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INHOMOGENEOUS QUADRATIC CONGRUENCES

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Abstract: For given positive integers a, b, q we investigate the density of solutions $(x, y) \in \mathbb{Z}^2$ to congruences $ax + by^2 \equiv 0 \bmod q$.

Keywords: quadratic congruences, Manin's conjecture, Gauss sums.

1. Introduction

Let a, b, q be non-zero integers with $q \geqslant 1$ and $(ab, q) = 1$. Let e, f be coprime positive integers with $e \neq f$ and let $X, Y \geqslant 1$. A broad array of problems in number theory can be reduced to estimating the number of solutions $(x, y) \in \mathbb{Z}^2$ to congruences of the shape

$$ax^e + by^f \equiv 0 \bmod q,$$

with $0 < x \leqslant X$ and $0 < y \leqslant Y$. It is often convenient to focus on those solutions which are coprime to q. Let $M_{e,f}(X, Y; a, b, q)$ denote the total number of such solutions. A trivial upper bound is given by

$$M_{e,f}(X, Y; a, b, q) \ll q^{\varepsilon} \left(\frac{XY}{q} + \min\{X, Y\} \right),$$

for any $\varepsilon > 0$. Here the implied constant is allowed to depend at most upon the choice of ε, and upon the exponents e and f, a convention that we adhere to for the remainder of this work. One is usually concerned with situations for which either of the ranges X or Y is substantially smaller than the modulus q, where sharper estimates are sought.

This paper is inspired by work of Pierce [11], together with our own recent contribution [1] to the topic. In [11, Theorem 3], under the assumption that q is

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square-free and $\max\{X, 2Y\} \leqslant q$, it is shown that there is a positive constant A such that

$$M_{e,f}(X, Y; 1, -1, q) \ll \tau(q)^A \left(\frac{XY}{q} + \frac{X}{\sqrt{q}} + \sqrt{q}\log^2 2q\right), \tag{1.1}$$

where τ is the divisor function. This estimate is used by Pierce to obtain a non-trivial bound for the 3-part $h_3(D)$ of the class number of a quadratic number field $\mathbb{Q}(\sqrt{D})$, when $|D|$ admits a divisor of suitable magnitude. In [1] a substantial improvement is obtained when $(e, f) = (2, 3)$ and q is far from being square-free. This in turn is used to study the density of elliptic curves with square-free discriminant and to count $\mathbb{Q}$-rational points on some singular del Pezzo surfaces. The goal of this paper is to undertake a careful analysis of the easier quantity $M_{1,2}(X, Y; a, b, q)$ with a view to providing a useful technical tool in future resolutions of the Manin conjecture for singular del Pezzo surfaces.

The above investigations of $M_{e,f}(X, Y; a, b, q)$ use the orthogonality of additive characters to encode the divisibility condition in the congruence. The resulting complete exponential sums can be estimated using the Weil bound when the modulus is square-free. When $(e, f) = (1, 2)$ the exponential sums that arise are particularly simple to handle, being quadratic Gauss sums. We will establish the following refinement of (1.1).

Theorem 1. *Let a, b, q be non-zero integers with $q \geqslant 1$ and $(ab, q) = 1$ and let $X, Y \geqslant 1$. Then we have*

$$\begin{aligned} M_{1,2}(X, Y; a, b, q) = &\frac{\varphi(q)}{q^2} \cdot XY \\ &+ O\left(\frac{X}{q} \cdot \tau(q) + L(q)\sigma_{-1/2}(q)\left(\frac{Y}{\sqrt{q}} \cdot \tau(q) + \sqrt{q}L(q)\right)\right), \end{aligned}$$

where $L(n) := \log(n+1)$, $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ and φ is the Euler totient function.

In recent years there has emerged a particularly fruitful approach to the Manin conjecture for singular del Pezzo surfaces X defined over $\mathbb{Q}$. There are two basic stages:

— one constructs an explicit bijection between rational points of bounded height on X and integral points in a region on a universal torsor $\mathcal{T}_X$; and
— one estimates the number of integral points in this region on the torsor by its volume and shows that the volume has the predicted asymptotic growth rate.

A geometrically driven approach to the first part has been developed by Derenthal and Tschinkel [4, §4]. The second part mainly relies on analytic number theory and has been put on a general footing by Derenthal [3], whenever the torsor is a hypersurface. In this case the torsor equation typically takes the form

$$\alpha_0^{a_0}\alpha_1^{a_1}\cdots\alpha_i^{a_i} + \beta_0^{b_0}\beta_1^{b_1}\cdots\beta_j^{b_j} + \gamma_0\gamma_1^{c_1}\cdots\gamma_k^{c_k} = 0, \tag{1.2}$$

with $(a_0,\dots,a_i)\in\mathbb{N}^{i+1}$, $(b_0,\dots,b_j)\in\mathbb{N}^{j+1}$ and $(c_1,\dots,c_k)\in\mathbb{N}^k$. Work of Hassett [7, Theorem 5.7] shows that there is a natural realisation of a universal torsor as an open subset via $\mathcal{T}_X\hookrightarrow \mathrm{Spec}(\mathrm{Cox}(\widetilde{X}))$, where the coordinates of $\mathcal{T}_X$ correspond to generators of the Cox ring of the minimal desingularisation $\widetilde{X}$ of X. Torsor equations such as (1.2) are usually handled by viewing them as a congruence modulo $q=\gamma_1^{c_1}\cdots\gamma_k^{c_k}$. An example of this is provided by our work [1] on $M_{2,3}(X,Y;a,b,q)$, which is pivotal in the resolution of the Manin conjecture for a singular del Pezzo surface of degree 2. Experience suggests that there are several examples of singular del Pezzo surfaces whose torsor equations produce congruences of the shape

$$ru^lx+sv^my^2=0 \mod tw,$$

for fixed $l,m\in\mathbb{N}$. A case in point is the cubic surface with $\mathbf{D}_5$ singularity type which is studied jointly by the first author and Derenthal [2]. Here the relevant congruence that emerges is precisely of this form with $l=2$ and $m=1$. Using a result of similar strength to Theorem 1 the Manin conjecture is established for this surface but only with a modest logarithmic saving in the error term.

Returning to the counting function $M_{1,2}(X,Y;a,b,q)$, one might hope to do better by averaging the counting function over suitably constrained values of a,b and q. That such an approach is available to us follows from the fact that the underlying quadratic Gauss sums arising in the proof of Theorem 1 satisfy explicit formulae. This will allow us to study quite general expressions of the form

$$\mathcal{S}:=\sum_{(a,b,q)\in S}c_{a,b,q}\sum_{\substack{y\in J\\(y,q)=1}}\sum_{\substack{x\in I(a,b,q,y)\\ax+by^2\equiv 0 \bmod q}}1, \tag{1.3}$$

for $c_{a,b,q}\in\mathbb{C}$. Here $S\subset\mathbb{Z}^2\times\mathbb{N}$ is a finite set of triples (a,b,q) such that $(ab,q)=1$, $J=(y_0,y_0+Y]$ is a fixed interval of length $Y\geqslant 1$, and

$$I(a,b,q,y)=\big(f^-(a,b,q,y),f^+(a,b,q,y)\big] \tag{1.4}$$

is an interval depending on a,b,q,y. Theorem 1 will be an easy consequence of a general estimate for $\mathcal{S}$, which is presented in §3. There are two main ingredients at play here: Vaaler's trigonometric formula for the saw-tooth function $\psi(x):=\{x\}-1/2$, where $\{x\}=x-[x]$ denotes the fractional part of x, and the explicit formulae for the quadratic Gauss sum. These will be recalled in §2.

When further restrictions are placed on S and $c_{a,b,q}$ one can go even further. Motivated by our discussion above we set

$$S=\left\{(ru^l,sv^m,tw)\ :\ U<u\leqslant 2U,\ V<v\leqslant 2V,\ W<w\leqslant 2W,\ (rsuv,tw)=1\right\}, \tag{1.5}$$

where $U,V,W\geqslant 1/2$ and l,m,r,s,t are fixed non-zero integers for which $l,m,t\geqslant 1$ and $(rs,t)=1$. We shall think of r,s,t as being parameters, whose dependence we want to keep track of, but l and m are fixed once and for all. We further assume that $c_{a,b,q}$ factorises in the form

$$c_{a,b,q} = c_{ru^l,sv^m,tw} = d_{u,v}e_w, \qquad \text{with } |d_{u,v}|, |e_w| \leqslant 1. \tag{1.6}$$

We also entertain the possibility that there is a further factorisation

$$d_{u,v} = d'_u\tilde{d}_v, \qquad \text{with } |d'_u|, |\tilde{d}_v| \leqslant 1. \tag{1.7}$$

Moreover, we set

$$\tilde{f}^{\pm}(u,v,w,y) := f^{\pm}(ru^l, sv^m, tw, y).$$

We make the assumption that $\tilde{f}^{\pm}(u,v,w,y)$ are continuous functions and have piecewise continuous partial derivatives with respect to the variables u,v,w. We further assume that $\tilde{f}^+ \geqslant \tilde{f}^-$ in the whole domain $(U,2U]\times(V,2V]\times(W,2W]\times J$, with

$$\left|\frac{\partial^{i+j+k}\tilde{f}^{\pm}}{\partial u^i\partial v^j\partial y^k}(u,v,w,y)\right| \leqslant \rho^i\sigma^j\tau^k F \tag{1.8}$$

there, for $i,j,k \in \{0,1\}$ such that $i+j+k \neq 0$, where ρ,σ,τ,F are suitable non-negative numbers. For any $H > 0$ we set

$$\Delta_H := \left(1 + \frac{HF\rho U}{tW}\right)\left(1 + \frac{HF\sigma V}{tW}\right)\left(1 + \frac{HF\tau Y}{tW}\right) \tag{1.9}$$

and

$$\mathcal{Z} := \begin{cases} (tW+U)^{1/2}(tW+V)^{1/2}(UV)^{1/2}W, & \text{if (1.7) holds and } UV \geqslant tW, \\ (tW)^{1/2}UVW, & \text{in general.} \end{cases}$$

We may now record the outcome of our analysis of the sum $\mathcal{S}$ in (1.3) in this setting.

Theorem 2. *Let $\varepsilon > 0$ and assume that*

$$H \geqslant \frac{tW}{F}. \tag{1.10}$$

Then under the above hypotheses we have

$$\mathcal{S} = \sum_{U<u\leqslant 2U}\ \sum_{\substack{V<v\leqslant 2V \\ (rsuv,tw)=1}}\ \sum_{W<w\leqslant 2W} \frac{d_{u,v}e_w}{tw} \sum_{\substack{y_0<y\leqslant y_0+Y \\ (y,tw)=1}} \tilde{X}(u,v,w,y)$$
$$+ O\left(\frac{UVWY}{H}\right) + O(\mathcal{T}),$$

where $\tilde{X}(u,v,w,y) := \tilde{f}^+(u,v,w,y) - \tilde{f}^-(u,v,w,y)$ and

$$\mathcal{T} := \Delta_H\left(\frac{Y}{(tW)^{1/2}}\left(U^{1-\{l/2\}}V^{1-\{m/2\}}W + UVW^{1/2}\right) + \mathcal{Z}\right)(HtUVW)^{\varepsilon}.$$

Theorem 2 will be established in §4. The character sums that arise from the explicit formulae for Gauss sums used in Theorem 1 are handled using a mixture of the ordinary large sieve and the large sieve for real characters developed by Heath-Brown [8]. A review of favourable conditions under which the main term dominates the error term in Theorem 2 is saved for §4.3.

2. Technical tools

In this section we collect together the technical lemmas that will feature in our proof of Theorems 1 and 2. We will use the following approximation of the function $\psi(x)$ using trigonometric polynomials due to Vaaler (see Graham and Kolesnik [6, Theorem A.6], for example).

Lemma 1. *Let $H > 0$. Then there exist coefficients $a_h \in \mathbb{R}$ satisfying $a_h \ll 1/|h|$, such that*

$$\left|\psi(x) - \sum_{1 \leqslant |h| \leqslant H} a_h \mathrm{e}(hx)\right| \leqslant \frac{1}{H+1} \sum_{|h| \leqslant H} \left(1 - \frac{|h|}{H+1}\right) \mathrm{e}(hx).$$

This result will lead to the intervention of exponential sums, which once evaluated will also produce certain types of character sums. To handle these we will require the following variant of Heath-Brown's large sieve for real characters [8, Corollary 4].

Lemma 2. *Let $\varepsilon > 0$, let $M, N \in \mathbb{N}$, and let $a_1, ..., a_M$ and $b_1, ..., b_N$ be arbitrary complex numbers satisfying $|a_m|$, $|b_n| \leqslant 1$. Then*

$$\sum_{\substack{m \leqslant M \\ (m,2)=1}} \sum_{n \leqslant N} a_m b_n \left(\frac{n}{m}\right) \ll (MN)^{\varepsilon} \left(MN^{1/2} + M^{1/2}N\right).$$

We end this section with an explicit evaluation of the quadratic Gauss sum

$$\mathcal{G}(s,t;u) := \sum_{n=1}^{u} \mathrm{e}\left(\frac{sn^2 + tn}{u}\right), \tag{2.1}$$

for given non-zero integers s, t, u such that $u \geqslant 1$. Let

$$\delta_n := \begin{cases} 0, & \text{if } n \equiv 0 \bmod 2, \\ 1 & \text{if } n \equiv 1 \bmod 2, \end{cases} \qquad \epsilon_n := \begin{cases} 1, & \text{if } n \equiv 1 \bmod 4, \\ i, & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

The next lemma gives the value of $\mathcal{G}(s,t;u)$ if $(s,u) = 1$.

Lemma 3. *Suppose that* $(s,u)=1$. *Then we have the following.*

(i) *If* u *is odd, then*

$$\mathcal{G}(s,t;u)=\epsilon_u\sqrt{u}\left(\frac{s}{u}\right)\mathrm{e}\left(-\frac{\overline{4s}t^2}{u}\right). \tag{2.2}$$

(ii) *If* $u=2v$ *with* v *odd, then*

$$\mathcal{G}(s,t;u)=2\delta_t\epsilon_v\sqrt{v}\left(\frac{2s}{v}\right)\mathrm{e}\left(-\frac{\overline{8s}t^2}{v}\right). \tag{2.3}$$

(iii) *If* $4\mid u$, *then*

$$\mathcal{G}(s,t;u)=(1+i)\epsilon_s^{-1}(1-\delta_t)\sqrt{u}\left(\frac{u}{s}\right)\mathrm{e}\left(-\frac{\overline{s}t^2}{4u}\right). \tag{2.4}$$

Proof. (i) Let u be odd and assume $(s,u)=1$. Then, by Lemmas 3 and 9 in [5], we have

$$\mathcal{G}(s,t;u)=\mathrm{e}\left(-\frac{\overline{4s}t^2}{u}\right)\left(\frac{s}{u}\right)\mathcal{G}(1,0;u).$$

Gauss proved (see Nagell [10, Theorem 99], for example) that

$$\mathcal{G}(1,0;n)=\begin{cases}(1+i)\sqrt{n}, & \text{if } n\equiv 0 \bmod 4,\\ \sqrt{n}, & \text{if } n\equiv 1 \bmod 4,\\ 0, & \text{if } n\equiv 2 \bmod 4,\\ i\sqrt{n}, & \text{if } n\equiv 3 \bmod 4,\end{cases} \tag{2.5}$$

from which (2.2) follows.

(ii) Let $2\|u$ and assume $(s,u)=1$. Write $u=2v$ and note that $2\nmid v$. If $2\mid t$ then

$$\mathcal{G}(s,t;2v)=\mathrm{e}\left(-\frac{\overline{s}t^2}{4u}\right)\mathcal{G}(s,0;2v)=0$$

by Lemmas 4 and 9 in [5]. If $2\nmid t$, then

$$\mathcal{G}(s,t;2v)=2\mathrm{e}\left(-\frac{\overline{8s}t^2}{v}\right)\mathcal{G}(2s,0;v)$$

by Lemma 6 in [5]. Now applying (2.2) gives (2.3).

(iii) Let $4\mid u$ and assume $(s,u)=1$. If $2\nmid t$, then $\mathcal{G}(s,t;u)=0$ by Lemma 5 in [5]. Assume that $2\mid t$. Then, by Lemma 4 in [5], we have

$$\mathcal{G}(s,t;u)=\mathrm{e}\left(-\frac{\overline{s}t^2}{4u}\right)\mathcal{G}(s,0;u).$$

For $(s,u)=1$, the Gauss sum satisfies the reciprocity law

$$\mathcal{G}(s,0;u)\mathcal{G}(u,0;s)=\mathcal{G}(1,0;su).$$

Noting that s is odd and $4\mid su$, and applying (2.2) to $\mathcal{G}(u,0;s)$ and (2.5) to $\mathcal{G}(1,0;su)$, we deduce (2.4). ∎

3. Analysis of $\mathcal{S}$

In this section we begin in earnest our investigation of the sum $\mathcal{S}$ presented in (1.3). Recall that $c_{a,b,q}$ are arbitrary complex numbers and $S \subset \mathbb{Z}^2 \times \mathbb{N}$ is a finite set of triples (a,b,q) such that $(ab,q)=1$, with $J := (y_0, y_0+Y]$ and $I(a,b,q,y)$ given by (1.4). We henceforth stipulate that

$$\mathrm{domain}(f^+) = \mathrm{domain}(f^-) = \mathcal{R},$$

where

$$\mathcal{R} = (a_0, a_0+A] \times (b_0, b_0+B] \times (q_0, q_0+Q] \times (y_0, y_0+Y] \tag{3.1}$$

is a half-open cuboid in $\mathbb{R}^4$ such that $S \times J \subset \mathcal{R}$. We further suppose that $f^{\pm}(a,b,q,y)$ are continuous, have piecewise continuous partial derivatives with respect to the variables a,b,y, and satisfy $f^+ \geqslant f^-$ in the whole domain $\mathcal{R}$. Moreover, we set

$$X(a,b,q,y) := |I(a,b,q,y)| = f^+(a,b,q,y) - f^-(a,b,q,y).$$

Our first step is to rewrite the congruence $ax+by^2 \equiv 0 \bmod q$ in $\mathcal{S}$ as

$$x + \overline{a}by^2 \equiv 0 \bmod q,$$

where $\overline{a}$ denotes the multiplicative inverse of a modulo q. It follows that

$$\begin{aligned}\sum_{\substack{x\in I(a,b,q,y)\\ ax+by^2\equiv 0 \bmod q}} 1 &= \left[\frac{f^+(a,b,q,y)}{q} + \frac{\overline{a}by^2}{q}\right] - \left[\frac{f^-(a,b,q,y)}{q} + \frac{\overline{a}by^2}{q}\right] \\ &= \frac{X(a,b,q,y)}{q} - \psi\left(\frac{f^+(a,b,q,y)}{q} + \frac{\overline{a}by^2}{q}\right) \\ &\quad + \psi\left(\frac{f^-(a,b,q,y)}{q} + \frac{\overline{a}by^2}{q}\right).\end{aligned}$$

We may therefore write

$$\mathcal{S} = \mathcal{M} - \mathcal{E}^+ + \mathcal{E}^-, \tag{3.2}$$

where

$$\mathcal{M} := \sum_{(a,b,q)\in S} \frac{c_{a,b,q}}{q} \sum_{\substack{y\in J\\ (y,q)=1}} X(a,b,q,y) \tag{3.3}$$

is the main term and

$$\mathcal{E}^{\pm} := \sum_{(a,b,q)\in S} c_{a,b,q} \sum_{\substack{y\in J\\ (y,q)=1}} \psi\left(\frac{f^{\pm}(a,b,q,y)}{q} + \frac{\overline{a}by^2}{q}\right)$$

are error terms. The next result is an easy consequence of Lemma 1 and transforms these error terms into exponential sums.

Lemma 4. *Let $H > 0$. Then we have $|\mathcal{E}^{\pm}| \ll \mathcal{E} + \mathcal{F}^{\pm}$, where*

$$\mathcal{E} := \frac{Y}{H} \sum_{(a,b,q)\in S} |c_{a,b,q}|, \tag{3.4}$$

$$\mathcal{F}^{\pm} := \sum_{1\leqslant h\leqslant H} \frac{1}{h} \left| \sum_{(a,b,q)\in S} C_{a,b,q} S_h^{\pm}(a,b,q) \right|, \tag{3.5}$$

with $C_{a,b,q} := c_{a,b,q} + |c_{a,b,q}|$ and

$$S_h^{\pm}(a,b,q) := \sum_{\substack{y\in J\\ (y,q)=1}} \mathrm{e}\left(h \cdot \frac{f^{\pm}(a,b,q,y)}{q}\right) \mathrm{e}\left(h \cdot \frac{\overline{a}by^2}{q}\right).$$

We proceed to reduce our exponential sums $S_h^{\pm}(a,b,q)$ to complete quadratic Gauss sums. First we remove the factor $\mathrm{e}\,(h \cdot f^{\pm}(a.b,q,y)/q)$ using partial summation, obtaining

$$\begin{aligned} S_h^{\pm}(a,b,q) = {} & \mathrm{e}\left(h \cdot \frac{f^{\pm}(a,b,q,y_0+Y)}{q}\right) T_h(a,b,q,y_0+Y) \\ & - \frac{2\pi i h}{q} \int_{y_0}^{y_0+Y} \left(\frac{\partial}{\partial t} f^{\pm}(a,b,q,t)\right) \mathrm{e}\left(h \cdot \frac{f^{\pm}(a,b,q,t)}{q}\right) T_h(a,b,q,t)\mathrm{d}t, \end{aligned}$$

where

$$T_h(a,b,q,t) := \sum_{\substack{y_0<y\leqslant t\\ (y,q)=1}} \mathrm{e}\left(h \cdot \frac{\overline{a}by^2}{q}\right).$$

Next we remove the coprimality condition $(y,q)=1$ using Möbius inversion, getting

$$T_h(a,b,q,t) := \sum_{e|q} \mu(e) \sum_{y_0/e<y\leqslant t/e} \mathrm{e}\left(he \cdot \frac{\overline{a}by^2}{q/e}\right).$$

We remove common factors by writing

$$q' = \frac{q/e}{(he,q/e)}, \qquad h' = \frac{he}{(he,q/e)} \tag{3.6}$$

and observing that

$$T_h(a,b,q,t) = \sum_{e|q} \mu(e) \sum_{y_0/e<y\leqslant t/e} \mathrm{e}\left(\frac{h'\overline{a}by^2}{q'}\right),$$

with $(h', q') = 1$. Here we note that q' and h' depend on e, q and h. The inner sum is an incomplete quadratic Gauss sum which we complete by writing

$$\sum_{y_0/e<y\leqslant t/e} \mathrm{e}\left(\frac{h'\overline{a}by^2}{q'}\right) = \sum_{n=1}^{q'} \mathrm{e}\left(\frac{h'\overline{a}bn^2}{q'}\right) \cdot \frac{1}{q'} \cdot \sum_{k=1}^{q'} \sum_{y_0/e<l\leqslant t/e} \mathrm{e}\left(k \cdot \frac{n-l}{q'}\right)$$
$$= \frac{1}{q'} \cdot \sum_{k=1}^{q'} r_e(k, q'; t)\mathcal{G}(h'\overline{a}b, k; q'),$$

where $\mathcal{G}(h'\overline{a}b, k; q')$ is given by (2.1) and

$$r_e(k, q'; t) := \sum_{y_0/e<l\leqslant t/e} \mathrm{e}\left(-\frac{kl}{q'}\right) \ll \min\left\{Y/e, \|k/q'\|^{-1}\right\},$$

if $y_0 \leqslant t \leqslant y_0 + Y$.

Let

$$g_h^{\pm}(a, b, q, t) := \left(\frac{\partial}{\partial t} f^{\pm}(a, b, q, t)\right) \mathrm{e}\left(h \cdot \frac{f^{\pm}(a, b, q, t)}{q}\right).$$

Our work so far has shown that

$$S_h^{\pm}(a, b, q) = \sum_{e|q} \frac{\mu(e)}{q'} \cdot \sum_{k=1}^{q'} \mathcal{G}(h'\overline{a}b, k; q')B(e, k),$$

with

$$B(e, k) := \mathrm{e}\left(h \cdot \frac{f^{\pm}(a, b, q, y_0 + Y)}{q}\right) r_e(k, q'; y_0 + Y)$$
$$- \frac{2\pi i h}{q} \int_{y_0}^{y_0+Y} g_h^{\pm}(a, b, q, t) r_e(k, q'; t)\mathrm{d}t.$$

Returning to the error terms $\mathcal{F}^{\pm}$ in (3.5), we deduce that

$$\mathcal{F}^{\pm} \ll \sum_{h\leqslant H} \sum_{q} \sum_{e|q} \frac{1}{hq'} \sum_{k=1}^{q'} \min\left\{Y/e, \|k/q'\|^{-1}\right\} \left(R_1(e, h, q, k) + R_2(e, h, q, k)\right),$$

with

$$R_1(e, h, q, k) := \left| \sum_{\substack{a,b \\ (a,b,q)\in S}} C_{a,b,q} \mathcal{G}(h'\overline{a}b, k; q') \mathrm{e}\left(h \cdot \frac{f^{\pm}(a, b, q, y_0 + Y)}{q}\right) \right|,$$

$$R_2(e, h, q, k) := \frac{h}{q} \int_{y_0}^{y_0+Y} \left| \sum_{\substack{a,b \\ (a,b,q)\in S}} C_{a,b,q} \mathcal{G}(h'\overline{a}b, k; q') g_h^{\pm}(a, b, q, t) \right| \mathrm{d}t.$$

Now we are ready to evaluate R_1 and R_2 using the formulae for Gauss sums in Lemma 3. Since we get slightly different formulae in the cases (i), (ii), (iii), it is reasonable to break the term on the right-hand side of our estimate for $\mathcal{F}^{\pm}$ into $\mathcal{F}_1^{\pm}$, $\mathcal{F}_2^{\pm}$ and $\mathcal{F}_4^{\pm}$, where $\mathcal{F}_1^{\pm}$ denotes the contribution of odd moduli q', $\mathcal{F}_2^{\pm}$ denotes the contribution of moduli with $2\|q'$, and $\mathcal{F}_4^{\pm}$ denotes the contribution of moduli with $4 \mid q'$. For $i = 1, 2, 4$, we define

$$\xi_i(q') := \begin{cases} 1, & \text{if } i = 1 \text{ and } q' \text{ is odd,} \\ 1, & \text{if } i = 2 \text{ and } 2\|q', \\ 1, & \text{if } i = 4 \text{ and } 4 \mid q', \\ 0, & \text{otherwise.} \end{cases}$$

We may therefore write

$$\mathcal{F}_i^{\pm} = \sum_{h \leqslant H} \sum_{q} \sum_{e|q} \frac{\xi_i(q')}{hq'} \sum_{k=1}^{q'} \min\left\{Y/e, \|k/q'\|^{-1}\right\} \left(R_1(e,h,q,k) + R_2(e,h,q,k)\right), \tag{3.7}$$

for $i = 1, 2, 4$.

For brevity, we only evaluate R_1 and R_2 when q' is odd, which is the relevant case for the treatment of $\mathcal{F}_1^{\pm}$. The cases $2\|q'$ and $4 \mid q'$ can each be handled similarly. If $(q', 2h') = 1$, then Lemma 3(i) yields

$$\mathcal{G}(h'\overline{a}b, k; q') = \epsilon_{q'} \sqrt{q'} \cdot \left(\frac{h'ab}{q'}\right) \mathrm{e}\left(-\frac{\overline{4bh'} \cdot ak^2}{q'}\right).$$

Hence, in this case we have

$$R_1 = \sqrt{q'} \left| \sum_{\substack{a,b \\ (a,b,q) \in S}} C_{a,b,q} \left(\frac{ab}{q'}\right) \mathrm{e}\left(-\frac{\overline{4bh'} \cdot ak^2}{q'}\right) \mathrm{e}\left(h \cdot \frac{f^{\pm}(a,b,q,y_0+Y)}{q}\right) \right| \tag{3.8}$$

and

$$R_2 = \frac{h}{q} \cdot \sqrt{q'} \int_{y_0}^{y_0+Y} \left| \sum_{\substack{a,b \\ (a,b,q) \in S}} C_{a,b,q} \left(\frac{ab}{q'}\right) \mathrm{e}\left(-\frac{\overline{4bh'} \cdot ak^2}{q'}\right) g_h^{\pm}(a,b,q,t) \right| \mathrm{d}t, \tag{3.9}$$

where $R_1 = R_1(e,h,q,k)$ and $R_2(e,h,q,k)$. To proceed further, we need to remove the weight functions $f^{\pm}$ and $g_h^{\pm}$.

Recall (3.1). We are now ready to impose a suitable constraint on the partial derivatives of $f^{\pm}$, wherever they are defined. We will assume that

$$\left| \frac{\partial^{i+j+k} f^{\pm}}{\partial a^i \partial b^j \partial y^k}(a,b,q,y) \right| \leqslant \alpha^i \beta^j \tau^k F \tag{3.10}$$

in $\mathcal{R}$ for $i,j,k \in \{0,1\}$ such that $i+j+k \neq 0$, where α,β,γ,F are suitable non-negative numbers. We shall also suppose that

$$H \geqslant \frac{q_0}{F} \tag{3.11}$$

and set

$$\Delta_H := \left(1+\frac{HF\alpha A}{q_0}\right)\left(1+\frac{HF\beta B}{q_0}\right)\left(1+\frac{HF\tau Y}{q_0}\right). \tag{3.12}$$

We now repeatedly apply partial summation with respect to a and b to remove the weight functions $f^{\pm}$ and $g_h^{\pm}$ in (3.8) and (3.9). Then we interchange the integrals arising in this process with the sums on the right-hand side of (3.7). Finally, we estimate the resulting integrals by multiplying their lengths with the supremums of their integrands, which we bound using (3.10). Taking (3.11) into consideration, we arrive at the bound for $\mathcal{F}_1^{\pm}$ in the following Theorem. Likewise, we obtain the corresponding bounds for $\mathcal{F}_2^{\pm}$ and $\mathcal{F}_4^{\pm}$.

Theorem 3. *Assume the condition* (3.10) *and let H satisfy* (3.11). *Then we have*

$$\mathcal{F}^{\pm} \ll \mathcal{F}_1^{\pm} + \mathcal{F}_2^{\pm} + \mathcal{F}_4^{\pm},$$

where

$$\mathcal{F}_i^{\pm} \ll \Delta_H \sup_{(\eta,\theta)\in\mathbb{R}^2} \sum_{h\leqslant H}\sum_{q}\sum_{e|q} \frac{\xi_i(q')}{h\sqrt{q'}} \sum_{k=0}^{q'-1} \min\left\{Y/e, \|k/q'\|^{-1}\right\} \left|R^{(i)}(\eta,\theta;e,h,q,k)\right|$$

for $i = 1,2,4$, with

$$R^{(1)}(\eta,\theta;e,h,q,k) := \sum_{\substack{a\leqslant\eta,\ b\leqslant\theta\\ (a,b,q)\in S}} C_{a,b,q}\left(\frac{ab}{q'}\right) \mathrm{e}\left(-\frac{\overline{4bh'}\cdot ak^2}{q'}\right), \tag{3.13}$$

$$R^{(2)}(\eta,\theta;e,h,q,k) := \delta_k \sum_{\substack{a\leqslant\eta,\ b\leqslant\theta\\ (a,b,q)\in S}} C_{a,b,q}\left(\frac{ab}{q'/2}\right) \mathrm{e}\left(-\frac{\overline{8bh'}\cdot ak^2}{q'/2}\right), \tag{3.14}$$

$$R^{(4)}(\eta,\theta;e,h,q,k) := (1-\delta_k) \sum_{\substack{a\leqslant\eta,\ b\leqslant\theta\\ (a,b,q)\in S}} \epsilon_{h'ab}^{-1} C_{a,b,q}\left(\frac{q'}{ab}\right) \mathrm{e}\left(-\frac{\overline{bh'}\cdot ak^2}{4q'}\right). \tag{3.15}$$

We are now in a position to deduce the bound in Theorem 1 for fixed non-zero integers a,b,q such that $q \geqslant 1$ and $(ab,q)=1$. In fact there is little extra effort required to handle a more general quantity. Let $J=(y_0,y_0+Y]$ be an interval with $Y \geqslant 1$ and assume that $f^{\pm}: J \to \mathbb{R}$ are continuously differentiable functions with $f^{+}(y) \geqslant f^{-}(y)$ for all $y \in J$. Set $I(y) := (f^{-}(y), f^{+}(y)]$ and $X(y) := f^{+}(y) - f^{-}(y)$. Assume that $|\frac{df^{\pm}}{dy}(y)| \leqslant T$ for all $y \in J$. Then we have the following result.

Corollary. *Let $H > 0$ and $\Delta_H := 1 + HTY/q$. We have*

$$\sum_{\substack{y\in J\\(y,q)=1}} \sum_{\substack{x\in I(y)\\ ax+by^2\equiv 0 \bmod q}} 1 = \frac{1}{q}\sum_{\substack{y\in J\\(y,q)=1}} X(y) + O\left(\frac{Y}{H}\right)$$
$$+O\left(\Delta_H L(H)\sigma_{-1/2}(q)\left(\frac{Y}{\sqrt{q}}\cdot\tau(q)+\sqrt{q}L(q)\right)\right).$$

where L and $\sigma_{-1/2}$ are as in the statement of Theorem 1.

Proof. Recall (3.3) and (3.4). We set $f^{\pm}(a,b,q,y) = f^{\pm}(y)$, $q_0 = q$, $F = q$, $\tau = T/F$ and $\alpha = \beta = 0$ in the build-up to Theorem 3 . Estimating $R^{(i)}(\eta,\theta;e,h,q,k)$ trivially by $O(1)$, and combining this with our work so far, we readily obtain the asymptotic estimate

$$\frac{1}{q}\sum_{\substack{y\in J\\(y,q)=1}} X(y) + O\left(\frac{Y}{H}\right)$$
$$+O\left(\Delta_H\sum_{h\leqslant H}\frac{1}{h}\sum_{e\mid q}\frac{e^{1/2}(he,q/e)^{1/2}}{q^{1/2}}\sum_{k=0}^{q-1}\min\left\{\frac{Y}{e},\frac{q}{e(he,q/e)k}\right\}\right)$$

for the double sum in the statement. The second O-term here is seen to be

$$\ll \Delta_H\cdot\frac{Y}{q^{1/2}}\sum_{h\leqslant H}\frac{1}{h}\sum_{e\mid q}\frac{(he,q/e)^{1/2}}{e^{1/2}} + \Delta_H(\log H+1)(\log q+1)\sigma_{-1/2}(q)\sqrt{q},$$

where the first term comes from the contribution of $k = 0$ and the second one from the contribution of $k \neq 0$. Since $(he,q/e)^{1/2} \leqslant (h,q)^{1/2}e^{1/2}$, we have

$$\sum_{h\leqslant H}\frac{1}{h}\sum_{e\mid q}\frac{(he,q/e)^{1/2}}{e^{1/2}} \leqslant \tau(q)\sum_{h\leqslant H}\frac{(h,q)^{1/2}}{h} \ll \tau(q)\sigma_{-1/2}(q)\log(H+1).$$

This therefore completes the proof of the corollary. ■

For Theorem 1 we take $J = (0,Y]$ and $I = (0,X]$, so that $f^{\pm}$ are constant and we can set $T = 0$ and $\Delta_H = 1$ in the corollary. Taking $H = q$ we therefore obtain

$$M_{1,2}(X,Y;a,b,q) = \frac{X}{q}\sum_{\substack{y\in J\\(y,q)=1}} 1 + O\left(L(q)\sigma_{-1/2}(q)\left(\frac{Y}{\sqrt{q}}\cdot\tau(q)+\sqrt{q}L(q)\right)\right).$$

On noting that

$$\sum_{\substack{y\in J\\(y,q)=1}} 1 = \frac{\varphi(q)}{q}\cdot Y + O\left(\tau(q)\right),$$

this completes the proof of Theorem 1 .

4. Proof of Theorem 2

We now place ourselves in the setting of Theorem 2, which is concerned with estimating $\mathcal{S}$ in (1.3) when S is given by (1.5) for fixed non-zero integers l, m, r, s, t for which $l, m, t \geqslant 1$ and $(rs, t) = 1$. Assume furthermore that (1.6) holds. Now we can set

$$\begin{aligned} a_0 &:= rU^l, & A &:= (2^l - 1)rU^l, & b_0 &:= sV^m, \\ B &:= (2^m - 1)sV^m, & q_0 &:= tW, & Q &:= tW \end{aligned}$$

in (3.1). With $\tilde{f}^{\pm}$ as in §1, we also set

$$\tilde{I}(u,v,w,y) := I(ru^l, sv^m, tw, y), \qquad \tilde{X}(u,v,w,y) := X(ru^l, sv^m, tw, y)$$

and

$$D_{u,v} = d_{u,v} + |d_{u,v}|. \tag{4.1}$$

Next we observe that (3.10) is equivalent to (1.8) in $(U, 2U] \times (V, 2V] \times (W, 2W] \times J$ for $i, j, k \in \{0, 1\}$ such that $i + j + k \neq 0$, where

$$\rho U = \frac{l}{2^l - 1} \cdot \alpha A, \qquad \sigma V = \frac{m}{2^m - 1} \cdot \beta B.$$

In particular (3.12) has the same order of magnitude as (1.9) under this assumption, where we recall that l and m are viewed as absolute constants.

We may now write

$$\mathcal{S} = \sum_{U<u\leqslant 2U} \sum_{\substack{V<v\leqslant 2V \\ (rsuv,tw)=1}} \sum_{W<w\leqslant 2W} d_{u,v} e_w \sum_{\substack{y_0<y\leqslant y_0+Y \\ (y,tw)=1}} \sum_{\substack{x\in \tilde{I}(u,v,w,y) \\ ru^l x + sv^m y^2 \equiv 0 \bmod tw}} 1,$$

and recall the decomposition in (3.2). Using (3.3), the main term equals

$$\mathcal{M} = \sum_{U<u\leqslant 2U} \sum_{\substack{V<v\leqslant 2V \\ (rsuv,tw)=1}} \sum_{W<w\leqslant 2W} \frac{d_{u,v} e_w}{tw} \sum_{\substack{y_0<y\leqslant y_0+Y \\ (y,tw)=1}} \tilde{X}(u,v,w,y). \tag{4.2}$$

Using (3.4) and (1.6), the error term $\mathcal{E}$ is bounded by

$$\mathcal{E} = \frac{Y}{H} \sum_{U<u\leqslant 2U} \sum_{\substack{V<v\leqslant 2V \\ (rsuv,tw)=1}} \sum_{W<w\leqslant 2W} |d_{u,v} e_w| \ll \frac{UVWY}{H}. \tag{4.3}$$

We now turn to the error term $\mathcal{F}_1^{\pm}$. Using (1.6), Theorem 3 and (4.1), we see that

$$\begin{aligned} \mathcal{F}_1^{\pm} \ll \Delta_H \sup_{\substack{U\leqslant\eta\leqslant 2U \\ V\leqslant\theta\leqslant 2V}} \sum_{h\leqslant H} \sum_{\substack{W<w\leqslant 2W \\ (2rs,tw)=1}} \sum_{e|tw} \frac{1}{h\sqrt{q'}} \\ \times \sum_{k=0}^{q'-1} \min\left\{Y/e, \|k/q'\|^{-1}\right\} |R(\eta,\theta;h',q',k)|, \end{aligned}$$

where

$$R(\eta,\theta;h',q',k)=\sum_{\substack{U<u\leqslant\eta\\(uv,tw)=1}}\sum_{V<v\leqslant\theta}D_{u,v}\left(\frac{u^lv^m}{q'}\right)\mathrm{e}\left(-\frac{\overline{4sv^mh'}\cdot ru^lk^2}{q'}\right).$$

An application of (3.6) therefore yields

$$\begin{aligned}\mathcal{F}_1^{\pm}\ll\ &\frac{\Delta_H}{(tW)^{1/2}}\sup_{\substack{U\leqslant\eta\leqslant 2U\\V\leqslant\theta\leqslant 2V}}\sum_d\sum_e\sum_{\substack{h\leqslant H\\d\mid he}}\frac{d^{1/2}e^{1/2}}{h}\\&\times\sum_{\substack{W<w\leqslant 2W\\(2rs,tw)=1\\de\mid tw\\(he,tw/e)=d}}\sum_{k=0}^{q'-1}\min\left\{Y/e,\|k/q'\|^{-1}\right\}|R(\eta,\theta;h',q',k)|,\end{aligned}\tag{4.4}$$

where

$$d=(he,tw/e),\qquad q'=\frac{tw}{de},\qquad h'=\frac{he}{d}.\tag{4.5}$$

One derives similar bounds for $\mathcal{F}_2^{\pm}$ and $\mathcal{F}_4^{\pm}$ using (3.14) and (3.15) instead of (3.13). It will suffice to estimate $\mathcal{F}_1^{\pm}$ since the treatments of $\mathcal{F}_2^{\pm}$ and $\mathcal{F}_4^{\pm}$ will essentially be the same. We note that the right-hand side of (4.4) is empty if t is even, so we may assume that t is odd.

In the next sections, we shall treat the contributions of $k=0$ and $k\neq 0$ to the right-hand side of (4.4) separately. To this end, we define

$$\mathcal{K}_0:=\frac{\Delta_HY}{(tW)^{1/2}}\sup_{\substack{U\leqslant\eta\leqslant 2U\\V\leqslant\theta\leqslant 2V}}\sum_d\sum_e\sum_{\substack{h\leqslant H\\d\mid he}}\frac{d^{1/2}}{e^{1/2}h}\sum_{\substack{W<w\leqslant 2W\\(2rs,w)=1\\de\mid tw}}\left|\sum_{\substack{U<u\leqslant\eta\\(uv,tw)=1}}\sum_{V<v\leqslant\theta}D_{u,v}\left(\frac{u^lv^m}{q'}\right)\right|\tag{4.6}$$

and

$$\mathcal{K}_1:=\Delta_H(tW)^{1/2}\sup_{\substack{U\leqslant\eta\leqslant 2U\\V\leqslant\theta\leqslant 2V}}\sum_d\sum_e\sum_{\substack{h\leqslant H\\d\mid eh}}\frac{1}{d^{1/2}e^{1/2}h}\sum_{\substack{W<w\leqslant 2W\\(2rs,w)=1\\de\mid tw\\(he,tw/e)=d}}\sum_{k=1}^{[q'/2]}\frac{1}{k}|R(\eta,\theta;h',q',k)|.\tag{4.7}$$

Note that we have dropped the condition $(he,tw/e)=d$ in $\mathcal{K}_0$ but kept it in $\mathcal{K}_1$ since $R(\eta,\theta;h',q',k)$ is not well-defined if $(h',q')>1$.

As a rule of thumb we expect $\mathcal{K}_0$ to dominate if Y is large compared to q_0 and $\mathcal{K}_1$ to dominate otherwise. Therefore, one would like to obtain non-trivial bounds for $\mathcal{K}_0$ if Y is large and non-trivial bounds for $\mathcal{K}_1$ if Y is small. Here we are mainly interested in the case of large Y.

4.1. The contribution of $k = 0$

We aim to exploit cancellations coming from the Jacobi symbol. Our result will clearly depend on the parities of the exponents l and m. We will establish the following bound.

Proposition 1. *We have*

$$\mathcal{K}_0 \ll \frac{\Delta_H Y}{(tW)^{1/2}} \cdot (HtUVW)^{\varepsilon} \left(UVW^{1/2} + U^{1-\{l/2\}}V^{1-\{m/2\}}W\right).$$

We will achieve this result by considering four different cases. Suppose first that l and m are odd. In this case, we shall treat the term $\mathcal{K}_0$ using Heath-Brown's large sieve for real characters. First, we recall our assumption that t is odd and note that de is also necessarily odd by our summation conditions $(w, 2) = 1$ and $de \mid tw$. Now, using the oddness of the exponents l and m, the multiplicativity of the Jacobi symbol and (4.5), we observe that

$$\left(\frac{u^l v^m}{q'}\right) = \left(\frac{uv}{tde}\right)\left(\frac{uv}{w}\right)$$

since $(uv, tw) = 1$. Furthermore we write

$$\beta_z := \left(\frac{z}{tde}\right) \sum_{\substack{U<u\leqslant\eta \\ V<v\leqslant\theta \\ uv=z}} D_{u,v}.$$

Then it follows that

$$\sum_{\substack{U<u\leqslant\eta \\ (uv,tw)=1}} \sum_{V<v\leqslant\theta} D_{u,v}\left(\frac{u^l v^m}{q'}\right) = \sum_{UV<z\leqslant 4UV} \beta_z \left(\frac{z}{w}\right),$$

where we note that the coprimality condition $(uv, tw) = 1$ is implied by the Jacobi symbol. We further note that $\beta_z = O(z^{\varepsilon})$ by (1.6) and (4.1). Next we write

$$\left|\sum_{UV<z\leqslant 4UV} \beta_z \left(\frac{z}{w}\right)\right| = \alpha_w \sum_{UV<z\leqslant 4UV} \beta_z \left(\frac{z}{w}\right),$$

where α_w is a suitable complex number with $|\alpha_w| = 1$. The inner triple sum in (4.6) now takes the form

$$\sum_{\substack{W<w\leqslant 2W \\ (2rs,w)=1 \\ de|tw}} \left|\sum_{\substack{U<u\leqslant\eta \\ (uv,tw)=1}} \sum_{V<v\leqslant\theta} D_{u,v}\left(\frac{u^l v^m}{q'}\right)\right| = \sum_{\substack{W<w\leqslant 2W \\ (2rs,w)=1 \\ de|tw}} \alpha_w \sum_{UV<z\leqslant 4UV} \beta_z \left(\frac{z}{w}\right).$$

We observe that $tw \equiv 0 \bmod de$ if and only if $w \equiv 0 \bmod de/(de,t)$. Hence

$$\sum_{\substack{W<w\leqslant 2W\\(2rs,w)=1\\de\mid tw}} \alpha_w \sum_{UV<z\leqslant 4UV} \beta_z \left(\frac{z}{w}\right) = \sum_{\substack{W/j<w\leqslant 2W/j\\(2rs,jw)=1}} \tilde{\alpha}_w \sum_{UV<z\leqslant 4UV} \tilde{\beta}_z \left(\frac{z}{w}\right),$$

where

$$j=\frac{de}{(de,t)}, \qquad \tilde{\alpha}_w = \alpha_{jw}, \qquad \tilde{\beta}_z = \beta_z \cdot \left(\frac{z}{j}\right).$$

Recalling that $\beta_z = O(z^\varepsilon)$ and applying Lemma 2, we deduce that

$$\sum_{\substack{W/j<w\leqslant 2W/j\\(2rs,jw)=1}} \tilde{\alpha}_w \sum_{UV<z\leqslant 4UV} \tilde{\beta}_z \left(\frac{z}{w}\right) \ll (UVW)^\varepsilon \left(\frac{UVW^{1/2}}{j^{1/2}} + \frac{U^{1/2}V^{1/2}W}{j}\right).$$

Combining our work in (4.6), and noting that $de \mid tw$, we obtain the preliminary bound

$$\mathcal{K}_0 \ll \frac{\Delta_H Y}{(tW)^{1/2}} \cdot (UVWH)^\varepsilon \left(UVW^{1/2} + U^{1/2}V^{1/2}W\right) \sum_{\substack{d,e\\de\leqslant 2tW}} \sum_{\substack{h\leqslant H\\d\mid he}} \frac{d^{1/2}}{e^{1/2}hj^{1/2}}.$$

But

$$\begin{aligned}\sum_{\substack{d,e\\de\leqslant 2tW}} \sum_{\substack{h\leqslant H\\d\mid he}} \frac{d^{1/2}}{e^{1/2}hj^{1/2}} = \sum_{\substack{d,e\\de\leqslant 2tW}} \sum_{\substack{h\leqslant H\\d\mid he}} \frac{(de,t)^{1/2}}{eh} &\ll (HtW)^\varepsilon \sum_{e\leqslant 2tW} \sum_{h\leqslant H} \frac{(he^2,t)^{1/2}}{eh}\\ &\leqslant (HtW)^\varepsilon \sum_{e\leqslant 2tW} \frac{(e,t)}{e} \sum_{h\leqslant H} \frac{(h,t)^{1/2}}{h}\\ &\ll (HtW)^{2\varepsilon}.\end{aligned}$$

This therefore gives

$$\mathcal{K}_0 \ll \frac{\Delta_H Y}{(tW)^{1/2}} \cdot (HtUVW)^\varepsilon \left(UVW^{1/2} + U^{1/2}V^{1/2}W\right), \tag{4.8}$$

which is satisfactory for Proposition 1.

Next suppose that m is odd and l is even. Then we have

$$\left(\frac{u^l v^m}{q'}\right) = \chi_0(u) \left(\frac{v}{q'}\right),$$

where χ_0 is the principal character modulo q'. Hence, it is not possible to exploit the summation over u. Therefore, we sum over u trivially and estimate the term

$$\sum_{\substack{W<w\leqslant 2W\\(2rs,w)=1\\de\mid tw}} \left| \sum_{\substack{V<v\leqslant\theta\\(v,tw)=1}} D_{u,v} \left(\frac{v}{q'}\right) \right|$$

using Lemma 2, just as above. In this way we arrive at the same bound for $\mathcal{K}_0$, where the term $U^{1/2}$ in (4.8) is replaced by U, as required. If l is odd and m is even then the situation is the same, with the roles of u and v being interchanged. Thus, in this case, the term $V^{1/2}$ in (4.8) needs to be replaced by V.

Finally suppose that l and m are both even. Then

$$\left(\frac{u^l v^m}{q'}\right) = \chi_0(uv),$$

where χ_0 is the principal character modulo q'. Hence, in this case we have no cancellations at all in $\mathcal{K}_0$, and the only possibility is to estimate trivially. Here the term $UVW^{1/2} + U^{1/2}V^{1/2}W$ in (4.8) needs to be replaced by UVW.

This completes the proof of Proposition 1. We note from (3.15) that when dealing with the contribution corresponding to $\mathcal{K}_0$ in $\mathcal{F}_4^{\pm}$, the roles of ab and q in the Jacobi symbol are flipped. The oddness condition on $m = ab$ in Lemma 2 will be satisfied since $(ab, q)=1$ and $4 \mid q$, whence $(ab, 2) = 1$ in this case.

4.2. The contribution of $k \neq 0$

We first estimate the contribution $\mathcal{K}_1$ of $k \neq 0$ trivially, by bounding all coefficients $D_{u,v}$ and e_w and the characters occurring in $R(\eta, \theta; h', q', k)$ by $O(1)$. Rearranging summations and dropping several summation conditions, we obtain

$$\mathcal{K}_1 \ll \Delta_H (tW)^{1/2} UV \sum_{h \leqslant H} \frac{1}{h} \sum_{W < w \leqslant 2W} \sum_{k \leqslant tw} \frac{1}{k} \sum_{\substack{d,e \\ de \mid tw}} \frac{1}{d^{1/2} e^{1/2}},$$

which therefore implies the following bound.

Proposition 2. *We have* $\mathcal{K}_1 \ll \Delta_H (tW)^{1/2} UVW (HtW)^{\varepsilon}$.

A non-trivial saving can be obtained if UV is large compared to q_0 and, furthermore, $d_{u,v}$ factorises in the form (1.7), which we now assume. By (4.1) we have

$$R(\eta, \theta; h', q', k) = R_1(\eta, \theta; h', q', k) + R_2(\eta, \theta; h', q', k),$$

where

$$R_1(\eta, \theta; h', q', k) := \sum_{\substack{U < u \leqslant \eta \\ (uv, tw)=1}} \sum_{V < v \leqslant \theta} d'_u \tilde{d}_v \left(\frac{u^l v^m}{q'}\right) \mathrm{e}\left(-\frac{\overline{4sv^m h'} \cdot ru^l k^2}{q'}\right),$$

$$R_2(\eta, \theta; h', q', k) := \sum_{\substack{U < u \leqslant \eta \\ (uv, tw)=1}} \sum_{V < v \leqslant \theta} |d'_u| \cdot |\tilde{d}_v| \left(\frac{u^l v^m}{q'}\right) \mathrm{e}\left(-\frac{\overline{4sv^m h'} \cdot ru^l k^2}{q'}\right).$$

We focus here on bounding R_1, the estimation of R_2 being similar.

We begin by writing

$$\mathrm{e}\left(-\frac{\overline{4sv^m h'}\cdot ru^l k^2}{q'}\right)=\mathrm{e}\left(-\frac{\overline{4sv^m h'}\cdot ru^l k'}{q''}\right),$$

where

$$k':=\frac{k^2}{(q',k^2)},\qquad q''=\frac{q'}{(q',k^2)}. \tag{4.9}$$

Now we write the additive character in terms of multiplicative characters via

$$\begin{aligned}\mathrm{e}\left(-\frac{\overline{4sv^m h}\cdot ru^l k'}{q''}\right)&=\frac{1}{\varphi(q'')}\sum_{\chi \bmod q''}\overline{\chi}(-\overline{4sv^m h}\cdot ru^l k')\tau(\chi)\\&=\frac{1}{\varphi(q'')}\sum_{\chi \bmod q''}\chi(-4sh\overline{rk'})\overline{\chi}^l(u)\chi^m(v)\tau(\chi).\end{aligned}$$

It follows that

$$R_1(\eta,\theta;h',q',k)=\frac{1}{\varphi(q'')}\sum_{\chi \bmod q''}\chi(-4sh\overline{rk'})\tau(\chi)\sum_{\substack{U<u\leqslant\eta\\(u,tw)=1}}d''_u\overline{\chi}^l(u)\sum_{\substack{V<v\leqslant\theta\\(v,tw)=1}}\tilde{\tilde{d}}_v\chi^m(v),$$

where $d''_u:=d'_u(\frac{u}{q'})^l$ and $\tilde{\tilde{d}}_v:=\tilde{d}_v(\frac{v}{q'})^m$. Note that for every fixed $n\in\mathbb{N}$ and every character $\chi_1 \bmod q''$ there are at most $O\left(q''^{\varepsilon}\right)$ characters $\chi \bmod q''$ with $\chi_1=\chi^n$. Therefore, using Cauchy–Schwarz and the well-known bounds $|\tau(\chi)|\leqslant\sqrt{q''}$ and $\varphi(q'')\gg q''^{1-\varepsilon}$, we deduce that

$$\begin{aligned}|R_1(\eta,\theta;h',q',k)|\ll q''^{-1/2+\varepsilon}&\left(\sum_{\chi \bmod q''}\left|\sum_{\substack{U<u\leqslant\eta\\(u,tw)=1}}d''_u\overline{\chi}(u)\right|^2\right)^{1/2}\\&\times\left(\sum_{\chi \bmod q''}\left|\sum_{\substack{V<v\leqslant\theta\\(v,tw)=1}}\tilde{\tilde{d}}_v\chi(v)\right|^2\right)^{1/2}.\end{aligned}$$

Now using the large sieve for fixed modulus (see Iwaniec and Kowalski [9, page 179], for example), together with $|d''_u|,|\tilde{\tilde{d}}_v|\leqslant 1$, we deduce that

$$|R_1(\eta,\theta;h',q',k)|\ll q''^{-1/2+\varepsilon}(q'+U)^{1/2}(q'+V)^{1/2}(UV)^{1/2}.$$

The same estimate holds for $R_2(\eta,\theta;h',q',k)$ on redefining d''_u and $\tilde{\tilde{d}}_v$ accordingly. Hence, using (4.5) and (4.9), it follows that

$$\begin{aligned}\sum_{k=1}^{[q'/2]}\frac{1}{k}|R(\eta,\theta;h',q',k)|&\ll q'^{-1/2+\varepsilon}(q'+U)^{1/2}(q'+V)^{1/2}(UV)^{1/2}\sum_{k=1}^{[q'/2]}\frac{(q',k^2)^{1/2}}{k}\\&\ll d^{1/2}e^{1/2}(tW)^{-1/2+2\varepsilon}(tW+U)^{1/2}(tW+V)^{1/2}(UV)^{1/2},\end{aligned}$$

where we have estimated the k-sum by $O\left(q'^{\varepsilon}\right)$. Plugging the last line into (4.7), rearranging the summations and dropping several summation conditions, we obtain

$$\mathcal{K}_1 \ll \Delta_H (tW)^{\varepsilon}(tW+U)^{1/2}(tW+V)^{1/2}(UV)^{1/2} \sum_{h\leqslant H} \frac{1}{h} \sum_{W<w\leqslant 2W} \sum_{\substack{d,e\\ de|tw}} 1.$$

This yields the following result, which improves Proposition 2 if UV is larger than $q_0 = tW$.

Proposition 3. *We have* $\mathcal{K}_1 \ll \Delta_H (tW+U)^{1/2}(tW+V)^{1/2}(UV)^{1/2}W(HtW)^{\varepsilon}$, *if* (1.7) *holds.*

4.3. Conclusion

Now we are ready to prove our final asymptotic estimate for $\mathcal{S}$. First, combining Propositions 1, 2 and 3, we get

$$\mathcal{F}_1^{\pm} \ll \mathcal{K}_0 + \mathcal{K}_1 \ll \mathcal{T},$$

where $\mathcal{T}$ is as in the statement of Theorem 2. The same bound holds for $\mathcal{F}_2^{\pm}$ and $\mathcal{F}_4^{\pm}$. Hence, using Theorem 3, we obtain $\mathcal{F}^{\pm} \ll \mathcal{T}$. Combining this with (3.2), (4.2) and (4.3), we arrive at the statement of Theorem 2.

We end this section by discussing conditions under which we may expect the main term to dominate the error term in Theorem 2. In many applications, the length $\tilde{X}(u,v,w,y)$ of the x-interval will be of size $\tilde{X}(u,v,w,y) \asymp X \leqslant q_0 = tW$, for some fixed $X>0$, and the parameters in (1.8) will satisfy

$$F \asymp X, \quad \rho \asymp U^{-1}, \quad \sigma \asymp V^{-1}, \quad \tau \asymp Y^{-1}. \tag{4.10}$$

Moreover, in generic applications U and V will be shorter than the modulus, and so we further suppose that $U \leqslant tW$ and $V \leqslant tW$.

If there is not much cancellation in the sums over the coefficients, then the expected size of the main term in (4.2) is

$$\mathcal{M} \asymp \frac{UVWXY}{q_0}.$$

For the first O-term on the right side of the asymptotic formula in Theorem 2 to be dominated by this we need H just slightly larger than q_0/X. The choice

$$H = \frac{q_0^{1+\varepsilon}}{X}$$

would be satisfactory. Then $\Delta_H \ll q_0^{\varepsilon}$, by (1.9) and (4.10). Now, for $\mathcal{T}$ to be smaller than $\mathcal{M}$, we need

$$q_0^{1+\varepsilon} \leqslant \min\left\{U^{2\{l/2\}}V^{2\{m/2\}}X^2, Z\right\} \qquad \text{and} \qquad q_0^{\varepsilon}t^{1/2} \leqslant X,$$

where

$$Z := \begin{cases} (UV)^{1/4}(XY)^{1/2}, & \text{if (1.7) holds and } UV \geqq tW, \\ (XY)^{2/3}, & \text{in general.} \end{cases}$$

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