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THE DUAL OF THE LOCALLY CONVEX SPACE $C_p(X)$

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: If X is an infinite Tichonov space, we show that the weak dual $L_p(X)$ of the continuous function space $C_p(X)$ cannot be barrelled, bornological, or even quasibarrelled. Indeed, of the fourteen standard weak barrelledness properties between Baire-like and primitive, $L_p(X)$ enjoys precisely the four between property (C) and primitive if X is a P-space, and none otherwise. Since $L_p(X)$ is S_σ, it must admit an infinite-dimensional separable quotient. Under its Mackey topology, $L_p(X)$ enjoys eleven of the properties if X is discrete, nine if X is a nondiscrete P-space, and none otherwise.

Keywords: weak barrelledness, P-spaces, separable quotients.

1. Introduction

In the sequel, X will be an infinite Tichonov (completely regular Hausdorff) space and $C_p(X)$ will denote the space $C(X)$ of all continuous real-valued functions defined on X provided with the pointwise convergence topology. $L(X)$ denotes the topological dual of $C_p(X)$ and $L_p(X)$ the linear space $L(X)$ equipped with the weak topology $\sigma(L(X), C(X))$. A locally convex space (lcs) E is [*barrelled*] ⟨*quasibarrelled*⟩ if every [$\sigma(E', E)$-bounded] ⟨$\beta(E', E)$-bounded⟩ set in E' is equicontinuous, and E is *bornological* if every absolutely convex set which absorbs the bounded sets of E is a neighborhood of the origin.

Folklore says $L_p(X)$ is realcompact if and only if X is realcompact, and $L_p(X)$ is a Lindelöf space if and only if $C_p(X)$ is countably tight. Very likely the following question is still open: If $L_p(X)$ is countably tight, must $C_p(X)$ be a Lindelöf space?

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But our study of $L_p(X)$ will follow the study of $C_p(X)$ by Buchwalter, Schmets, Cascales, and others who proved in [3], [5, Cor.2.8], [11, Cor.11.7.3], [26] that (*i*) $C_p(X)$ is always quasibarrelled, (*ii*) $C_p(X)$ is bornological if and only if X is realcompact, (*iii*) $C_p(X)$ is barrelled if and only if every bounding set in X is finite, (*iv*) $C_p(X)$ is quasicomplete if and only if X is discrete, (*v*) $C_p(X)$ is sequentially complete if and only if X is a P-space, and (*vi*) $C_p(X)$ belongs to class $\mathfrak{G}$ if and only if X is countable.

We will quickly see that $L_p(X)$ is never barrelled, quasibarrelled, bornological, or in class $\mathfrak{G}$, then consider all fourteen properties usually associated with weak barrelledness [22], from Baire-like to primitive. We show that, of these fourteen, $L_p(X)$ enjoys precisely the four duality invariant properties between property (C) and primitive, or none at all, depending on whether X is a P-space or not. Result (*v*) is an immediate corollary. Recall [27, p.103] that a property is *duality invariant* if whenever it holds for a lcs $(E, \mathcal{T})$ it also holds for $(E, \mathcal{T}_1)$, where $\mathcal{T}_1$ is any locally convex topology on E such that $(E, \mathcal{T})' = (E, \mathcal{T}_1)'$.

Thus our analysis of $L_p(X)$ *vis-à-vis* weak barrelledness is complete, albeit monochromatic. Letting $L_m(X)$ denote $L_p(X)$ endowed with its Mackey topology $\tau(L(X), C(X))$, we obtain for $L_m(X)$ an analysis more colorful but somewhat less than complete: $L_m(X)$ is barrelled if and only if X is discrete, and then has precisely the eleven weak barrelledness properties between barrelled and primitive. If X is not a P-space, then $L_m(X)$ enjoys none of the fourteen properties. When X is a nondiscrete P-space, we show that $L_m(X)$ has all fourteen properties except Baire-like (BL), quasi-Baire (QB), barrelled, non-S_σ, and possibly $\aleph_0$-barrelled, so that $L_m(X)$ enjoys either nine or ten of the fourteen properties. For X a nondiscrete *Lindelöf* P-space, we prove that $L_m(X)$ possesses exactly nine of the properties (is not $\aleph_0$-barrelled). The only question left open[1] is whether $L_m(X)$ fails to be $\aleph_0$-barrelled for *every* nondiscrete P-space X.

Several times we rely on the fact that $C_p(X)$ contains a dense subspace $C^b(X) = \{f \in C(X) : f(X) \text{ is bounded in } \mathbb{R}\}$ dominated by that Banach space, denoted $C_u^b(X)$, which it becomes under the uniform convergence norm. This same fact and Baire's theorem prove $C_p(X)$ is always a *non-S_σ space*; i.e., a lcs that cannot be covered by an increasing sequence of proper closed subspaces (cf. [15]). Nonetheless, we prove $L_p(X)$ is never non-S_σ, i.e., is always S_σ. Consequently, every $L_p(X)$ admits infinite-dimensional separable quotients (that are $\aleph_0$-dimensional, even). Similarly for the strong dual $C_p(X)'_\beta$ of $C_p(X)$. The strong dual $C_c(X)'_\beta$, where $C_c(X)$ denotes $C(X)$ endowed with the compact-open topology, always admits infinite-dimensional separable quotients [16], as do the strong duals of infinite-dimensional Banach spaces [1]. Only very recently have we, jointly with Todd [16], identified a class of infinite-dimensional strong duals that *disallow* such quotients.

[1] The positive answer was given by the last-named author in the following paper: S.A. Saxon, *Weak barrelledness vs. P-spaces*, to appear in Vol. 80 of the Springer Proceedings in Mathematics and Statistics, titled *Descriptive Topology and Functional Analysis* (added in press).

2. $L_p(X)$: not bornological, not (quasi)barrelled, not in class $\mathfrak{G}$

Jarchow's unequivocal (i), the classical Buchwalter-Schmets characterizations (ii-v), and the Cascales-Kąkol-Saxon result (vi) cited above attest to the fact that interesting $C_p(X)$ spaces are always quasibarrelled and sometimes bornological, barrelled, or in class $\mathfrak{G}$. Never so for $L_p(X)$ spaces.

Theorem 1. *$L_p(X)$ is never barrelled, quasibarrelled, or bornological.*

Proof. The unit ball U in the Banach space $C_u^b(X)$ is infinite-dimensional and $\sigma(C(X), L(X))$-bounded. The Banach-Mackey theorem [18, §20.11(3)] ensures that U is strongly $\beta(C(X), L(X))$-bounded. But U cannot be equicontinuous on $L_p(X)$, for no lcs E with its $\sigma(E, E')$ topology admits equicontinuous infinite-dimensional sets in E'. Therefore $L_p(X)$ is not quasibarrelled, hence neither barrelled nor bornological. ■

We see (vi) more simply from our joint paper [7] with López-Pellicer, which shows that a lcs E having its weak $\sigma(E, E')$ topology is in Cascales and Orihuela's class $\mathfrak{G}$ if and only if $\dim E' \leqslant \aleph_0$. Indeed, if $\dim E' \leqslant \aleph_0$, then E is metrizable, hence in $\mathfrak{G}$, and the converse is Proposition 2 of [7], whose essential argument Talagrand knew (cf. [12, Prop.3.7]).

Theorem 2. *$L_p(X)$ is never in class $\mathfrak{G}$.*

Proof. $C(X)$ contains an infinite-dimensional subspace $C^b(X)$ dominated by a Banach space, and thus $\dim L_p(X)' \geqslant \dim C^b(X) \geqslant 2^{\aleph_0} > \aleph_0$, so that $L_p(X)$ is not in $\mathfrak{G}$ by the preceding paragraph. ■

3. Weak barrelledness and $L(X)$

A lcs E is *inductive* if E is the inductive limit of every increasing sequence of subspaces that cover E (cf. [8]). A lcs E is *primitive* if every linear form on E with continuous restrictions to the members of some increasing, covering sequence of subspaces must, itself, be continuous (cf. [23]). E is [*ℓ^∞-barrelled*] ⟨*c_0-barrelled*⟩ if every [$\sigma(E', E)$-bounded] ⟨$\sigma(E', E)$-null⟩ sequence is equicontinuous. Clearly, inductive ⇒ primitive and ℓ^∞-barrelled ⇒ c_0-barrelled.

Theorem 3. *$L_p(X)$ is never non-S_σ, inductive, ℓ^∞- or c_0-barrelled.*

Proof. There exists a sequence $(U_n)_n$ of disjoint non-empty open sets in X. For each n, choose $x_n \in U_n$ and $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = \{0\}$, and set

$$M_n = \bigcap_{m>n} \{v \in L(X) : \langle v, f_m \rangle = 0\}.$$

One easily checks that $L_p(X)$ is the union of the properly increasing sequence $(M_n)_n$ of closed subspaces, and thus, by definition, is an S_σ space. Also, $L_p(X)$ is not the inductive limit of the subspaces M_n and not c_0-barrelled, for otherwise the linearly independent and weakly null sequence $(f_n)_n$ would be equicontinuous on $L_p(X)$. ■

Since BL $\Rightarrow$ QB $\Rightarrow$ non-S_σ $\Rightarrow$ inductive, and QB $\Rightarrow$ barrelled $\Rightarrow$ $\aleph_0$-barrelled $\Rightarrow$ C-barrelled $\Rightarrow$ property (L) $\Rightarrow$ inductive, it is clear that $L_p(X)$ enjoys none of these eight weak barrelledness properties between Baire-like and inductive, nor can it be ℓ^∞- or c_0-barrelled. (Barrelledness does not generally imply BL, QB, or non-S_σ, but does, e.g., under metrizability.) The remaining four successively weaker barrelledness properties are property (C), property (S), dual locally complete (dlc), and primitive. All four are duality invariant and, as Theorem 6 will show, each is realized in $L_p(X)$ precisely when X is a P-space. Thus $L_p(X)$ reduces weak barrelledness to a single notion expressed in four standard ways.

A lcs E has [*property (C)*] ⟨*property (S)*⟩ if in the weak dual $(E', \sigma(E', E))$, [bounded sets are relatively countably compact] ⟨Cauchy sequences converge⟩ [19]. E is *locally complete* if and only if for each bounded sequence $(x_n)_n$ in E and each absolutely summable scalar sequence $(a_n)_n$, the series $\sum_n a_n x_n$ converges in E; and E is *dlc* if its weak dual is locally complete [20].

The [weak] ⟨Mackey⟩ topology on a lcs E is the [coarsest] ⟨finest⟩ locally convex topology on E that reproduces the given dual. On the quasibarrelled (cf. (i)) space $C_p(X)$ these extreme topologies, weak and Mackey, coincide. Never so for $L_p(X)$: Let S be a linearly independent null sequence in the Banach space $C_u^b(X)$; the absolutely convex closed hull of S is compact in $C_p(X)$, therefore Mackey equicontinuous, but not weakly equicontinuous. In short, $L_p(X) \neq L_m(X)$.

A lcs E is *feral* [15] if every infinite-dimensional set in E is unbounded. Clearly, a lcs with its weak topology is quasibarrelled if and only if its strong dual is feral, so (i) merely says that $L(X)$ *with its strong* $\beta(L(X), C(X))$ *topology is always feral.*

The proof is not hard: Let A be an infinite-dimensional subset of $L(X)$. With regard to the Hamel basis X, we choose sequences $(z_n)_n \subset A$ and $(u_n)_n \subset X$ of distinct elements such that, in the expansion of z_n, the coefficient c_n of u_n is nonzero. Let $(x_n)_n$ be a subsequence of $(u_n)_n$ for which there is a sequence $(U_n)_n$ of disjoint open sets in X with $x_n \in U_n$, and let $(y_n)_n$ and $(b_n)_n$ be the corresponding subsequences of $(z_n)_n$ and $(c_n)_n$. For each n, choose an open set $V_n \subset U_n$ such that $x_n \in V_n$ and V_n misses the finitely many other x in X having nonzero coefficients in the expansion of y_n. Finally, choose $g_n \in C(X)$ such that $g_n(x_n) = n/b_n$ and $g_n(X \backslash V_n) = \{0\}$. We observe that $g_n(y_n) = g_n(b_n x_n) = n$ and, since the V_n are pairwise disjoint, $\{g_n : n \in \mathbb{N}\}$ is bounded in $C_p(X)$. It follows that $(y_n)_n$, and thus A, is not strongly bounded; i.e., $(L(X), \beta(L(X), C(X)))$ is feral, Q.E.D.

We note other instructive translations: (v) simply says that $L_p(X)$ has property (S) if and only if X is a P-space, a weak barrelledness result we maximally extend in Corollary 1. Our next example consolidates important $L_p(X)$ facts from $C_p(X)$ weak barrelledness results due mainly to Buchwalter and Schmets (cf. [15]).

Theorem 4. *The following six statements are equivalent.*

(1) X *admits no infinite bounding set.*
(2) $C_p(X)$ *is barrelled* (Buchwalter and Schmets).
(3) $L_p(X)$ *is quasicomplete.*

(4) $L_p(X)$ *is sequentially complete.*
(5) $L_p(X)$ *is locally complete.*
(6) $L_p(X)$ *is feral.*

Proof. From [18, §23.6(4)] we have (2) $\Rightarrow$ (3). Trivially, (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

If $L_p(X)$ is locally complete, then each bounded absolutely convex closed set is a Banach disc [4, 5.1.6] that is $\beta(L(X), C(X))$-bounded by the Banach-Mackey theorem, and thus, by our interpretation above of (i), is finite-dimensional; i.e., $L_p(X)$ is feral, and (5) $\Rightarrow$ (6).

Trivially, (6) $\Rightarrow$ (2), proving equivalence of (2)-(6).

Since the bounding subsets of X may be viewed as linearly independent bounded subsets of $L_p(X)$, it is apparent that (6) $\Rightarrow$ (1).

We sketch a proof that (1) $\Rightarrow$ (6). Let $(z_n)_n$ be a linearly independent sequence in $L(X)$. Use (1) to inductively find a subsequence $(y_n)_n$ of $(z_n)_n$, a locally finite sequence $(U_n)_n$ of disjoint nonempty open sets in X, and a sequence $(f_n)_n \subset C(X)$ such that, for all $n \in \mathbb{N}$,

$$f_m(y_n) = 0 \quad \text{for} \quad m > n, \qquad f_n(y_n) = 1, \qquad \text{and} \qquad f_n(X \backslash U_n) = \{0\}.$$

By definition of local finiteness, every point in X has a neighborhood that meets only finitely many of the U_n, so the pointwise sum f of the series $\sum_n a_n f_n$ is in $C(X)$ for every scalar sequence $(a_n)_n$. If we set $a_1 = 1$ and $a_{n+1} = (n+1) - \sum_{k \leqslant n} a_k f_k(y_{n+1})$, then $f(y_n) = n$ for each n, which implies $(y_n)_n$ is not bounded in $L_p(X)$. Therefore (6) holds. ■

We may restate (iv) in light of [18, §23.6(4)], adding parts (3) and (4) via the Banach-Mackey theorem.

Theorem 5. *The following four assertions are equivalent.*

(1) X *is discrete.*
(2) $L_m(X)$ *is barrelled* (Buchwalter and Schmets).
(3) $L_m(X)$ *is quasibarrelled.*
(4) $L_m(X)$ *is bornological.*

Proof. The strongest locally convex topology ensures (1) $\Rightarrow$ (2,3,4).

Conversely, if (2), (3), or (4) holds, then $L_m(X)$ is quasibarrelled. As before, the unit ball U in $C_u^b(X)$ is $\beta(C(X), L(X))$-bounded, thus equicontinuous on $L_m(X)$, and is obviously closed in $C_p(X)$. By equicontinuity, U is closed in the algebraic dual $\mathbb{R}^X$ with its product topology. Given any $f \in \mathbb{R}^X$ with $|f| \leqslant 1$ and finite $S \subset X$, there exists $g \in U$ that agrees with f on S, and thus $f \in U$. In particular, the characteristic function of $\{x\}$ is in $C(X)$ for each $x \in X$, which means X is discrete. ■

We expand Arkhangel'skii's notion of strict τ-continuity [2]. If $\mathcal{M} = \{X_\alpha\}_{\alpha \in A}$ is a family of subsets of X covering X, we shall say that a real-valued function f on X is *strictly $\mathcal{M}$-continuous* if, for each $\alpha \in A$, there is some $g_\alpha \in C(X)$ with $f|X_\alpha = g_a|X_\alpha$. The definitions of strict $\mathcal{M}$-continuity and strict τ-continuity (τ an infinite cardinal) coincide when $\mathcal{M}$ consists of all subsets of X of cardinality τ.

From [9]: X is a P-space if and only if, for each $x \in X$, every countable intersection of neighborhoods of x is, itself, a neighborhood of x. To (v) we add

Theorem 6. *The following five assertions are equivalent.*

1. *(1) $L_p(X)$ is primitive.*
2. *(2) For each increasing sequence $\mathcal{M} = (X_n)_n$ of subsets of X covering X, every strictly $\mathcal{M}$-continuous function f is continuous.*
3. *(3) X is a P-space.*
4. *(4) The Mackey space $L_m(X)$ is ℓ^∞-barrelled.*
5. *(5) $L_p(X)$ has property (C).*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{M}$ and f be given as in (2). By definition, for each $n \in \mathbb{N}$ some $g_n \in C(X)$ satisfies $g_n|X_n = f|X_n$. With the usual embedding of X into $L_p(X)$ as a Hamel basis and the linearization of g_n and f, the latter becomes a linear form on $L_p(X)$ which agrees with g_n on $\operatorname{sp} X_n$. Primitivity then puts f in the continuous dual $C(X)$ of $L_p(X)$, and (2) holds.

(2) $\Rightarrow$ (3). Let $(U_n)_n$ be a sequence of neighborhoods of a point y in X. Since finite intersections of open sets are open, we readily choose by induction a decreasing sequence $(V_n)_n$ of open neighborhoods of y and a sequence $(g_n)_n \subset C(X)$ such that, for each $n \in \mathbb{N}$,

(a) $V_n \subset U_n$,
(b) $g_n(y) = 1$ and $g_n(X \backslash V_n) = \{0\}$, and
(c) $V_{n+1} \subset \bigcap_{i=1}^{n} \{x \in X : |g_i(x) - 1| < 1/n\}$.

Set

$$Y = \bigcap_{n=1}^{\infty} g_n^{-1}(1) \quad \text{and} \quad X_n = (X \backslash V_n) \bigcup Y \qquad \text{for all} \quad n \in \mathbb{N}.$$

Since $(V_n)_n$ is decreasing, $(X_n)_n$ is increasing. Moreover, $(X_n)_n$ covers X, for it obviously covers Y, and if $x \in X \backslash Y$, then there exist i and j such that $|g_i(x) - 1| \geqslant 1/j$. Set $m = \max\{i, j\}$, so that $1 \leqslant i \leqslant m$ and $|g_i(x) - 1| \geqslant 1/m$, which by (c) implies that $x \in X \backslash V_{m+1} \subset X_{m+1}$.

Claim: Let f be the function which vanishes on $X \backslash Y$ and is identically 1 on Y, and let $\mathcal{M} = (X_n)_n$. Then f agrees with g_n on X_n for all n; i.e., f is strictly $\mathcal{M}$-continuous. Clearly, f and g_n are both identically 1 on Y. From (b), $X \backslash V_n \subset X \backslash Y$ and both f and g_n vanish on $X \backslash V_n$. Hence f and g_n agree on $Y \cup (X \backslash V_n) = X_n$, establishing the Claim.

From (2), f is continuous on X. By construction, y is in Y, and continuity yields a neighborhood U of y which is mapped by f into the open interval $(1/2, 3/2)$. Since f is nonzero on U, we have $U \subset Y$. Again, from (b) and the definition of Y, we observe $Y \subset V_n$. Combining these facts with (a), we have

$$U \subset Y \subset \bigcap_{n=1}^{\infty} V_n \subset \bigcap_{n=1}^{\infty} U_n,$$

which proves (3).

(3) $\Rightarrow$ (4). Let $(f_n)_n$ be a bounded sequence in $C_p(X)$ with (bounded) absolutely convex hull A. To show $(f_n)_n$ is Mackey-equicontinuous, it suffices to show that the closure $\overline{A}$ of A in the product space $\mathbb{R}^X$ is compact and contained in $C(X)$. For each $x \in X$ we set

$$N(x) = \bigcap_{m,n \in \mathbb{N}} \{z \in X : |f_n(z) - f_n(x)| < 1/m\}.$$

Each $N(x)$ is a neighborhood of x by (3). By construction, each f_n has the constant value $f_n(x)$ on $N(x)$. Each g in A is a linear combination of the f_n and thus has constant value $g(x)$ on $N(x)$. Also, $\overline{A}$ is bounded in $\mathbb{R}^X$, thus compact by Tichonov's theorem. Now suppose $h \in \mathbb{R}^X$ and $h(y) \neq h(x)$ for some $x \in X$ and $y \in N(x)$. Set $\varepsilon = |h(x) - h(y)|$ and define a neighborhood U of h in $\mathbb{R}^X$ by writing

$$U = \left\{g \in \mathbb{R}^X : |g(x) - h(x)|, |g(y) - h(y)| < \varepsilon/2\right\}.$$

If $g \in U$, then it follows from the triangle inequality that $g(x) \neq g(y)$. And since g is not constant on $N(x)$, it is not in A; i.e., U is a neighborhood of h which misses A. We conclude that $\overline{A}$ can contain only functions that are constant on a neighborhood $N(x)$ of x for each $x \in X$. But such functions are clearly continuous on X, proving $\overline{A} \subset C(X)$.

(4) $\Rightarrow$ (5) $\Rightarrow$ (1). Evidently, any ℓ^∞-barrelled lcs with its weak topology has property (C), and property (C) always implies primitive. ■

Corollary 1. *The space $L_p(X)$ enjoys precisely the four properties between property (C) and primitive if X is a P-space, and none otherwise.*

Corollary 2. *If X is discrete, then $L_m(X)$ has precisely the eleven weak barrelledness properties between barrelled and primitive. These eleven, with the exception of barrelled and possibly $\aleph_0$-barrelled, become the ten or nine properties enjoyed by $L_m(X)$ when X is a nondiscrete P-space. If X is not a P-space, then $L_m(X)$ has none of the fourteen weak barrelledness properties.*

Proof. Theorem 5 ensures that $L_m(X)$ has at least the eleven properties promised when X is discrete. Theorem 3 and duality invariance insure that $L_m(X)$ can never be non-S_σ, nor can have either of the other two properties, BL and QB, that imply non-S_σ; thus $14 - 3 = 11$ is the precise number in this case.

If X is a nondiscrete P-space, then $L_m(X)$ is S_σ and not barrelled (Theorems 3 and 5), so $L_m(X)$ can have at most the $11-1 = 10$ properties between $\aleph_0$-barrelled and primitive. Now $L_m(X)$ is Mackey and ℓ^∞-barrelled by Theorem 6, hence by [21] enjoys all ten of these properties with the possible exception of $\aleph_0$-barrelled. (We have ℓ^∞-barrelled $\Rightarrow$ c_0-barrelled, and from [21], [Mackey $\wedge$ c_0-barrelled] $\Rightarrow$ C-barrelled.)

When X is not a P-space, $L_m(X)$ is not primitive by Theorem 6 and duality invariance of primitivity. Thus neither can $L_m(X)$ enjoy any of the stronger conditions. ■

X is a *Lindelöf space* if every open cover of X can be reduced to a countable subcover. Many X [2, IV.2.15] satisfy the popular hypothesis (cf. [2, II.1.5], [10, p.157], [12, Ch.14]) of our next Theorem since, clearly, a Lindelöf P-space is nondiscrete if and only if it is uncountable.

Theorem 7. *If X is a nondiscrete Lindelöf P-space, then $L_m(X)$ is ℓ^∞-barrelled but not $\aleph_0$-barrelled.*

Proof. The previous Theorem ensures that $L_m(X)$ is ℓ^∞-barrelled.

Let y be a non-isolated point in X. Zorn's lemma produces a maximal collection $\mathfrak{C}$ of nonempty open sets whose closures are pairwise disjoint and miss y.

Claim: $\mathfrak{C}$ is uncountable. Otherwise, since X is a P-space, the intersection of the complements of the closures of sets in $\mathfrak{C}$ would be a neighborhood V of y. Since y is not isolated, there is some $y' \neq y$ such that V is a neighborhood of y'. Hence there exists an open set $V' \subset V$ such that $y' \in V'$ and $y \notin \overline{V'} \subset V$. But then the collection $\mathfrak{C} \bigcup \{V'\}$ would contradict the maximallity of $\mathfrak{C}$. The Claim follows.

For each $U \in \mathfrak{C}$ choose $x_U \in U$ and set $S = \{x_U : U \in \mathfrak{C}\}$. The closed set $\overline{S}$ is Lindelöf by hypothesis, whereas our Claim shows that S is not Lindelöf. Consequently, there exists some $z \in \overline{S} \backslash S$. Furthermore, since the elements of $\mathfrak{C}$ are pairwise disjoint, $z \notin \bigcup \mathfrak{C}$.

Next, we construct a special Hamel basis for $L(X)$. Let $A = X \backslash \bigcup \mathfrak{C}$. For each U select $f_U \in C(X)$ such that

$$f_U(x_U) = 1 \qquad \text{and} \qquad f_U(X \backslash U) = \{0\},$$

and set

$$B_U = \{x_U\} \bigcup \{u - f_U(u) \cdot x_U : u \in U \backslash \{x_U\}\}.$$

Each B_U is a Hamel basis for the span of U in $L(X)$. Clearly, the set

$$B = \bigcup \{B_U : U \in \mathfrak{C}\} \bigcup A$$

is a Hamel basis for all of $L(X)$, and the coefficient functional for each x_U is just the continuous f_U.

Now $z \in A \subset B$, and the coefficient functional h corresponding to z is discontinuous, since it vanishes on S ($\subset B \backslash \{z\}$) and not on $\overline{S}$. Assume the Mackey space $L_m(X)$ is $\aleph_0$-barrelled. From the Saxon-Tweddle splitting theorem [24, Th.3.1], then, it is the direct sum of E_C and E_D, where E_C is the span of those basis elements having continuous coefficient functionals and E_D is the span of the rest. Therefore E_C is closed in $L_m(X)$ and contains S and not z. The Hahn-Banach theorem provides a linear form in the continuous dual $C(X)$ which is 1 at z and vanishes on E_C, hence on S, which is clearly absurd. ■

The plentiful [9] nondiscrete P-spaces X yield spaces $C_p(X)$ that are sequentially complete but not quasicomplete, and Mackey spaces $L_m(X)$ that are ℓ^∞-barrelled but not barrelled. Uncountable Lindelöf P-spaces X all yield Mackey spaces $L_m(X)$ that are ℓ^∞-barrelled but not $\aleph_0$-barrelled (cf. [21]). Whether this holds for arbitrary uncountable P-spaces X is an open question whose answer would determine (A) whether $L_m(X)$ spaces reduce weak barrelledness to two distinct notions or three, and (B) whether $L_m(X)$ spaces can augment our meager supply of Mackey $\aleph_0$-barrelled spaces that are not barrelled [24].

4. Separable quotients

By "separable quotient" we will mean "infinite-dimensional separable quotient by a closed subspace". A most famous open question of Banach space theory is whether every infinite-dimensional Banach space admits a separable quotient. In the broader search to determine which lcs's admit separable quotients we, jointly with Todd [16], recently proved that, among the GM-spaces of Eberhardt and Roelcke, precisely those that are S_σ spaces (equiv., that are not QB) admit separable quotients. Since infinite-dimensional Baire (hence QB) GM-spaces exist, these provide the first examples of infinite-dimensional lcs's that disallow separable quotients.

The main result of [25] is that a Banach space admits a separable quotient if and only if it admits a dense subspace which is non-barrelled. It is quite easy to prove that (a) A lcs admits a separable quotient if and only if it admits a dense subspace which is an S_σ space (cf. [25, P_6]), and (b) A lcs is an S_σ space if and only if it admits a closed $\aleph_0$-codimensional subspace. It then follows from Theorem 3 that

Theorem 8. *$L_p(X)$ admits a separable quotient; even more, admits an $\aleph_0$-dimensional quotient.*

Corollary 3. *$L_m(X)$ and the strong dual $C_p(X)'_\beta$ of $C_p(X)$ must admit separable quotients, as, indeed, must $L_p(X)$ with any finer locally convex topology.*

The dual $C_c(X)'$ is generally larger than $L(X)$, but $L(X)$ is clearly dense in the weak dual $C_c(X)'_\sigma$, and thus by (a) and Theorem 3, we see immediately that $C_c(X)'_\sigma$ admits separable quotients. In [16] we prove much more: $C_c(X)'$ admits separable quotients when it is given any locally convex topology between the weak topology $\sigma\left(C_c(X)', C(X)\right)$ and the strong topology $\beta\left(C_c(X)', C(X)\right)$. In particular, the strong dual $C_c(X)'_\beta$ always admits separable quotients. This is the $C_c(X)$ analog of a spectacular Banach space result in [1].

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