Functiones et Approximatio
50.2 (2014), 207–214
doi: 10.7169/facm/2014.50.2.1

ON LEFT DEMOCRACY FUNCTION

Przemysław Wojtaszczyk

To Lech Drewnowski, with thanks for
many years of nice mathematics

Abstract: We continue the study undertaken in [1] of left democracy function

$$h_l(N) = \inf_{\#\Lambda=N} \left\| \sum_{n\in\Lambda_N} x_n \right\|$$

of an unconditional basis in a Banach space X. We provide an example of a basis with h_l non-doubling. Then we show that for bases with non-doubling h_l the greedy projection is not optimal. Together with results from [1] improved by C. Cabrelli, G. Garrigós, E. Hernandez and U. Molter we get that the basis is greedy if and only if the greedy projection is optimal.

Keywords: non-linear approximation, greedy algorithm, democratic basis.

1. Introduction

The aim of this note is to settle some problems left open in [1]. Suppose we have a Banach space X with a normalised basis $(x_n)_{n=1}^{\infty}$. For $x = \sum_{n=1}^{\infty} a_n x_n \in X$ and $N = 1, 2, \dots$ we define a non-linear operator

$$\mathcal{G}_N(x) = \sum_{n\in\Lambda_N} a_n x_n \tag{1}$$

where Λ_N is any N-element subset of indices such that $\min_{n\in\Lambda_N} |a_n| \geqslant \max_{n\notin\Lambda_N} |a_n|$. Note that the set Λ_N may not be uniquely defined; in such a case we are allowed to take arbitrary choice. This is a theoretical model of many practically important tresholding operators. Systematic study of such operators was undertaken in the last years of the XX century (see e.g. [4, 3, 5]) and is an active area of research. It became apparent already in [5] that quantities like $\|\sum_{n\in A} x_n\|$

The author was partially supported by the “HPC Infrastructure for Grand Challenges of Science and Engineering Project, co-financed by the European Regional Development Fund under the Innovative Economy Operational Programme” and Polish NCN grant DEC2011/03/B/ST1/04902.

2010 Mathematics Subject Classification: primary: 41A17; secondary: 42C40

are important for the properties of this operator. The basis is called *democratic* [3] if those quantities depend essentially only on the number of elements of A, more precisely if there exists a constant C such that for all sets A, B with $\#A = \#B$ we have

$$\left\| \sum_{n \in A} x_n \right\| \leqslant C \left\| \sum_{n \in B} x_n \right\| \tag{2}$$

The main result of [3] asserts that a basis is unconditional and democratic if and only if it is *greedy* what means that $\mathcal{G}_N(x)$ is (up to a constant) a best N–term approximation of x by elements $\{x_n\}_{n=1}^{\infty}$; more precisely there exists a constant C such that for all $x \in X$ and $N = 1, 2, \ldots$ we have $\|x - \mathcal{G}_N(x)\| < C\sigma_N(x)$, ($\sigma_N$ is defined in (11)).

A more detailed study resulted in the definition [2] of the left democracy function

$$h_l(N) = \inf_{\#\Lambda = N} \left\| \sum_{n \in \Lambda} x_n \right\| \tag{3}$$

and right democracy function

$$h_r(N) = \sup_{\#\Lambda = N} \left\| \sum_{n \in \Lambda} x_n \right\|. \tag{4}$$

The detailed study of the role of those functions in approximation properties of the basis $(x_n)_{n=1}^{\infty}$ was recently undertaken in [1].

In the rest of this note we will always assume that $(x_n)_{n=1}^{\infty}$ is a lattice unconditional basis i.e.

$$\left\| \sum_{n=1}^{\infty} \lambda_n a_n x_n \right\| \leqslant \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \tag{5}$$

whenever $|\lambda_n| \leqslant 1$. Since every space with an unconditional basis can be renormed so that the basis will satisfy (5) we really consider unconditional bases here. We will use standard Banach space conventions and results, c.f. [6].

Acknowledgements. I would like to express my gratitude to professors C. Cabrelli, G. Garrigós, E. Hernández and U. Molter for sharing their ideas with me and for kind permission to present some of their unpublished results in this paper.

2. Space with nondoubling left democracy function

A positive function $\phi(n)$ defined for $n = 1, 2, \ldots$ is doubling if there exists a C such that $\phi(2n) \leqslant C\phi(n)$ for all n. Such functions appear in many places in analysis. It was observed in [1, Prop. 2.4] that $h_r(N)$ is doubling and that both h_l and h_r are increasing. The question if h_l is always doubling was left open [1, Remark 2.5] and in some results an assumption that h_l is doubling appears.

Now we are ready to state one of the main results of this note

Theorem 2.1. *There exists a Banach space X with the basis $(e_j)_{j=1}^{\infty}$ (satisfying* (5)*) such that the left democracy function h_l of this basis is not doubling.*

We will say that the basis $(x_n)_{n=1}$ is 1-symmetric if for every permutation of indices π and all sequences $(\epsilon_n)_{n=1}$ of numbers with absolute value one and all sequences $(a_n)_{n=1}$ of coefficients we have

$$\left\|\sum_{n=1} a_n x_n\right\| = \left\|\sum_{n=1} \epsilon_n a_n x_{\pi(n)}\right\|. \tag{6}$$

For natural numbers $n \leqslant N$ let $\mathcal{X}(n, N, 2)$ be a Banach space with 1-symmetric basis $(e_\mu)_{\mu=1}^{N}$ such that

$$\left\|\sum_{j\in\Gamma} e_j\right\| = \begin{cases} \sqrt{\#\Gamma} & \text{when } \#\Gamma \leqslant n \\ \sqrt{n} & \text{when } \#\Gamma > n. \end{cases}$$

One example of such a space can be defined as

$$\left\|\sum_{j=1}^{N} x_j e_j\right\| := \sup\left\{\sum_{j\in\Gamma} x_j v_j\right\}$$

where the supremum is taken over all subsets $\Gamma \subset \{1, \ldots, N\}$ of cardinality $\leqslant n$ and all sequences $(v_j)_{j\in\Gamma}$ with $\sum_{j\in\Gamma} |v_j|^2 \leqslant 1$. It is easy to see that it is a norm and the norm of a vector is the ℓ_2 norm of its n biggest (up to absolute value) coefficients. It also immediately follows from the definition that it is 1-symmetric.

Given an increasing sequence of natural numbers a_j for $j = 1, 2, \ldots$ with $a_1 \geqslant 4$ and $\lim_{j\to\infty} a_j = \infty$ we define $n_k = \prod_{j=1}^{k} a_j$. This implies $n_{k+1}/n_k \geqslant 4$.

Now let us define the space

$$\mathcal{X} =: \left(\sum_{k=1}^{\infty} \mathcal{X}(n_k, n_{k+1}, 2)\right)_2 .$$

This space has a natural basis $(e_\mu)_{\mu\in Y}$ where $Y = \bigcup_{k=1}^{\infty} Y_k$ where $\#Y_k = n_{k+1}$ and $\operatorname{span}(e_\mu)_{\mu\in Y_k} = \mathcal{X}(n_k, n_{k+1}, 2)$.

Lemma 2.2. *For the space $\mathcal{X}$ defined above the function $h_l(n)$ is not doubling.*

Proof. We will show that $\sup_n \frac{h_l(2n)}{h_l(n)} = \infty$. Let us take Γ with $\#\Gamma = n_{k+1}$. If $\Gamma = Y_k$ we get $\|\sum_{j\in\Gamma} e_j\| = \sqrt{n_k}$ so $h_l(n_{k+1}) \leqslant \sqrt{n_k}$.

Now let us take Γ with $\#\Gamma = 2n_{k+1}$. We have

$$\#\bigcup_{j=1}^{k} Y_j = n_2 + n_3 + \cdots + n_k + n_{k+1} \tag{7}$$

$$\leqslant n_{k+1}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^{k-1}}\right) \leqslant \frac{4}{3} n_{k+1}. \tag{8}$$

This means that at least $\frac{2}{3}n_{k+1}$ elements from Γ are in $\bigcup_{j=k+1}^{\infty} Y_j$. Let Γ^1 be a fixed set of such elements with $\frac{2}{3}n_{k+1} \leqslant \#\Gamma^1 \leqslant n_{k+1}$ and let us write $\Gamma^1 = \bigcup_{s=k+1}^{\infty} A_s$ where $A_s = \Gamma^1 \cap Y_s$. Since each of A_s's has at most n_{k+1} elements we get

$$\left\|\sum_{j\in\Gamma} e_j\right\| \geqslant \left\|\sum_{j\in\Gamma^1} e_j\right\| = \sqrt{\sum_{s=k+1}^{\infty}\left\|\sum_{j\in A_s} e_j\right\|^2} \tag{9}$$

$$= \sqrt{\sum_{s=k+1}^{\infty} \#A_s} = \sqrt{\#\Gamma^1} \geqslant \sqrt{\frac{2}{3}n_{k+1}}. \tag{10}$$

So $h_l(2n_{k+1}) \geqslant \sqrt{\frac{2}{3}n_{k+1}}$ and we get

$$\frac{h_l(2n_{k+1})}{h_l(n_{k+1})} \geqslant \frac{\sqrt{\frac{2}{3}n_{k+1}}}{\sqrt{n_k}} = \sqrt{\frac{2}{3}}\sqrt{a_{k+1}}.$$

Since a_k tends to infinity we get the claim. ■

Remark 2.1. A more careful analysis should show that $h_l(n)$ is exactly equal to the norm of the sum of the first n unit vectors.

Remark 2.2. Clearly we can use other values of p in place of 2.

3. Approximation spaces

It is standard in approximation theory to define spaces of elements which admit some rate of approximation. In our context two spaces are esential. We define them for a fixed Banach space with the basis $(x_n)_{n=1}^{\infty}$.

1. Non-linear approximation space $\mathcal{A}_q^\alpha$ with $\alpha > 0$ and $0 < q < \infty$ defined as

$$\mathcal{A}_q^\alpha = \left\{ x \in X \ : \ \|x\|_{\mathcal{A}_q^\alpha} = \|x\| + \left[\sum_{N=1}^{\infty} (N^\alpha \sigma_N(x))^q \tfrac{1}{N}\right]^{1/q} < \infty \right\}$$

   and for $q = \infty$ we define

$$\mathcal{A}_\infty^\alpha = \left\{ x \in X \ : \ \|x\|_{\mathcal{A}_q^\alpha} = \|x\| + \sup_{N\geqslant 1} N^\alpha \sigma_N(x) < \infty \right\}$$

   where $\sigma_N(x)$ is the error of the best N-term approximation i.e.

$$\sigma_N(x) = \inf\left\{ \|x - \sum_{n\in\Lambda} b_n x_n\| \ : \ \#\Lambda = N \text{ and } b_n\text{'s are arbitrary} \right\}. \tag{11}$$

2. Greedy classes $\mathcal{G}_q^\alpha$ are defined in the same way but we replace $\sigma_N(x)$ by error of a greedy approximation which is defined as $\gamma_N(x) = \max \|x - \mathcal{G}_N(x)\|$. The maximum is taken over all $\mathcal{G}_N(x)$'s in case it is not uniquely defined.

It is well known that $\mathcal{A}_q^\alpha$ are quasi-Banach spaces with the quasi-norm $\|.\|_{\mathcal{A}_q^\alpha}$. For the spaces $\mathcal{G}_q^\alpha$ the situation is not so clear–we do not know if it is a linear space. Clearly if the basis is greedy then $\sigma_N \sim \gamma_N$ and the spaces are equal. Also, since always $\sigma_N(x) \leqslant \gamma_N(x)$, we have $\mathcal{G}_q^\alpha \subset \mathcal{A}_q^\alpha$. The problem whether the equality $\mathcal{G}_q^\alpha = \mathcal{A}_q^\alpha$ characterise greedy bases was considered in [1]. Actually it turned out to be quite difficult so the authors considered the problem of equivalence of quantities $\|x\|_{\mathcal{A}_q^\alpha}$ and $\|x\|_{\mathcal{G}_q^\alpha}$. Let us say that *greedy approximation is optimal*[1] for α and q if there exists a constant C such that for every $x \in \mathcal{A}_q^\alpha$ we have

$$\|x\|_{\mathcal{G}_q^\alpha} \leqslant C\|x\|_{\mathcal{A}_q^\alpha}.$$

The main result of this section is the following

Theorem 3.1. *If* (x_n) *in unconditional, the following are equivalent*

1. (x_n) *is democratic*
2. $\|x - \mathcal{G}_N(x)\| \leqslant C\sigma_N(x)$ *for all* x
3. $\|x\|_{\mathcal{G}_q^\alpha} \leqslant C\|x\|_{\mathcal{A}_q^\alpha}$ *for all (some)* $\alpha, q > 0$.

Remark 3.3. This Theorem for bases with doubling h_l was proved by C. Cabrelli, G. Garrigós, E. Hernández and U. Molter and stated without proof in a note *Added in proof* in [1]. Below I present their proof with their kind permission.

Proof. That for unconditional bases 1. is equivalent to 2. was proved by Konyagin–Temlyakov [3]. 2. ⇒ 3. is clear and was already mentioned above. We will prove that for a non-greedy unconditional basis 3. fails. We will distinguish two cases: when h_l is doubling and when h_l is not doubling. To prove the first case we need to recall Proposition 7.1 from [1]

Proposition 3.2. *Suppose that there exist integers* $n_\mu \geqslant k_\mu \geqslant 1$ *for* $\mu = 1, 2, \ldots$ *such that*

$$\lim_{\mu\to\infty} \frac{n_\mu}{k_\mu} = \infty \qquad \text{and} \qquad \frac{h_r(k_\mu)}{h_l(n_\mu)} \geqslant C\left(\frac{n_\mu}{k_\mu}\right)^\alpha \tag{12}$$

for some $C > 0$ *and* $\alpha > 0$. *Then greedy approximation is not optimal for* α *and any* $q \in (0, \infty]$.

Lemma 3.3 (C. Cabrelli, G. Garrigós, E. Hernández, U. Molter). *Let* $\alpha > 0$ *and* $h_r, h_l : \mathbb{N} \to (0, \infty)$ *be* any *two increasing functions such that* h_l *is doubling and* $\limsup_{\mu\to\infty} \frac{h_r(\mu)}{h_l(\mu)} = \infty$. *Then there exists integers* $n_\mu \geqslant k_\mu \geqslant 1$ *for* $\mu = 1, 2, \ldots$ *such that (12) holds.*

[1] In [1] this notion was expressed as "the inclusion $\mathcal{A}_q^\alpha \hookrightarrow \mathcal{G}_q^\alpha$ holds".

Proof. We easily see that there exists an increasing sequence of integers $\{w_\mu\}_{\mu=1}^\infty$ such that

$$\lim_{\mu\to\infty} h_r(w_\mu)/h_l(w_\mu) = \infty. \tag{13}$$

Given w_μ we fix an integer $r(\mu)$ such that $2^{r(\mu)-1} \leqslant w_\mu < 2^{r(\mu)}$. Since h_l is doubling, for any $M, \mu \in \mathbb{N}$ we have

$$h_l(w_\mu M) \leqslant h_l(2^{r(\mu)}M) \leqslant C^{r(\mu)} h_l(M). \tag{14}$$

Using (13) we fix an increasing sequence $(k_\mu)_{\mu=1}^\infty$ such that each k_μ is some $w_{\mu'}$ such that

$$\frac{h_r(k_\mu)}{h_l(k_\mu)} \geqslant C^{r(\mu)} w_\mu^\alpha \tag{15}$$

and we define $n_\mu = w_\mu k_\mu$, so the first part of (12) holds . Using (14) and (15), we obtain

$$\frac{h_r(k_\mu)}{h_l(n_\mu)} = \frac{h_r(k_\mu)}{h_l(w_\mu k_\mu)} \geqslant \frac{h_r(k_\mu)}{C^{r(\mu)} h_l(k_\mu)} \geqslant w_\mu^\alpha = \left(\frac{n_\mu}{k_\mu}\right)^\alpha \qquad \blacksquare$$

To settle the first case we note that a non-greedy basis with doubling h_l satisfies the assumptions of Lemma 3.3 so using Proposition 3.2 we get the claim.

Now let us assume that we have a normalised, 1–unconditional basis $(e_j)_{j=1}^\infty$ with the function $h_l(n)$ *not doubling.* For each s there exists n_s such that $h_l(2n_s) \geqslant (s+1)h_l(n_s)$. For simplicity in what follows we will write $\|S\| = \|\sum_{j\in S} e_j\|$. Let us fix a set M_s such that $\#M_s = n_s$ and $\|M_s\| \geqslant h_l(n_s) \geqslant \|M_s\| - \frac{1}{s+1}$. Then for any set D disjoint from M_s with $\#D = n_s$ we have

$$\|D\| + \|M_s\| \geqslant \|M_s \cup D\| \geqslant h_l(2n_s) \geqslant (s+1)h_l(n_s) \geqslant (s+1)\Big(\|M_s\| - \frac{1}{s+1}\Big)$$

so for every such D we have $\|D\| \geqslant s\|M_s\| - 1 \geqslant s h_l(n_s) - 1$.

Note that h_l is unbounded (because bounded is doubling).

Given M_s let us take $r =: \lfloor\sqrt{s}\rfloor$ disjoint sets V_j also disjoint with M_s, such that $\#\bigcup_{j=1}^r V_j = n_s$ each of cardinality $\lfloor n_s/r\rfloor$ or $\lceil n_s/r\rceil$. Denote the set V_j with the biggest $\|V_j\|$ as V^s. Since $r\max\|V_j\| \geqslant \|\bigcup_{j=1}^r V_j\| \geqslant s\|M_s\| - 1$ we see that

$$\|V^s\| \geqslant \frac{s}{r}\|M_s\| - \frac{1}{r}. \tag{16}$$

Put $x_s = \sum_{V^s} e_j + 2\sum_{M_s} e_j$. We have $\|x_s\| \leqslant \|V^s\| + 2\|M_s\| \leqslant 3\|V^s\|$. The number of non-zero coefficients of x_s equals $\#V_s + \#M_s \leqslant 2\#M_s$. In what follows we are only interested in $k \leqslant 2\#M_s$ because for $k > 2\#M_s$ we have $\mathcal{G}_k(x_s) = x_s$ and $\sigma_k(x_s) = 0$.

For $k \leqslant \#M_s$ we have

$$\|x_s - \mathcal{G}_k(x_s)\| \geqslant \|V^s\|$$

so for $0 < q < \infty$ we have

$$\|x_s\|_{\mathcal{G}_q^\alpha} \geqslant \left[\sum_{k=1}^{\#M_s} (k^\alpha \|x_s - \mathcal{G}_k(x_s)\|)^q \frac{1}{k}\right]^{1/q} \geqslant C\|V^s\|(\#M_s)^\alpha. \tag{17}$$

and for $q = \infty$ we have

$$\|x_s\|_{\mathcal{G}_\infty^\alpha} \geqslant \max_{k \leqslant \#M_s} k^\alpha \|x - \mathcal{G}_k(x)\| \geqslant (\#M_s)^\alpha \|V_s\|. \tag{18}$$

On the other hand for $k \geqslant \#V^s$ using (16) we have

$$\sigma_k(x_s) \leqslant 2\|M_s\| \leqslant 2\frac{r\|V^s\| + 1}{s} \leqslant 3\frac{r\|V^s\|}{s} \tag{19}$$

and for $k < \#V^s$

$$\sigma_k(x_s) \leqslant \|x_s\| \leqslant 3\|V^s\|. \tag{20}$$

Therefore using (20) and (19), for $q < \infty$ we have

$$\begin{aligned}
\|x_s\|_{\mathcal{A}_q^\alpha} &= \|x_s\| + \left[\sum_{k=1}^{2\#M_s} (k^\alpha \sigma_k(x_s))^q \frac{1}{k}\right]^{1/q} \\
&\leqslant 3\|V^s\| + \left[(3\|V^s\|)^q \sum_{k=1}^{\#V^s-1} k^{q\alpha-1} + \left(3\frac{r\|V^s\|}{s}\right)^q \sum_{k=\#V^s}^{2\#M_s} k^{q\alpha-1}\right]^{1/q} \\
&\leqslant 3\|V^s\| + [C\|V^s\|^q(\#V^s)^{q\alpha} + C(r/s)^q\|V^s\|^q(\#M_s)^{q\alpha}]^{1/q} \\
&\leqslant C(\#M_s)^\alpha\|V^s\| \left(r^{-q\alpha} + (r/s)^q\right)^{1/q} \\
&\leqslant C\|x_s\|_{\mathcal{G}_q^\alpha} \left(r^{-q\alpha} + (r/s)^q\right)^{1/q} \\
&\leqslant C\|x_s\|_{\mathcal{G}_q^\alpha} \left(s^{-q\alpha/2} + s^{-q/2}\right)^{1/q}.
\end{aligned} \tag{21}$$

Analogously for $q = \infty$ we have

$$\begin{aligned}
\|x_s\|_{\mathcal{A}_\infty^\alpha} &= \|x_s\| + \sup_{k \geqslant 1} k^\alpha \sigma_k(x_s) \\
&\leqslant 3\|V^s\| + \max_{k < \#V^s} 3k^\alpha \|V^s\| + \max_{\#V_s \leqslant k \leqslant 2\#M_s} 3k^\alpha \frac{r\|V^s\|}{s} \\
&\leqslant \|V^s\| \left(3 + 3(\#V^s)^\alpha + 3(2\#M_s)^\alpha r/s\right) \\
&\leqslant C\|x_s\|_{\mathcal{G}_\infty^\alpha} \left((\#M_s)^{-\alpha} + s^{-\alpha/2} + s^{-1/2}\right).
\end{aligned} \tag{22}$$

Since s is arbitrary, from (21) and (22) we infer that the greedy approximation is not optimal for any α and q also in the nondoubling case. ■

References

[1] G. Garrigós, E. Hernández and M. De Natividade, *Democracy functions and optimal embeddings for approximation spaces*, Adv. Comput. Math. **37**(2) (2012), 255–283.

[2] A. Kamont, V.N. Temlyakov, *Greedy Approximation and the multivariate Haar system*, Studia Math. **161**(3) (2004), 199–223.

[3] S.V. Konyagin, V.N. Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approximation **5** (1999), 365–379.

[4] V.N. Temlyakov, *The best m-term approximation and greedy algorithms*, Advances in Computational Mathematics **8** (1998), 249–265.

[5] P. Wojtaszczyk, *Greedy algorithm for biorthogonal general systems*, J. Approx. Theory **107** (2000), 293–314.

[6] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge University Press, Cambridge 1991

Addresses: Przemysław Wojtaszczyk: Interdisciplinary Centre for Mathematical and Computational Modelling, University of Warsaw, 02-838 Warszawa, ul. Prosta 69, Poland; and Institut of Mathematics, Polish Academy of Sciences
00-956 Warszawa, ul. Ŝniadeckich 8, Poland.

E-mail: wojtaszczyk@icm.edu.pl

Received: 27 February 2013; **revised:** 12 April 2013