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SPECTRAL APPROXIMATIONS OF UNBOUNDED OPERATORS OF THE TYPE "NORMAL PLUS COMPACT"

Michael Gil'

Abstract: Let B be a compact operator in a Hilbert space H and S an unbounded normal one in H, having a compact resolvent. We consider operators of the form $A = S + B$. Numerous integro-differential operators A can be represented in this form. The paper deals with approximations of the eigenvalues of the considered operators by the eigenvalues of the operators $A_n = S + B_n$ ($n = 1, 2, \ldots$), where B_n are n-dimensional operators. Besides, we obtain the error estimate of the approximation.

Keywords: Hilbert space, linear operators, eigenvalues, approximation, integro-differential operators, Schatten-von Neumann operators.

1. Introduction and statement of the main result

Let H be a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and the unit operator I. Let S be a normal operator in H, having the compact resolvent, and B be a compact operator in H. Our main object in this paper is the operator

$$A = S + B. \tag{1.1}$$

Numerous integro-differential operators A can be represented in the form (1.1), cf. [1, 3, 6]. This paper deals with the spectral approximation of operator A. The literature devoted to approximations of the eigenvalues of various concrete operators is very rich, cf. the interesting papers [2, 4, 5, 10, 15] and references given therein. At the same time, to the best of our knowledge, the spectrum approximations of the operators of the form (1.1) were not investigated in the available literature.

Introduce the notations. For a linear unbounded operator A in H, $Dom(A)$ is the domain, A^* is the adjoint of A; $\sigma(A)$ denotes the spectrum of A and A^{-1} is the inverse to A, $R_\lambda(A) = (A - I\lambda)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent; $\lambda_k(A)$ are the eigenvalues of A taken with their multiplicities. $\rho(A, \lambda) = \inf_{s \in \sigma(A)} |\lambda - s|$ is the

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distance between $\lambda \in \mathbb{C}$ and $\sigma(A)$. If A is bounded, then $\|A\|$ means its operator norm. For an integer $p \geqslant 1$, SN_p is the Schatten-von Neumann ideal of compact operators K in H with the finite norm $N_p(K) = [\mathrm{Trace}(KK^*)^{p/2}]^{1/p}$.

Let $\{e_k\}_{k=1}^{\infty}$ be the normalized eigenvectors of S, and B be represented in the basis $\{e_k\}_{k=1}^{\infty}$ by a matrix $(b_{jk})_{j,k=1}^{\infty}$. So A is represented by the matrix (a_{jk}) with $a_{jj} = \lambda_j(S) + b_{jj}$ and $a_{jk} = b_{jk}$ $(j \neq k)$.

For an integer $n < \infty$, put $\hat{b}_{jk}^{(n)} = b_{jk}$ if $1 \leqslant j,k \leqslant n$ and $\hat{b}_{jk}^{(n)} = 0$ overwise. Denote by B_n the operator represented in the basis $\{e_k\}_{k=1}^{\infty}$ by matrix $(b_{jk}^{(n)})_{j,k=1}^{\infty}$. So B_n has the range no more than n. We will approximate the spectrum of A by the spectrum of the operators $A_n = S + B_n$ $(n = 1, 2, \ldots)$. So $A_n = S_n \oplus C_n$, where

$$C_n = (b_{jk})_{j,k=1}^{n} + \mathrm{diag}(\lambda_k(S))_{k=1}^{n} \qquad \text{and} \qquad S_n = \mathrm{diag}(\lambda_k(S))_{k=n+1}^{\infty}. \tag{1.2}$$

Consequently, C_n has in the basis $\{e_k\}_{k=1}^{n}$ the entries $c_{jj} = \lambda_j(S) + b_{jj}$ and $c_{jk} = b_{jk}$ $(j \neq k; 1 \leqslant j,k \leqslant n)$.

Note that the resolvent

$$(A-\lambda I)^{-1} = (S+B-\lambda I)^{-1} = (S-I\tau)^{-1}(I+(B+\tau-\lambda I)(S-I\tau)^{-1})^{-1} \quad (\tau \notin \sigma(S))$$

is compact for any regular λ of A, and therefore, the spectrum of A is discrete. Since B is compact we have

$$q_n := \|A_n - A\| = \|B_n - B\| \to 0 \qquad \text{as} \ \ n \to \infty.$$

To formulate the result, denote

$$g(C_n) = [N_2^2(C_n) - \sum_{k=1}^{n} |\lambda_k(C_n)|^2 \,]^{1/2}.$$

The following relations are checked in [7, Section 2.1].

$$g^2(C_n) \leqslant N_2^2(C_n) - |\,\mathrm{Trace}\, C_n^2|$$

and

$$g^2(C_n) \leqslant \frac{N_2^2(C_n - C_n^*)}{2} = 2N_2^2(C_{nI}), \tag{1.3}$$

where $C_{nI} = (C_n - C_n^*)/2i$.

If C_n is a normal matrix: $C_n C_n^* = C_n^* C_n$, then $g(C_n) = 0$.

Denote by $r(q_n)$ the unique positive root of the algebraic equation

$$z^n = q_n \sum_{j=0}^{n-1} \frac{z^{n-j-1} g^j(C_n)}{\sqrt{j!}}. \tag{1.4}$$

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let A be defined by (1.1), C_n and S_n be defined by (1.2). Then for any eigenvalue $\mu(A)$ of A and a natural n, either there is an eigenvalue $\lambda(C_n)$ of the $n \times n$-matrix C_n, such that $|\mu(A) - \lambda(C_n)| \leqslant r(q_n)$, or $|\mu(A) - \lambda_j(S)| \leqslant r(q_n)$ for some $j > n$. Moreover, if*

$$S = S^* \tag{1.5}$$

and

$$B - B^* \in SN_2, \tag{1.6}$$

then $r(q_n) \to 0$ as $n \to \infty$.

This theorem is proved in the next section. In addition, in Section 3, we suggest another error estimate, which tends to zero, provided $B - B^* \in SN_{2p}, p \geqslant 1$. That estimate is generally rougher than that in Theorem 1.1.

Note that approximations of unbounded self-adjoint Jacobi matrices acting in l^2 by the use of finite submatrices were investigated in the very interesting paper [11] (see also the papers [12, 13]).

Put

$$P_n(x) = \sum_{j=0}^{n-1} \frac{g^j(C_n)x^{j+1}}{\sqrt{j!}} \qquad (x \geqslant 0).$$

Thanks to [7, Lemma 1.6.1] $r(q_n) \leqslant \zeta(q_n)$, where

$$\zeta(q_n) = \begin{cases} \sqrt[n]{q_n P_n(1)} & \text{if } q_n P_n(1) \leqslant 1, \\ q_n P_n(1) & \text{if } q_n P_n(1) \geqslant 1. \end{cases}$$

Thus in Theorem 1.1 one can replace $r(q_n)$ by $\zeta(q_n)$.

2. Proof of Theorem 1.1

First, let us prove that $r(q_n) \to 0$, provided (1.5) and (1.6) hold. To this end note that $N_2(C_n - C_n^*) = N_2(B_n - B_n^*)$. Thus by (1.3) we obtain

$$g(C_n) \leqslant \sqrt{1/2} N_2(B_n - B_n^*) \leqslant \sqrt{1/2} N_2(B - B^*).$$

Rewrite (1.4) as

$$1 = q_n \sum_{j=0}^{n-1} \frac{g^j(C_n)}{z^{j+1}\sqrt{j!}}.$$

Hence, it follows that

$$1 \leqslant q_n \sum_{j=0}^{\infty} \frac{\sqrt{1/2} N_2(B - B^*)}{r^{j+1}(q_n)\sqrt{j!}}.$$

Since $q_n \to 0$, we have $r(q_n) \to 0$.

Furthermore, put $Q_n = \sum_{k=1}^n (., e_k)e_k$. Then $C_n = Q_nAQ_n$ and $S_n = (I - Q_n)S = S(I - Q_n)$. Clearly, $S_nC_n = C_nS_n = 0$ and

$$\sigma(A_n) = \sigma(C_n) \cup \{\lambda_k(S)\}_{k=n+1}^\infty. \tag{2.1}$$

Thus

$$\|R_\lambda(A_n)\| = \max\{\|Q_nR_\lambda(C_n)\|, \ \|(I - Q_n)R_\lambda(S_n)\|\}. \tag{2.2}$$

Thanks to Corollary 2.1.2 [7] we have

$$\|Q_nR_\lambda(C_n)\| \leqslant \sum_{k=0}^{n-1} \frac{g^k(C_n)}{\sqrt{k!}\rho^{k+1}(C_n, \lambda)} \qquad \text{for any regular point } \lambda \text{ of } C_n. \tag{2.3}$$

Rewrite (2.3) as

$$\|Q_nR_\lambda(C_n)\| \leqslant P_n(1/\rho(C_n, \lambda)). \tag{2.4}$$

Now (2.4) and (2.2) imply the inequality

$$\|R_\lambda(A_n)\| \leqslant \max\{P_n(1/\rho(C_n, \lambda), 1/\rho(S_n, \lambda)\}. \tag{2.5}$$

But due to (2.1) $\rho(C_n, \lambda) \geqslant \rho(A_n, \lambda)$ and $\rho(S_n, \lambda) \geqslant \rho(A_n, \lambda)$. In addition, $P_n(x) \geqslant x$ for $x \geqslant 0$. Thus

$$\|R_\lambda(A_n)\| \leqslant P_n(1/\rho(A_n, \lambda)). \tag{2.6}$$

Furthermore, for two operators A and $\tilde{A}$, the spectral variation $sv_A(\tilde{A})$ of $\tilde{A}$ with respect to A is defined by

$$sv_A(\tilde{A}) := \sup_{\mu \in \sigma(\tilde{A})} \inf_{\lambda \in \sigma(A)} |\lambda - \mu|.$$

Lemma 2.1. *Let Dom* $(A) =$ *Dom* $(\tilde{A})$ *and* $\hat{q} := \|A - \tilde{A}\| < \infty$. *In addition, assume that*

$$\|R_\lambda(A)\| \leqslant \phi(1/\rho(A, \lambda)) \qquad \textit{for all regular } \lambda \textit{ of } A,$$

where $\phi(x)$ *is a monotonically increasing non-negative continuous function of a non-negative variable* x, *such that* $\phi(0) = 0$ *and* $\phi(\infty) = \infty$. *Then the inequality*

$$sv_A(\tilde{A}) \leqslant z(\phi, \hat{q})$$

is true, where $z(\phi, \hat{q})$ *is the a unique positive root of the equation*

$$1 = \hat{q}\phi(1/z). \tag{2.7}$$

Proof. For a $\lambda \in \mathbb{C}$, let $\hat{q}\phi(1/\rho(A, \lambda)) < 1$. Then $\hat{q}\|R_\lambda(A)\| < 1$. But due to the Hilbert identity

$$R_\lambda(\tilde{A}) - R_\lambda(A) = R_\lambda(\tilde{A})(A - \tilde{A})R_\lambda(A)$$

the latter inequality implies that $\lambda \notin \sigma(\tilde{A})$. So for all $\mu \in \sigma(\tilde{A})$ we have.

$$\hat{q}\phi(1/\rho(A, \mu)) \geqslant 1.$$

Since $\phi(x)$ monotonically increases, we have $\rho(A, \mu) \leqslant z(\phi, \hat{q})$. This, proves the required inequality. ■

The previous lemma and (2.6) imply

$$sv_{A_n}(A) \leqslant r(q_n). \tag{2.8}$$

According to (2.1) this proves the theorem.

3. The case $B - B^* \in SN_{2p}$ $(p > 1)$

Assume that

$$B - B^* \in SN_{2p} \qquad (p = 2, 3, \ldots). \tag{3.1}$$

Let c_m $(m = 1, 2, \ldots)$ be a sequence of positive numbers defined by by the recursive relation

$$c_1 = 1, \qquad c_m = c_{m-1} + \sqrt{c_{m-1}^2 + 1} \qquad (m = 2, 3, \ldots).$$

To formulate the result, for a $p \in [2^m, 2^{m+1}]$ $(m = 1, 2, \ldots)$, put

$$b_p = c_m^t c_{m+1}^{1-t} \qquad \text{with} \;\; t = 2 - 2^{-m} p.$$

The following inequality is valid:

$$b_p \leqslant \frac{pe^{1/3}}{2} \leqslant p \qquad (p \geqslant 2),$$

cf. [8, Corollary 1.3].

For instance,

$$b_2 = c_1 = 1, \qquad b_3 = \sqrt{c_1 c_2} = \sqrt{1 + \sqrt{2}} \leqslant 1.554, \qquad b_4 = c_2 \leqslant 2.415,$$
$$b_5 = c_2^{3/4} c_3^{1/4} \leqslant 2.900; \qquad b_6 = (c_2 c_3)^{1/2} \leqslant 3.485;$$
$$b_7 = c_2^{1/4} c_3^{1/4} \leqslant 4.185 \qquad \text{and} \qquad b_8 = c_3 \leqslant 5.027.$$

Put $\beta_p = 2(1 + b_{2p})$ and take $n = jp$ for an integer $j \geqslant 2$. Denote by $\hat{r}_p(q_n)$ the unique positive root of the algebraic equation

$$z^n = q_n \sum_{m=0}^{p-1} \sum_{k=0}^{j-1} z^{n-pk-m-1} \frac{N_{2p}^{kp+m}(\beta_p C_{nI})}{\sqrt{k!}}. \tag{3.2}$$

Recall that $C_{nI} = (C_n - C_n^*)/2i$.

Theorem 3.1. *Let conditions (1.5) and (3.1) hold. Then for any $\mu \in \sigma(A)$ and an $n = pj$ with an integer $j \geqslant 2$, either there is an eigenvalue $\lambda(C_n)$ of matrix C_n satisfying $|\mu - \lambda(C_n)| \leqslant \hat{r}_p(q_n)$, or $|\mu - a_{jj}| \leqslant \hat{r}_p(q_n)$ for some $j > n$. Moreover, $\hat{r}_p(q_n) \to 0$ as $n \to \infty$.*

To prove this theorem we need the following result.

Lemma 3.2. *Let T be a linear operator acting in a Euclidean space $\mathbb{C}^n$ with $n = jp$ and integers $p \geqslant 2, j \geqslant 2$. Then*

$$\|R_\lambda(T)\| \leqslant \sum_{m=0}^{p-1}\sum_{k=0}^{j-1} \frac{N_{2p}^{kp+m}(\beta_p T_I)}{\rho^{pk+m+1}(T,\lambda)\sqrt{k!}} \qquad (\lambda \notin \sigma(T)), \tag{3.3}$$

where $T_I = (T - T^*)/2i$.

Proof. Due to the Schur theorem, cf. [14], $T = D + V$ $(\sigma(T) = \sigma(D))$, where D is a normal matrix and V is a nilpotent matrix. Besides, D and V have the same invariant subspaces, and V is called the nilpotent part of T. Thanks to [7, Lemma 6.8.3],

$$\|R_\lambda(T)\| \leqslant \sum_{m=0}^{p-1}\sum_{k=0}^{j-1} \frac{N_{2p}^{kp+m}(V)}{\rho^{pk+m+1}(T,\lambda)\sqrt{k!}} \qquad (\lambda \notin \sigma(T)), \tag{3.4}$$

where V is the nilpotent part of T. Making use Lemma 2.2 from [8], we get the inequality

$$N_{2p}(V) \leqslant (1 + b_{2p})N_{2p}(V_I) \qquad (1 \leqslant p < \infty;\ V_I = (V - V^*)/2i)$$

But by the Weyl inequality we have $N_{2p}(D_I) \leqslant N_{2p}(T_I)$ with $(D_I = (D - D^*)/2i)$. Hence we obtain

$$N_{2p}(V) \leqslant (1 + b_{2p})N_{2p}(V_I) = (1 + b_{2p})N_{2p}(T_I - D_I) \leqslant 2(1 + b_{2p})N_{2p}(T_I).$$

Thus $N_{2p}(V) \leqslant \beta_p N_{2p}(T_I)$. This and (3.4) proves the lemma. ■

Proof of Theorem 3.1. First let us prove that $\hat{r}_p(q_n) \to 0$ as $n \to \infty$, provided (1.5) and (3.1) hold. To this end rewrite (3.3) as

$$1 = q_n \sum_{m=0}^{p-1}\sum_{k=0}^{j-1} \frac{N_{2p}^{kp+m}(\beta_p C_{nI})}{z^{pk+m+1}\sqrt{k!}}.$$

Hence

$$1 \leqslant q_n \sum_{m=0}^{p-1}\sum_{k=0}^{\infty} \frac{N_{2p}^{kp+m}(\beta_p C_{nI})}{r_p^{pk+m+1}(q_n)\sqrt{k!}}.$$

Since $q_n \to 0$, we have $\hat{r}_p(q_n) \to 0$.

Furthermore, due to Lemma 3.2,

$$\|Q_n R_\lambda(C_n)\| \leqslant \hat{P}_{n,p}(1/\rho(C_n,\lambda)), \tag{3.5}$$

where

$$\hat{P}_{n,p}(x) = \sum_{m=0}^{p-1}\sum_{k=0}^{j-1} x^{pk+m+1} \frac{N_{2p}^{kp+m}(\beta_p C_{nI})}{\sqrt{k!}} \qquad (x \geqslant 0).$$

Now (3.5) and (2.2) imply the inequality

$$\|R_\lambda(A_n)\| \leqslant \max\{\hat{P}_{n,p}(1/\rho(C_n,\lambda)), 1/\rho(S_n,\lambda)\}.$$

But due to (2.1) , we have $\rho(C_n,\lambda) \geqslant \rho(A_n,\lambda)$ and $\rho(S_n,\lambda) \geqslant \rho(A_n,\lambda)$. In addition, $\hat{P}_{n,p}(x) \geqslant x$ for $x \geqslant 0$. Thus,

$$\|R_\lambda(A_n)\| \leqslant \hat{P}_{n,p}(1/\rho(A_n,\lambda)). \tag{3.6}$$

Due to Lemma 2.1, the inequality $sv_{A_n}(A) \leqslant \hat{z}_p(q_n)$ holds. According to (2.1) this proves the theorem. ■

Furthermore, again use [7, Lemma 1.6.1], we obtain $\hat{r}_p(q_n) \leqslant \hat{\zeta}_p(q_n)$, where

$$\hat{\zeta}_p(q_n) = \begin{cases} \sqrt[n]{q_n\hat{P}_{n,p}(1)} & \text{if } q_n\hat{P}_{n,p}(1) \leqslant 1, \\ q_n\hat{P}_{n,p}(1) & \text{if } q_n\hat{P}_{n,p}(1) \geqslant 1. \end{cases}$$

Thus in Theorem 3.1 one can replace $\hat{r}_p(q_n)$ by $\hat{\zeta}_p(q_n)$.

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Address: Michael Gil': Department of Mathematics, Ben Gurion University of the Negev, P.0. Box 653, Beer-Sheva 84105, Israel.

E-mail: gilmi@bezeqint.net

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