt[3]{a^2 b}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3^2 + 8\zeta_3}) := F_3, \\ E_{\langle g_1^2 g_2\rangle} &= F(\sqrt[3]{ab}) = \mathbb{Q}(\zeta_3, \sqrt[3]{\zeta_3 + 8}) := F_4, \\ E_{\langle g_1^3\rangle} &= F(\sqrt[3]{a}, \sqrt[3]{b}) = \mathbb{Q}(\zeta_9, \sqrt[3]{1+8\zeta_3^2}) := F_5, \end{aligned}$$

and

$$E_{\langle g_2\rangle} \cong E_{\langle g_1^3 g_2\rangle} \cong E_{\langle g_1^6 g_2\rangle} = F(\zeta_9, \sqrt[3]{\zeta_9^{-1}\omega}) = \mathbb{Q}(\zeta_9, \sqrt[3]{\zeta_9^{-1}\omega}) := F_6.$$

From Theorem 3, we get, for every prime $p \neq 3$,

$$\begin{aligned} |K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_2})|^3 &=_p |K_2(\mathcal{O}_{F_6})|^3|K_2(\mathcal{O}_{F_5})|, \\ |K_2(\mathcal{O}_{F_5})| &=_p |K_2(\mathcal{O}_{F_1})||K_2(\mathcal{O}_{F_2})||K_2(\mathcal{O}_{F_3})||K_2(\mathcal{O}_{F_4})|, \end{aligned}$$

and

$$|K_2(\mathcal{O}_E)||K_2(\mathcal{O}_{F_5})|^2 =_p |K_2(\mathcal{O}_{F_1})|^3|K_2(\mathcal{O}_{F_6})|^3|K_2(\mathcal{O}_{F_3})|^3|K_2(\mathcal{O}_{F_4})|^3.$$

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