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BANACH LATTICE SUMS WITH HEREDITARY FATOU PROPERTIES

Mieczysław Mastyło, Enrique A. Sánchez-Pérez

Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: We present a characterization of Banach function lattices with the Fatou property generated by the interpolation sum applied to infinite families of Banach lattices with the Fatou property. We also discuss the Köthe duality between the sum and the intersection constructions for Banach spaces.

Keywords: Banach lattice, vector measure, Fatou property, Köthe dual space.

1. Introduction

In their seminal paper [1], Aronszajn and Gagliardo initiated the abstract theory of interpolation spaces. In this work they defined the so called minimal and maximal interpolation methods, which play a fundamental role in this theory. These methods are generated by two abstract constructions, the sum Σ and the intersection Δ, applied to compatible families of Banach spaces. Later Janson [3] (see also [2]) discovered that under mild assumptions the mentioned minimal and maximal methods are dual to each other in interpolation sense.

Although the abstract constructions can be given for general Banach spaces they are in fact relevant tools in the setting of the interpolation of Banach spaces when the spaces involved are Banach function lattices. The lattice properties of the resulting space are interesting from the point of view of applications. In particular, the Fatou property of the sum becomes fundamental for describing the duality relations between the sum and the intersection of a family of Banach function lattices (see [9]).

The primary purpose of this paper is to characterize the families of Banach function lattices $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ on a measure space for which the sums $\Sigma(X_\alpha)$ have

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the Fatou property. We point out however that we are not interested in showing concrete examples or applications but in a general description in terms of the lattice properties of the spaces involved that would be interesting by itself. This is the main motivation of the present paper, in which we show that the sum of Banach function lattices with the Fatou property has not in general the Fatou property, and this holds only when some families of representations of the functions in the space have nice order properties.

Our notation is standard. We will work with classes of Banach function lattices over the same σ-finite measure space $(\Omega, \mathcal{S}, \mu)$. As usual we denote by $L^0(\mu)$ the space of (equivalence classes of μ-a.e. equal) real valued measurable functions on Ω. We say that $(X, \|\cdot\|_X)$ is a *Banach function lattice* on $(\Omega, \mathcal{S}, \mu)$ if X is an ideal in $L^0(\mu)$ and whenever $f, g \in X$ and $|f| \leqslant |g|$ a.e., then $\|f\|_X \leqslant \|g\|_X$. For the aim of clarity, we will assume throughout the paper that the support of all the Banach function lattices X over $(\Omega, \mathcal{S}, \mu)$ is Ω (i.e., there exists $h \in X$ such that $h > 0$ a.e.).

Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$; X is said to have the *Fatou property* if for every sequence (f_n) in X and $f \in L^0(\mu)$ satisfying $0 \leqslant f_n \uparrow f$ a.e. and $\sup_{n\geqslant 1} \|f_n\|_X < \infty$ we have $f \in X$ and $\|f_n\|_X \to \|f\|_X$. Notice that Fatou property is equivalent to the statement that the unit ball B_X of X is closed in $L^0(\mu)$. Banach lattices with the Fatou property are also called maximal. We say that X is order continuous if for every sequence (f_n) such that $f_n \downarrow 0$ we have $\|f_n\|_X \to 0$.

The Köthe dual space X' of X is defined to be the space of all $f \in L^0(\mu)$ such that the *associate norm*

$$\|f\|_{X'} := \sup\Big\{ \int_\Omega |fg|\, d\mu;\ \|g\|_X \leqslant 1\Big\}$$

is finite. The Köthe dual $X' = (X', \|\cdot\|_{X'})$ is a Banach lattice on $(\Omega, \mathcal{S}, \mu)$. We set $X'' := (X')'$. Clearly $X \subset X''$ with $\|f\|_{X''} \leqslant \|f\|$. If $\|f\|_{X''} = \|f\|_X$ for all $f \in X$, then the norm of X is called *order semicontinuous.* A Banach lattice X on $(\Omega, \mathcal{S}, \mu)$ has an order semicontinuous norm if and only if X has the *weak Fatou property,* i.e. if for every sequence $(f_n)_n$ in X and $f \in X$ satisfying $0 \leqslant f_n \uparrow f$ a.e. and $\sup_{n\geqslant 1} \|f_n\|_X < \infty$ we have $\|f_n\|_X \to \|f\|_X$. It is well known that $X'' = X$ with equality of norms if and only if X has the Fatou property. Notice also that if X is order continuous, then the Banach dual X^* can be identified with the Köthe dual X'. For more information on Banach lattices we refer to [5, 7].

2. Main results

Let E, F be Banach spaces. If E and F are isomorphic Banach spaces, then we write $E \simeq F$. We write $E \overset{c}{\hookrightarrow} F$ if E is a subspace of F and the inclusion map $I\colon E \to F$ is continuous with $\|I\| \leqslant c$, and $E \hookrightarrow F$ if this happens and $c = 1$. $E \cong F$ will mean that $E \hookrightarrow F$ and $F \hookrightarrow E$.

In the present section we discuss the characterization of the maximal Banach lattices generated by Σ-construction applied to a family of Banach lattices on a given measure space. Let us start our discussion with the following fundamental definitions (see, e.g., [6, pp. 16-18]). Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of Banach spaces. It is said to be *compatible* if there exists a Hausdorff topological vector space $\mathcal{X}$ such that X_α is $\mathcal{X}$ algebraically and topologically embedded in $\mathcal{X}$. A family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is called *strongly compatible* if there exists a Banach space Y such that $X_\alpha \overset{c_\alpha}{\hookrightarrow} Y$ with $\sup_{\alpha\in\mathcal{A}} c_\alpha < \infty$.

Given a compatible family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ of Banach spaces we let

$$\Delta(X_\alpha) = \Big\{x \in \bigcap_{\alpha\in\mathcal{A}} X_\alpha;\ \|x\|_{\Delta(X_\alpha)} := \sup_{\alpha\in\mathcal{A}} \|x\|_{X_\alpha} < \infty\Big\}.$$

Then $(\Delta(X_\alpha), \|\cdot\|_{\Delta(X_\alpha)})$ is a Banach space with the following properties:

(i) $\Delta(X_\alpha) \hookrightarrow X_\alpha$ for every $\alpha \in \mathcal{A}$.

(ii) If F is a Banach space such that $F \overset{c}{\hookrightarrow} X_\alpha$, for every $\alpha \in \mathcal{A}$, then $F \overset{c}{\hookrightarrow} \Delta(X_\alpha)$.

If $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a strongly compatible family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ of Banach spaces, we define $\Sigma(X_\alpha)$ to be the Banach space of all $x \in Y$ that are representable in the form

$$x = \sum_{\alpha\in\mathcal{A}} x_\alpha \quad (x_\alpha \in X_\alpha) \qquad \text{where } \sum_{\alpha} \|x_\alpha\|_{X_\alpha} < \infty \tag{$*$}$$

equipped with the norm

$$\|x\|_{\Sigma(X_\alpha)} = \inf\Big\{\sum_{\alpha\in\mathcal{A}} \|x_\alpha\|_{X_\alpha};\ x = \sum_{\alpha\in\mathcal{A}} x_\alpha\Big\},$$

where the infimum is taken over all possible representations of x in the form $(*)$.

Notice that $\Sigma(X_\alpha)$ is the smallest Banach space with the property $X_\beta \hookrightarrow \Sigma(X_\alpha)$ for every $\beta \in \mathcal{A}$. We note that the constructions Δ and Σ play a fundamental role in abstract theory of interpolation (for details we refer to [1, 2, 6]).

In this paper we consider the Δ and Σ constructions for Banach function lattices over the same measure space and so we can assume that the family of spaces is compatible, since all its elements are continuously included into the space $\mathcal{X} = L^0(\mu)$.

As we have mentioned we are interested in finding a description of the families of Banach lattices $\{X_\alpha\}$ such that $\Sigma(X_\alpha)$ satisfies the Fatou property or the weak Fatou property. It should be pointed out here that even in the case of countable families $\{X_\alpha\}$ of reflexive (and so maximal) Banach lattices, $\Sigma(X_\alpha)$ is not maximal in general. We provide an example.

Example 2.1. We construct a space X_n for each $n \in \mathbb{N}$ by renorming ℓ_p $(1 \leqslant p < \infty)$ as follows:

$$\|x\|_{X_n} := \max_{1\leqslant i\leqslant n} |x_i| + \Big(\sum_{i=n+1}^{\infty} |x_i|^p\Big)^{1/p}, \qquad x = (x_i) \in \ell_p.$$

Then $X_n \simeq \ell_p$, by

$$n^{-1/p}\|x\|_{\ell_p} \leqslant 2^{1-1/p}\|x\|_{X_n} \leqslant \|x\|_{\ell_p}.$$

It is straightforward to see that X_n is a maximal Banach lattice and $X_n \hookrightarrow c_0$ for each n.

We claim that $\Sigma(X_n) \cong c_0$. To see this, let $(x_i)_i$ be a norm one element in c_0 and $\varepsilon > 0$. Consider an increasing sequence (i_j) such that $|x_k| \leqslant \varepsilon/2^j$ for all $k \geqslant i_j$. Take a decomposition of $(x_i)_i$ as follows,

$$(x_i)_i = \sum_{j=1}^{\infty} y_j,$$

where $y_j = \sum_{i=i_{j-1}+1}^{i_j} x_i e_i$ for each $j \in \mathbb{N}$ with $i_0 = 0$, and (e_i) is the canonical basis of ℓ^p. Then we have

$$\sum_{j=1}^{\infty} \|y_j\|_{X_{i_j}} \leqslant 1 + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = 1 + \varepsilon.$$

Since ε is arbitrary, $c_0 \hookrightarrow \Sigma(X_n)$. Clearly $\Sigma(X_n) \hookrightarrow c_0$ by $X_n \hookrightarrow c_0$ for each $n \in \mathbb{N}$ and so the claim is proved. To conclude we note that c_0 is not maximal.

Before we state and prove our main result, we introduce some specific definitions that we present below.

- Let $(\Omega, \mathcal{S}, \mu)$ be a measure space. We say that a sequence $(\varphi_n)_n$ of positive functions in $L^\infty(\mu)$ is a *partition of unity* if $\sum_{n=1}^{\infty} \varphi_n = 1$ a.e..
- If $0 \leqslant f \in \Sigma(X_m)$, a sequence $(\phi_m)_m$ is an *ε-representation* of f if $\phi_m \in X_m$ for each $m \in \mathbb{N}$, $f = \sum_{m=1}^{\infty} \phi_m$ a.e. and

$$\sum_{m=1}^{\infty} \|\phi_m\|_{X_m} < \|f\| + \varepsilon.$$

- Let $x^k = (x_{j,k})_j$ be a bounded real sequence for each $k \in \mathbb{N}$. A sequence $(x^k)_k$ is said to be *uniformly convergent* to $(y_k)_k$ if every $\varepsilon > 0$ there exist M and N such that $|x_{j,k} - y_k| < \varepsilon$ for each $j > M$, $k > N$.
- Let $\phi^k = (\phi_{j,k})_j$ be a sequence in $L^0(\Omega, \mathcal{S}, \mu)$ for each $k \in \mathbb{N}$. A sequence $(\phi^k)_k$ is said to be *uniformly convergent* to $(\psi_k)_k$ a.e. if there exists a μ-null set A such that for every $\varepsilon > 0$ there exist M, N such that $|\phi_{j,k}(\omega) - \psi_k(\omega)| < \varepsilon$ for every $\omega \in \Omega \setminus A$ and each $j > M$, $k > N$.

Before proceeding, we need a technical lemma.

Lemma 2.1. *Suppose that for each $m \in \mathbb{N}$, $(a_{m,k})_k$ is a sequence of nonnegative real numbers so that for each $k \in \mathbb{N}$ all the series $\sum_{m=1}^{\infty} a_{m,k}$ converge uniformly to their limits, and $\lim_{k\to\infty} \sum_{m=1}^{\infty} a_{m,k}$ exists. If for all $m \in \mathbb{N}$ and $p \in \mathbb{N}$ there exists $\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} a_{m,j+p}$ which does not depend on p, then*

$$\sum_{m=1}^{\infty} \lim_{k\to\infty} \Big(\frac{1}{k} \sum_{j=1}^{k} a_{m,j}\Big) = \lim_{k\to\infty} \sum_{m=1}^{\infty} a_{m,k}.$$

Proof. Fix $\varepsilon > 0$ and put $A := \lim_{k\to\infty} \sum_{m=1}^{\infty} a_{m,k}$. Our hypothesis on uniform convergence implies that there exist $p \in \mathbb{N}$ and a positive integer M such that

$$\sum_{m=1}^{M} a_{m,k+p} > A - \varepsilon, \qquad k \in \mathbb{N}.$$

This yields

$$\begin{aligned}\sum_{m=1}^{\infty} \lim_{k\to\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j}\Big) &= \sum_{m=1}^{\infty} \lim_{k\to\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j+p}\Big) \geqslant \sum_{m=1}^{M} \lim_{k\to\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j+p}\Big)\\ &= \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k}\Big(\sum_{m=1}^{M} a_{m,j+p}\Big) \geqslant A - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\sum_{m=1}^{\infty} \lim_{k\to\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j}\Big) \geqslant A.$$

Now by the Fatou lemma we have

$$\begin{aligned}\sum_{m=1}^{\infty} \lim_{k\to\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j}\Big) &\leqslant \liminf_{k\to\infty} \sum_{m=1}^{\infty} \Big(\frac{1}{k}\sum_{j=1}^{k} a_{m,j}\Big) = \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k}\Big(\sum_{m=1}^{\infty} a_{m,j}\Big)\\ &= \lim_{k\to\infty} \sum_{m=1}^{\infty} a_{m,k} = A,\end{aligned}$$

and this gives the required equality. ■

We also will need a "matrix form" variant of the well known Komlós Theorem that can be found in [4, Lemma 4.1].

Lemma 2.2. *Suppose that for each $m \in \mathbb{N}$, $(\phi_{m,n})_n$ is a sequence of nonnegative functions in $L^0(\mu)$ so that $\operatorname{conv}\{\phi_{m,n} : n \in \mathbb{N}\}$ is bounded. Then there is a sequence $(\phi_m)_m$ in $L^0(\mu)$ and a strictly increasing sequence of natural numbers $(n_k)_k$ such that for each $m \in \mathbb{N}$, and for every subsequence $(f_{m,r})_r$ of $(\phi_{m,n_k})_k$, and a.e.,*

$$\lim_{r\to\infty} \frac{1}{r}\big(f_{m,1} + \cdots + f_{m,r}\big) = \phi_m.$$

As we show in the result stated below, uniform convergence of the sequences of representations of norm bounded nonnegative and nondecreasing sequences $(f_n)_n$ of $\Sigma(X_m)$ when evaluated on almost every $\omega \in \Omega$ is the keystone for proving the Fatou property of $\Sigma(X_\alpha)$. Roughly speaking this is equivalent of having uniformly norm bounded nondecreasing representations of the elements of the sequence $(f_n)_n$.

In order to clarify the statement of the result, we say that a norm bounded nonnegative and nondecreasing sequence $(f_n)_n$ in $\Sigma(X_m)$ has *uniformly bounded ordered representations* if for every $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ and ε-representations $(\phi^\varepsilon_{m,n})_m$ of $(f_n)_{n \geqslant n_\varepsilon}$ such that for all $n \geqslant n_\varepsilon$, $(\phi^\varepsilon_{m,n})_m$ is nondecreasing.

Now we are ready to state our main result.

Theorem 2.1. *Let $\{X_m\}_{m\in\mathbb{N}}$ be a strongly compatible family of maximal Banach lattices on a measure space $(\Omega, \mathcal{S}, \mu)$. Then the following statements are equivalent.*

(i) *$\Sigma(X_m)$ is a maximal Banach lattice.*

(ii) *For every norm bounded nonnegative and nondecreasing sequence $(f_n)_n$ in $\Sigma(X_m)$ and every $\varepsilon > 0$, there exist a positive integer n_ε and ε-representations $(\phi_{m,n})_m$ for all f_n, $n \geqslant n_\varepsilon$, that are uniformly convergent to $(f_n)_n$ a.e., i.e., that the sequence of representations satisfy the condition*

$$\lim_{M,\, n_0 \to \infty} \Big(\sup_{n > n_0} \Big(\sum_{m=1}^{M} |\phi_{m,n} - f_n| \Big) = 0, \qquad a.e.$$

(iii) *Every nonnegative and nondecreasing sequence $(f_n)_n$ in the unit ball of $\Sigma(X_m)$ has uniformly bounded ordered representations.*

(iv) *For every nonnegative and nondecreasing sequence $(f_n)_n$ in the unit ball of $\Sigma(X_m)$ and every $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ and a partition of unity $(\varphi_m)_m$ such that $(\varphi_m f_n)_m$ is an ε-representation of f_n for all $n \geqslant n_\varepsilon$.*

Proof. (ii) $\Rightarrow$ (i). Put $\Sigma := \Sigma(X_m)$. Let $(f_n)_n$ be a nondecreasing sequence of nonnegative functions in the unit ball B_Σ. We need to show that $f := \sup_{n\in\mathbb{N}} f_n \in B_\Sigma$. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a sequence $(\phi_{m,n})_m$ such that $\phi_{m,n} \in X_m$ for each m,

$$f_n = \sum_{m=1}^{\infty} \phi_{m,n}, \qquad \text{a.e.}$$

and

$$\sum_{m=1}^{\infty} \|\phi_{m,n}\|_{X_m} \leqslant 1 + \varepsilon. \tag{$*$}$$

In particular we have that $\operatorname{conv}\{\phi_{m,n};\, n \in \mathbb{N}\}$ is bounded in Σ and so also in $L^0(\mu)$. By Lemma 2.2 it follows that there is a sequence $(\phi_m)_m$ in $L^0(\mu)$ and a strictly increasing sequence of natural numbers $(n_k)_k$ such that for each $m \in \mathbb{N}$, and for every subsequence $(f_{m,r})$ of $(\phi_{m,n_k})_k$, $\lim_{r\to\infty} \frac{1}{r}\big(f_{m,1} + \cdots + f_{m,r}\big)$ converges a.e. to ϕ_m for each $m \in \mathbb{N}$. In particular for each $m \in \mathbb{N}$,

$$\phi_m = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \phi_{m,n_j}, \qquad \text{a.e.}$$

Since X_m is maximal, $\phi_m \in X_m$ for each $m \in \mathbb{N}$ and

$$\|\phi_m\|_{X_m} \leqslant \liminf_{k\to\infty} \Big\| \frac{1}{k} \sum_{j=1}^{k} \phi_{m,n_j} \Big\|_{X_m} \leqslant \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \|\phi_{m,n_j}\|_{X_m}.$$

Combining above inequalities with $(*)$ yields

$$\sum_{m=1}^{\infty} \|\phi_m\|_{X_m} \leqslant \sum_{m=1}^{\infty} \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \|\phi_{m,n_j}\|_{X_m}$$
$$\leqslant \liminf_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \Big(\sum_{m=1}^{\infty} \|\phi_{m,n_j}\|_{X_m} \Big) \leqslant 1+\varepsilon.$$

Now it follows by Lemma 2.1 and (ii) (taking $(a_{m,k})_k := (\phi_{m,n_k}(\omega))_k$, where $\omega \in \Omega$ is such that $\phi_{m,n_k}(\omega) < \infty$ for all m, $k \in \mathbb{N}$) that we have (by $\lim_{n\to\infty} f_n = f$ a.e.)

$$\sum_{m=1}^{\infty} \phi_m = \lim_{k\to\infty} \sum_{m=1}^{\infty} \phi_{m,n_k} = \lim_{k\to\infty} f_{n_k} = f \qquad \text{a.e.};$$

from this we see that $f \in \Sigma$ and

$$\|f\|_\Sigma \leqslant 1+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (i) holds.

(i) $\Rightarrow$ (iv). Fix $\varepsilon > 0$ and take a norm bounded nonnegative and nondecreasing sequence $(f_n)_n$ in $\Sigma := \Sigma(X_m)$ such that $\sup_{n\geqslant 1} \|f_n\|_\Sigma = 1$. By the Fatou property of $\Sigma(X_m)$, $f := \lim_n f_n$ belongs to $\Sigma(X_m)$ and $\|f\| = 1$. Take $\varepsilon > 0$ and an $\varepsilon/2$-representation $(\phi_m)_m$ of f. Then $(\phi_m/f)_m$ is a partition of unity for f, since $\sum_{m=1}^{\infty}(\phi_m/f) = 1$ and so $f = \sum_{m=1}^{\infty}(\phi_m/f)f = f$, and $(\phi_m/f)f = \varphi_m \in X_m$ for each $m \in \mathbb{N}$.

Take n_0 such that for all $n > n_0$, $1 - \varepsilon/2 < \|f_n\|$. Note that, since $(f_n)_n$ is nondecreasing and order bounded by f, $(\phi_m/f))_m$ is also a partition of unity for each f_n, since for all m, $(\phi_m/f)f_n \leqslant \varphi_m \in X_m$. In fact, it is easy to see that $((\phi_m/f)f_n)_n$ defines an ε-representation of f_n for each $n > n_0$. This proves (iv).

(iv) $\Rightarrow$ (iii). It is enough to consider the nondecreasing representations $(\phi_{m,n})_m = (\varphi_m f_n)_m$ for each f_n for the adequate $n \geqslant n_\varepsilon$.

(iii) $\Rightarrow$ (ii). Fix $\omega \in \Omega$ which does not belong to the null set where the pointwise convergence does not hold. Fix $\varepsilon > 0$ and consider the corresponding nondecreasing ε-representations $(\phi^\varepsilon_{m,n})_m$ of f_n that exists for each $n > n_0$ from some positive integer n_0.

Since $(f_n)_n$ converges pointwise to f and for each fixed m the sequence $(\phi^\varepsilon_{m,n})_m$ is nondecreasing, we have that for a given $\delta > 0$ there exist positive integers $N_0 \geqslant n_0$ and M_0 such that

$$\Big| f(\omega) - \sum_{m=1}^{M} \phi^\varepsilon_{m,n}(\omega) \Big| < \delta/2$$

and

$$\big| f_n(\omega) - f(\omega) \big| < \delta/2, \qquad n > N_0,\ M > M_0.$$

The following inequalities show that the a.e. convergence requirement for this sequence of representations holds for all $n \geqslant N_0$ and $M \geqslant M_0$, since

$$\Big|f_n(\omega) - \sum_{m=1}^{M} \phi^{\varepsilon}_{m,n}(\omega)\Big| \leqslant \big|f(\omega) - f_n(\omega)\big| + \Big|f(\omega) - \sum_{m=1}^{M} \phi^{\varepsilon}_{m,n}(\omega)\Big| < \delta.$$

This shows that

$$\lim_{M,\, n_0 \to \infty} \Big(\sup_{n > n_0} \Big| \sum_{m=1}^{M} \phi^{\varepsilon}_{m,n} - f_n \Big| \Big) = 0, \qquad \text{a.e.}$$

and so (ii) is proved. ■

In order to give further characterizations of the Fatou property of $\Sigma(X_m)$ based on the proof of Theorem 2.1, in what follows we center our attention in a special class of sums of Banach function spaces. We analyze the sums of Banach lattices which satisfy the property: for every $0 \leqslant f \in \Sigma(X_m)$ there is a representation $(\phi_m)_m$ such that $\sum_m \|\phi_m\|_{X_m} = \|f\|_{\Sigma(X_m)}$. In what follows we call such a sequence $(\phi_m)_m$ a 0-representation of f.

Lemma 2.3. *Let $\{X_m\}_{m\in\mathbb{N}}$ be a strongly compatible sequence of maximal Banach lattices on a measure space $(\Omega, \mathcal{S}, \mu)$. Suppose that $\Sigma := \Sigma(X_m)$ has the weak Fatou property. Then the following statements are equivalent.*

- (i) *Every $0 \leqslant f \in \Sigma$ has a 0-representation.*
- (ii) *If $(f_n)_n$ is a nondecreasing sequence in Σ such that $f_n \to f$ a.e., where $f \in \Sigma$, then there exists a positive sequence $(\varepsilon(n))_n$ in c_0 for which there is an $\varepsilon(n)$-representation for each f_n such that for each m, $(\phi^{\varepsilon(n)}_{m,n})_n$ is nondecreasing.*

Proof. (i) $\Rightarrow$ (ii). Fix $f \in \Sigma(X_m)$, and let $(\phi_m)_m$ be a 0-representation for f. Consider a nondecreasing sequence $(f_n)_n$ such that $f_n \to f$ a.e.. Note that by the weak Fatou property, $\sup_n \|f_n\|_\Sigma = \|f\|_\Sigma$. Let $(\varepsilon(n))_n$ be a nonnegative sequence in c_0 given by $\varepsilon(n) = \|f\|_\Sigma - \|f_n\|_\Sigma$ for each $n \in \mathbb{N}$. Define the partition of unity $(\varphi_m)_m$ by $\varphi_m := \phi_m / f$ for each $m \in \mathbb{N}$. Take for each $n \in \mathbb{N}$ a family of representations of f_n defined by

$$\phi^{\varepsilon(n)}_{m,n} := \varphi_m f_n.$$

Since each X_m is maximal, we can easily get (by $\lim_{n\to\infty} \phi^{\varepsilon(n)}_{m,n} = \phi_m$ a.e.)

$$\lim_{n\to\infty} \|\phi^{\varepsilon(n)}_{m,n}\|_{X_m} = \|\phi_m\|_{X_m}.$$

Hence

$$\sum_{m=1}^{\infty} \|\phi^{\varepsilon(n)}_{m,n}\|_{X_m} \leqslant \sup_{n\geqslant 1} \|f_n\|_\Sigma = \|f\|_\Sigma = \|f_n\|_\Sigma + \varepsilon(n), \qquad n \in \mathbb{N}.$$

Consequently, for each n, $(\phi^{\varepsilon(n)}_{m,n})_m$ defines an $\varepsilon(n)$-representation for f_n.

(ii) $\Rightarrow$ (i). Fix $0 \leqslant f \in \Sigma(X_m)$. Take $f_n := f$ for each $n \in \mathbb{N}$. Then there is a sequence $(\varepsilon(n))_n$ and $\varepsilon(n)$-representations $(\phi_{m,n}^{\varepsilon(n)})_m$ of f that are nondecreasing for each fixed m. Since $f = \sum_{m=1}^{\infty} \phi_{m,n}^{\varepsilon(n)}$ for each n and for each n the representations are nondecreasing, we get that in fact the representation $\phi_{m,n}^{\varepsilon(n)}$ does not depend on $\varepsilon(n)$. Therefore

$$\|f\|_\Sigma \leqslant \sum_{m=1}^{\infty} \|\phi_{m,n}^{\varepsilon(n)}\| \leqslant \|f\| + \varepsilon(n), \qquad n \in \mathbb{N},$$

and so $\|f\| = \sum_{m=1}^{\infty} \|\phi_{m,n}^{\varepsilon(n)}\|$. ∎

Corollary 2.1. *Let $\{X_m\}_{m\in\mathbb{N}}$ be a strongly compatible sequence of maximal Banach lattices on a measure space $(\Omega, \mathcal{S}, \mu)$. Then the following assertions are equivalent.*

(i) *Every $0 \leqslant f \in \Sigma(X_m)$ has a 0-representation and $\Sigma(X_m)$ is maximal.*
(ii) *If $(f_n)_n$ is a nondecreasing and norm bounded sequence in $\Sigma(X_m)$ such that $f_n \to f$ a.e., then there exists a nonnegative sequence $(\varepsilon(n))_n$ in c_0 for which there is an $\varepsilon(n)$-representation for each f_n such that for each m, $(\phi_{m,n}^{\varepsilon(n)})_n$ is nondecreasing.*

Proof. (i) $\Rightarrow$ (ii). Let $(f_n)_n$ is a nondecreasing and bounded sequence in $\Sigma(X_m)$ such that $f_n \to f$ a.e. The Fatou property implies that $f \in \Sigma(X_m)$. Since $\Sigma(X_m)$ has also the weak Fatou property, Lemma 2.3 applies and so (ii) is proved.

(ii) $\Rightarrow$ (i). Combining Theorem 2.1 with (ii) implies that $X(\mu)$ is maximal. Moreover, by Lemma 2.3, we have that each f allows a 0-representation, and this completes the proof. ∎

3. Applications to the Köthe duality

In this section we apply the results of the previous one to derive the representation of the Köthe duals of the sums of Banach function lattices with the Fatou property generated by arbitrary families of spaces that are not necessarily countable. This requires to extend the results of the previous section to this case.

We note that the Fatou property of a sum does not imply the Fatou property of each of the Banach space that composes the sum, even in the finite case. To see this it is enough to note that the sum of ℓ^∞ and c_0 gives again ℓ^∞, that has the Fatou property while c_0 does not have it. This of course has its counterpart in the characterization using ordered representations of the elements in the space, and adds some difficulty to the natural extension of the concept of uniformly bounded representations to the noncountable case.

We recall that if $\{X_m\}_{m\in\mathbb{N}}$ is a sequence of Banach lattices on a measure space then a nonnegative and nondecreasing bounded sequence $(f_n)_n$ in $\Sigma(X_m)$ has uniformly bounded ordered representations if for each $\varepsilon > 0$ there is a positive integer n_ε and ε-representations $(\phi_{m,n}^{\varepsilon})_m$ of $(f_n)_{n\geqslant n_\varepsilon}$ such that for all $n \geqslant n_\varepsilon$, $(\phi_{m,n}^{\varepsilon})_m$ is nondecreasing.

We can extend this definition to the sum $\Sigma(X_\alpha)$ of an arbitrary family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ of Banach lattices on a given measure space in the natural way: a sequence $(f_n)_n$ in $\Sigma(X_\alpha)$ has *uniformly bounded ordered representations* if it has them in $\Sigma(X_{\alpha_m})$ for every countable family $\{X_{\alpha_m}\}$ of $\{X_\alpha\}$ such that $(f_n)_n \subseteq \bigcup_{m\in\mathbb{N}} X_{\alpha_m}$. Notice that it follows from Theorem 2.1 that under the assumption that every X_α is maximal, the statement that every nonnegative and nondecreasing norm bounded sequence has a uniformly bounded ordered representations is equivalent to the statement that all countable sums of the family $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ are maximal.

An immediate consequence of Lemma 3.2 (iii) in [9] and Theorem 2.1 is the following result stated below. Notice that if $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a strongly compatible family of maximal Banach lattices on a measure space $(\Omega, \mathcal{S}, \mu)$, then by duality there exists a constant $c > 0$ such that $X'_\alpha \overset{c}{\hookrightarrow} \Delta(X_m)'$ for all α and so $\{X'_\alpha\}_{\alpha\in\mathcal{A}}$ is also a strongly compatible family of Banach lattices on $(\Omega, \mathcal{S}, \mu)$.

Theorem 3.1. *Let $\{X_m\}_{m\in\mathbb{N}}$ be a strongly compatible family of maximal Banach lattices on a measure space $(\Omega, \mathcal{S}, \mu)$. Suppose that each norm bounded nonnegative and nondecreasing sequence $(f_n)_n$ in $\Sigma(X'_m)$ has uniformly bounded ordered representations (equivalently, $\Sigma(X'_m)$ is maximal). Then $\Delta(X_m)$ is maximal and the following Köthe duality formula holds*

$$\Delta(X_m)' \cong \Sigma(X'_m).$$

Now we introduce the notion of *union space* of a family of Banach function spaces. Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of strongly compatible Banach lattices and let $\bigcup(X_\alpha)$ denote the union of the sets of the family. It is easy to check (see [2, p. 215], [8, Theorem 1]) that a homogeneous functional defined by

$$\|x\|_{\bigcup(X_\alpha)} = \inf\{\|x\|_{X_\alpha};\, X_\alpha \ni x\}, \qquad x \in \bigcup(X_\alpha),$$

is a norm on $\bigcup(X_\alpha)$ if and only if $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a scale, i.e., $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a directed family of strongly compatible Banach spaces, i.e., such that for every $\alpha, \beta \in \mathcal{A}$ there exists $\gamma \in \mathcal{A}$ such that $X_\alpha \hookrightarrow X_\gamma$ and $X_\beta \hookrightarrow X_\gamma$.

Following [8], the scale $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be $\bigcup$-*complete* if $(\bigcup(X_\alpha), \|\cdot\|_{\bigcup(X_\alpha)})$ is a Banach space.

Combining the above results, we obtain the following one which seems to be of independent interest from the point of view of applications.

Theorem 3.2. *Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a strongly compatible scale of maximal Banach lattices. If $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is $\bigcup$-complete and each norm bounded nonnegative and nondecreasing sequence $(f_n)_n$ in $\Sigma(X_\alpha)$ has uniformly bounded ordered representations, then $\Sigma(X_\alpha)$ is a maximal Banach lattice.*

Proof. We will use Theorem 1 from [8], which states that for a strongly compatible scale of Banach spaces $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ to be $\bigcup$-complete is equivalent to

$$\bigcup(X_\alpha) \cong \Sigma(X_\alpha).$$

Let $(f_n)_n$ be a nondecreasing sequence of nonnegative functions in the unit ball B_Σ of $\Sigma := \Sigma(X_\alpha)$. Given $\varepsilon > 0$, for each $n \in \mathbb{N}$, we can find an $\alpha_n \in \mathcal{A}$ such that

$$\|f_n\|_{X_{\alpha_n}} \leqslant \|f_n\|_\Sigma + \varepsilon \leqslant 1 + \varepsilon.$$

In particular this implies that for the strongly compatible family $\{X_{\alpha_m}\}_{m\in\mathbb{N}}$ we have $f_n \in \Sigma(X_{\alpha_m})$ with

$$\|f_n\|_{\Sigma(X_{\alpha_m})} \leqslant 1 + \varepsilon, \qquad n \in \mathbb{N}.$$

Thus we have from Theorem 2.1 (by using the existence of uniformly bounded ordered representations of $(f_n)_n$) that $f \in \Sigma(X_{\alpha_m})$ with $\|f\|_{\Sigma(X_{\alpha_m})} \leqslant 1 + \varepsilon$. Combining with $\Sigma(X_{\alpha_m}) \hookrightarrow \Sigma$ yields $f \in \Sigma$ and

$$\|f\|_\Sigma \leqslant 1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $f \in B_\Sigma$ and this completes the proof. ∎

We need the following observation and for the sake of completeness we prove it.

Proposition 3.1. *Let X be a Banach lattice on $(\Omega, \mathcal{S}, \mu)$ and let $Z \subset X$ be an ideal subspace dense in X. If the norm of X is order semi-continuous on Z, then X has the weak Fatou property.*

Proof. Let $0 \leqslant f_n \uparrow f$, a.e., $f \in X$. By density we can find a sequence $(g_m)_m \subset Z$ such that $\|g_m - f\|_X \to 0$. Then $h_n^{(m)} := \min\{f_n, |g_m|\} \in Z$ and $0 \leqslant h_n^{(m)} \uparrow h^{(m)} = \min\{f, |g_m|\}$ a.e. as $n \to \infty$. Since $\||g_m| - f\|_X \to 0$, we get $\|h^{(m)} - f\|_X \to 0$ as $m \to \infty$. Our hypothesis gives

$$\|f\|_X = \lim_{n\to\infty} \|h^{(m)}\|_X, \qquad m \in \mathbb{N}.$$

Combining the comments above with $\|h_n^{(m)}\|_X \leqslant \|f_n\|_X \leqslant \|f\|_X$ and $\|f\|_X = \lim_{n\to\infty} \|h^{(m)}\|_X$, we arrive at $\|f\|_X = \lim_{n\to\infty} \|f_n\|_X$, and the proof is complete. ∎

Let us show now some applications.

Theorem 3.3. *Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a strongly compatible scale of maximal Banach lattices. Assume that $\Delta(X_\alpha)$ is dense in $\Sigma(X_\alpha)$ and X_α for every $\alpha \in \mathcal{A}$. Then $\Sigma(X_\alpha)$ has the weak Fatou property.*

Proof. We apply Corollary 2 from [8], which states that if $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is a strongly compatible scale and Z is a subspace which is dense in X_α for every $\alpha \in \mathcal{A}$, then for all $x \in Z$,

$$\|x\|_{\Sigma(X_\alpha)} = \|x\|_{\bigcup(X_\alpha)}.$$

By Lemma 3.1 in [9], we have that $\Delta(X_\alpha) = \Delta(X_\alpha'') = (\Sigma(X_\alpha'))'$, which in particular means that $\Delta(X_\alpha)$ has the Fatou property, and so it has order semi-continuous norm. Thus, applying Proposition 3.1 we obtain the result. ∎

An immediate consequence of the obtained results is the following one.

Corollary 3.1. *Let* $\{X_\alpha\}_\alpha$ *be a family of maximal Banach lattices on a given measure space such that* $\{X'_\alpha\}_{\alpha\in\mathcal{A}}$ *is a strongly compatible scale and each countable sum of* $\{X'_\alpha\}$ *is maximal (equivalently, every nonnegative norm bounded nondecreasing sequence in each countable sum of* $\{X'_\alpha\}$ *has uniformly bounded ordered representations). Then the following Köthe duality formula holds,*

$$\Delta(X_\alpha)' \cong \Sigma(X'_\alpha).$$

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Addresses: Mieczysław Mastyło: Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland;
Enrique A. Sánchez-Pérez: Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain.

E-mail: mastylo@amu.edu.pl, easancpe@mat.upv.es

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