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DIOPHANTINE APPROXIMATION IN $\mathbf{Q}(\sqrt{-30})$, $\mathbf{Q}(\sqrt{-33})$ AND $\mathbf{Q}(\sqrt{-57})$

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Abstract: For the imaginary quadratic fields with discriminants -120, -132 and -248, the first three, five and two points of the Lagrange and Markov spectra respectively are found.

Keywords: Diophantine approximation, hyperbolic geometry, Bianchi groups

1. Introduction

1.1. History

Let α be a real irrational number. In 1891 A. Hurwitz [7] showed that the inequality

$$|\alpha - a/c| < 1/(hc^2)$$

has infinitely many solutions in coprime integers a and c when $h = \sqrt{5}$, and $\sqrt{5}$ is the best constant possible. In 1917 the first geometric proof of this result was obtained by L. Ford in [3], where he makes use of properties of the modular group.

Let $d > 0$ be a square-free integer. Let $\mathcal{O}_d$ be the ring of integers of the field $\mathbf{Q}(\sqrt{-d})$. Let $\theta \in \mathbf{C} - \mathbf{Q}\left(\sqrt{-d}\right)$. Denote by $n(p, q)$ the norm of the ideal generated by p,$q \in \mathcal{O}_d$. Let

$$\nu_d(\theta) = \liminf \frac{|q(q\theta - p)|}{n(p, q)}, \tag{1.1}$$

where $p, q \in \mathcal{O}_d$, $q \neq 0$. Then the inequality

$$\left|\theta - \frac{p}{q}\right| < \nu_d(\theta)\frac{n(p, q)}{|q|^2} \tag{1.2}$$

has infinitely many solutions in $p, q \in \mathcal{O}_d$ with $n(p, q) < 2\sqrt{d}$ (see e.g. [5], \$17(5), XVI*, for the justification of this inequality). The set of numbers $\mathcal{L}_d = \left\{\nu_d(\theta), \theta \in \mathbf{C} - \mathbf{Q}\left(\sqrt{-d}\right)\right\}$ is the *Lagrange spectrum* for the imaginary quadratic

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field $\mathbf{Q}\left(\sqrt{-d}\right)$ and $C_d = \sup \mathcal{L}_d$ the *Hurwitz constant* for the field. If k_∞ is the highest limit point of $\mathcal{L}_d$, then $\mathcal{L}_d \cap (k_\infty, \infty)$ is called *the discrete part of* $\mathcal{L}_d$.

In 1925 Ford [4], applying his method to the Picard group $PSL_2(\mathcal{O}_1)$, showed that the Hurwitz constant for the Gauss field $C_1 = 1/\sqrt{3}$. For the fields with class number one with $d = 1$, 2, 3, 7, 11, and 19, the Hurwitz constants were found by Ford [4], Perron [9], [10], [11], Hofreiter [6], Poitou [12]), (see also A. Schmidt [14]). After Ford [4], none of these authors applied his geometric ideas to Diophantine approximation of complex numbers. In [27], these ideas, as they were developed in [29], were used to obtain an upper bound for the Hurwitz constant of an imaginary quadratic number field (see Theorem 3.1 below). In the cases of $d = 1$, 2, 5, 6, 30 and 33, this bound is sharp [27], [35]. For the class two field with $d = 15$, the Hurwitz constant was found in [30]. For $d = 2$ and 7, the second minimum is known [14]. For $d = 1$ (A. Schmidt [15], Vulakh [20], [21]), $d = 3$ (A. Schmidt [17]), $d = 5$, $d = 6$ (Vulakh [31]) and $d = 11$ (A. Schmidt [16]), the discrete part of the Lagrange spectrum (which coincides with the discrete part of the Markov spectrum) was found. Applying the results of [22], it can be shown that the Lagrange spectrum of an imaginary quadratic field is continuous in its lower part. There are known upper (see Hofreiter [6], Perron [11]) and lower [23], [35] bounds for the Hurwitz constants C_d. Lower bounds for the highest limit point of $\mathcal{L}_d$ for some values of d are given in [23] and [35].

1.2. Main results

The Hurwitz constants for the fields $\mathbf{Q}\left(\sqrt{-30}\right)$ and $\mathbf{Q}\left(\sqrt{-33}\right)$ are found in [35]. The Hurwitz constants for the field $\mathbf{Q}\left(\sqrt{-57}\right)$ is found in Section 5 of the present paper. Here, the method, which was used in [31] to find the discrete part of the Lagrange spectrum of the fields $\mathbf{Q}\left(\sqrt{-5}\right)$ and $\mathbf{Q}\left(\sqrt{-6}\right)$, is applied to the fields $\mathbf{Q}\left(\sqrt{-30}\right)$, $\mathbf{Q}\left(\sqrt{-33}\right)$ and $\mathbf{Q}\left(\sqrt{-57}\right)$. It is based on application of the Farey polygons associated with the extended Bianchi groups B_d, introduced in [30], to reduce the problem of finding the discrete part of the Markov spectrum for B_d to the corresponding problem for one of its maximal Fuchsian subgroups. Such reduction is used in Sections 3, 4 and 5 to show that

$$\begin{aligned}
\mathcal{L}_{30} \cap (2.2936, \infty) &= \left\{\sqrt{37/7},\ \sqrt{22}/2,\ \sqrt{23}/2\right\},\\
\mathcal{L}_{33} \cap \left[\sqrt{14}/2,\ \infty\right) &= \left\{\ \sqrt{14}/2,\ \sqrt{437/124},\ 2,\ \sqrt{17}/2,\ \sqrt{19}/2\right\},\\
\mathcal{L}_{57} \cap (3.1735, \infty) &= \left\{\sqrt{41}/2,\ \sqrt{46}/2\right\}
\end{aligned}$$

respectively.

1.3. Background and Terminology

The upper half-space $H^3 = \{(z,t) : z \in \mathbf{C},\ t > 0\}$ with the metric $ds^2 = t^{-2}(|dz|^2 + dt^2)$ can be used as a model of the 3-dimensional hyperbolic space.

$\mathrm{PSL}_2(\mathbf{C})$ is the group of orientation-preserving isometries of H^3. The action of $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbf{C})$ on $(z,t) \in H^3$ is given by

$$F(z,t) = \left(\frac{(az+b)(\overline{cz+d}) + a\bar{c}t^2}{|cz+d|^2 + |c|^2 t^2}, \frac{t}{|cz+d|^2 + |c|^2 t^2} \right) \tag{1.3}$$

(see e.g. [2], p. 58, or [18], p. 15). The Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$ is a geometrically finite discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$. We denote by B_d the maximal discrete subgroup of the group of isometries of H^3, which contains $\mathrm{PSL}_2(\mathcal{O}_d)$ (see [25]; in [26], this group is denoted by RB_d). The type of $g \in B_d$ is *elliptic, parabolic* or *loxodromic* depending on whether it has a fixed point in H^3, a single fixed point in $\mathbf{C}$, or exactly two fixed points in $\mathbf{C}$. If g is loxodromic, the geodesic connecting its fixed points is called the *axis* of g. The transformation g is *hyperbolic* if it is loxodromic and every plane containing its axis is g-invariant. The set of parabolic fixed points (*cusps*) of B_d can be identified with $\mathbf{Q}(\sqrt{-d})$.

Let P be a Dirichlet polygon of $G_\infty = \mathrm{Stab}(\infty, B_d)$ in $\mathbf{C}$. Denote $P_\infty = \{(z,t) \in H^3 : z \in P\}$. The region

$$\mathcal{D} = P_\infty \cap \{x \in H^3 : |g'(x)| < 1,\ g \in B_d\} \tag{1.4}$$

is an *isometric* fundamental domain for B_d in H^3 (see [1], p. 66, or [18], p.18). Here $g'(x)$ stands for the Jacobian of the transformation g.

Denote

$$K = K(\infty) = G_\infty \overline{\mathcal{D}}, \qquad K(u) = gK(\infty), \tag{1.5}$$

where $u = g(\infty)$. It is clear that $\cup K(u) = H^3$, $u \in B_d\infty$, and that $\dim(K(u) \cap K(u')) \leqslant 2$ if $u \neq u'$. We shall call the tessellation of H^3 by $K(u)$, $u \in B_d\infty$, the *K-tessellation.* Let ∂K be the boundary of K. We shall say that $\partial K \cap \overline{\mathcal{D}}$ is the *floor* of $\mathcal{D}$. The components of ∂K (and $\mathcal{D}$) of dimensions 0, 1, and 2 will be called the *vertices (or cusps), edges*, and *faces* of K respectively. The vertices (and edges) of K which belong to $\overline{\mathcal{D}}$ will be called the vertices (and edges) of $\mathcal{D}$. For any region R in H^3, the components of the boundary of R of dimension 2 which lie in vertical planes will be called the *vertical faces* of R. (Notice that, in general, according to these definitions, the components of the boundary of $\mathcal{D}$ of dimension 0 (or 1) which lie in the vertical faces of $\mathcal{D}$ are not vertices (or edges) of $\mathcal{D}$).

A geodesic in H^3 is a semicircle or a ray, which is orthogonal to $\mathbf{C}$. For a geodesic L with endpoints θ, θ' in $\mathbf{C}$, denote $k(L) = |\theta - \theta'|$ and $\nu(L) = \inf |g(\theta) - g(\theta')|^{-1}$, the infimum being taken over all $g \in B_d$. A geodesic L is said to be *extremal* with respect to B_d if $\nu(L) = 1/k(L)$. Note that an extremal L cuts $K(\infty)$. The set of numbers $\mathcal{M}(B_d) = \{\nu(L)\}$ is called the *Markov spectrum* for B_d.

Denote by $Cl(K)$ the class group of a field $K = \mathbf{Q}\left(\sqrt{-d}\right)$. There are 65 fields K such that $B_d(\infty) = K$, that is ∞ is the only cusp of a fundamental domain of B_d in H^3. The condition $B_d(\infty) = K$ holds for

1) $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$ with $Cl(K) = (1)$,
2) $d = 5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427$ with $Cl(K) = (2)$,
3) $d = 21, 30, 33, 42, 57, 70, 78, 85, 93, 102, 130, 133, 177, 190, 195, 253, 435, 483, 555, 595, 627, 715, 795, 1435$ with $Cl(K) = (2, 2)$,
4) $d = 105, 165, 210, 273, 330, 345, 357, 385, 462, 1155, 1995, 3003, 3315$ with $Cl(K) = (2, 2, 2)$,
5) $d = 1365$ with $Cl(K) = (2, 2, 2, 2)$.

Weinberger ([36], Theorem 2) showed that there is at most one imaginary quadratic field with exponent 2 and $d > 1365$. It follows that the condition $B_d(\infty) = K$ holds for at most one field K with $d > 1365$.

For all these values of d, and therefore for $d = 30$, 33 and 57, $\mathcal{M}(B_d)$ coincide with the Markov spectrum of the field $\mathbf{Q}(\sqrt{-d})$ (see [31], p. 41).

Let

$$\mathcal{M}_h(B_d) = \{\nu(L) \in \mathcal{M}(B_d),\ L \text{ is the axis of a hyperbolic } g \in B_d\}.$$

In all the known cases (see [20], [21], [31], [15], [16], [17]), almost all the points in the discrete part of $\mathcal{M}(B_d)$ (that is the part of $\mathcal{M}(B_d)$ that lies above its highest limit point) belong to $\mathcal{M}_h(B_d)$. Since every hyperbolic $g \in B_d$ belongs to some maximal Fuchsian subgroup of B_d, the problem of finding $\mathcal{M}_h(B_d)$ can be reduced to the problem of finding the Markov spectra for the maximal Fuchsian subgroups of B_d. The classification of such subgroups of B_d is known (see [24], [25], [33], [8]). They can be identified with the B_d-unit groups of indefinite integral binary Hermitian forms.

1.4. Outline

It is shown in [25] how the problem of classification of maximal Fuchsian subgroups of B_d can be reduced to the problem of classification of indefinite primitive Hermitian forms (see [31], Theorem 2.1, see also [33], [8]). Let S be a hemisphere in H^3 with center in $\mathbf{C}$. Denote $G_S = \mathrm{Stab}(S, B_d)$. Let L be a geodesic in S. Denote $\nu_S(L) = \inf |g(\theta) - g(\theta')|^{-1}$, the infimum being taken over all $g \in G_S$. We say that a geodesic $L \subset S$ is *extremal with respect to* G_S if $\nu_S(L) = 1/k(L)$. Denote $\mathcal{M}_S = \{\nu_S(L), L \subset S\}$. Theorem 2.2 from [31] contains a sufficient condition for a geodesic $L \subset S$, which is extremal with respect to G_S, to be extremal with respect to B_d.

Let $\mathcal{H}_d$ be the spectrum of minima of binary indefinite Hermitian forms over $\mathcal{O}_d$ (see e.g. [31], Chapter 2). It is shown in [34] that $(1/2)\mathcal{H}_d \subset \mathcal{M}(B_d)$ and that, for any point $\nu \in \mathcal{H}_d$, there is a one-parameter family of extremal geodesics L_θ, $0 \leqslant \theta < 2\pi$, such that $\nu(L_\theta) = \nu/2$ ([34], Theorem 1.1). Moreover, the geodesics L_θ, which are the axes of some elements in B_d, form a dense subset of this family (see [34] for more details).

For a one-parameter family of extremal geodesics L_θ, $0 \leqslant \theta < 2\pi$, introduced in Theorem 1.1 from [34], the point $\nu(L_\theta) = 1/(2R) = \nu(\Phi)/2$ in the Markov spectrum of B_d (and in the Lagrange spectrum $\mathcal{L}_d$) is called a *Hermitian point.* Let H_d be the largest Hermitian point in $\mathcal{M}(B_d)$. It is shown in [34], that $H_{30} = \sqrt{5}$, $H_{33} = \sqrt{11}/2$ and $H_{57} = \sqrt{38}/2$ (see Sections 3, 4 and 5 below).

In Section 2, we recall some definitions and results from [30] related to the Farey polygons associated with the groups B_d.

Denote by $S(b, R)$ the hemisphere in H^3 with center $b \in \mathbf{C}$ and radius R. In Subsection 3.1, the Farey polygons are used to show that, for $d = 30$, Theorem 2.2 from [31] is applicable to the hemisphere $S = S\left(1/2 - 12/\omega, 1/\sqrt{20}\right)$. For this hemisphere S, it is shown in Section 3 that

$$\mathcal{M}(B_{30}) \cap (\sqrt{5}, \sqrt{23}/2] = \mathcal{M}_S \cap (\sqrt{5}, \sqrt{23}/2]. \tag{1.6}$$

This reduction is used in Subsection 3.3 to prove Theorem 3.1, which is one of the three main results of this paper.

The outline of Section 4 is similar to that of Section 3. However, for $d = 33$, the reduction similar to (1.6) does not exist. In Subsection 4.1, the Farey polygons are used to show that, for $d = 33$, Theorem 2.2 from [31] is applicable to the hemisphere $S = S\left(2\omega/11, 1/\sqrt{11}\right)$. For this hemisphere S, by Lemma 4.1,

$$\mathcal{M}(B_{33}) \cap (\sqrt{14}/2, \sqrt{19}/2] = \mathcal{M}_S \cap (\sqrt{14}/2, \sqrt{19}/2] \cup \{2\},$$

where $2 \in \mathcal{M}(B_{33})$ is not attained at any geodesic $L \subset S$. However, this reduction is sufficient to prove Theorem 4.1, which is the second main result of this paper.

In Subsection 5.1, the generators (Theorem 5.1) and the isometric fundamental domain of B_{57} are found. Then the Farey polygons are used to show that, for $d = 57$, Theorem 2.2 from [31] is applicable to the hemisphere $S = S\left(1/2 - 33/(2\omega), 1/\sqrt{38}\right)$. For this hemisphere S, it is shown in Section 5 that

$$\mathcal{M}(B_{57}) \cap [\sqrt{38}/2, \sqrt{46}/2] = \mathcal{M}_S \cap [\sqrt{38}/2, \sqrt{46}/2].$$

This reduction is used in Subsection 5.3 to prove Theorem 5.2, which is the third main result of this paper.

In each of these three cases, a face of the isometric fundamental domain $\mathcal{D}$ of B_d lies in the hemisphere S, the reflection in S in H^3 belongs to B_d and the radius of S is the smallest among all the hemispheres containing the faces of $\mathcal{D}$, which makes the reduction mentioned above relatively easy. Not all the hemispheres defined by the Hermitian forms enumerated in the Tables 1-5 of [35] possess these properties. Thus, in the cases of $d = 19$, 43, 67 and 163, none of them holds.

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2. Farey polygons

Here we summarize some results from [30], Section 2. Assume that the summits of all the edges in the floor of $K(\infty)$ belong to $K(\infty)$. (Anke Pohl has indicated that this assumption should be made in the statement of Theorem 5.4 from [30]). Let $v \in H^3$ be a vertex of $K(u)$. Assume that v belongs to the edges e_j, $j = 1, ..., t$.

Let F_j be the plane through u which is orthogonal to e_j. Denote by $A(u,v)$ the part of $K(u)$ which is bounded by the planes F_j, $j = 1, ..., t$. Thus, each of the sets $A(u,v)$ has one cusp u and one vertex v. The union of all the sets $A(u,v)$ with the same cusp u is $K(u)$. The union of all $A(u,v)$ with the same vertex v is called the *v-cell* (see [30]). Denote the v-cell by $N(v)$. The faces of $N(v)$ are called *hyperbolic Farey polygons* and the vertical projections of the faces of $N(v)$ from ∞ into $\mathbf{C}$ the *Farey polygons*. Let B be a face of $N(v)$ with vertices at the cusps u_m, $m = 1, ..., n$. Let $h(B)$ be the largest value of k such that the horoballs bounded by the horospheres $Q(u_m, k)$, $m = 1$, ..., n, cover B. Recall that the horosphere $Q(p/q, k)$, where $p, q \in \mathcal{O}_d$, is a euclidean sphere in H^3 with center $(p/q, r)$ and radius $r = n(p,q)/(k|q|^2)$. We shall call the number $h(B)$ the *Farey constant* of B. Denote by k_d the smallest value of $h(B)$ over all the faces B of all the v-cells. By Theorem 3.1 from [31], the Hurwitz constant for the field $\mathbf{Q}(\sqrt{-d})$, $C_d \leqslant 1/k_d$. It is shown in [27], [35] and in the present paper that this bound is sharp for $d = 1$, 2, 5, 6, 30, 33 and 57.

3. Diophantine approximation in $\mathbf{Q}(\sqrt{-30})$

3.1. Reduction

Let $d = 30$ and $\omega = \sqrt{-30}$. Then $\{1, \omega\}$ is the standard basis of the ring of integers $\mathcal{O}_{30}$ of the field $\mathbf{Q}(\sqrt{-30})$. The group $G_\infty = \mathrm{Stab}(\infty, B_{30})$ is generated by reflections in the vertical plains in H^3 through the lines $\mathrm{Re}\, z = 0$, $\mathrm{Re}\, z = 1/2$, $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = \sqrt{30}/2$ in $\mathbf{C}$, which will be denoted by S_{11}, S_8, S_9, and S_{10} respectively. It is shown in [13] that the group B_{30} is generated by reflections in the faces of its fundamental domain $\mathcal{D}$, whose four faces lie in these vertical planes and the floor of $\mathcal{D}$ lies in seven hemispheres

$$S_1 = S(0,1), \qquad S_2 = S\left(\frac{-30}{\sqrt{D}}, \frac{1}{\sqrt{2}}\right), \qquad S_3 = S\left(\frac{-20}{\sqrt{D}}, \frac{1}{\sqrt{3}}\right),$$
$$S_4 = S\left(\frac{-12}{\sqrt{D}}, \frac{1}{\sqrt{5}}\right), \qquad S_5 = S\left(\frac{1}{2} - \frac{15}{\sqrt{D}}, \frac{1}{2\sqrt{2}}\right),$$
$$S_6 = S\left(\frac{1}{2} - \frac{10}{\sqrt{D}}, \frac{1}{2\sqrt{3}}\right), \qquad S_7 = S\left(\frac{1}{2} - \frac{24}{\sqrt{D}}, \frac{1}{2\sqrt{5}}\right).$$

For $S_i = S(b_i, R_i)$, let $\Phi_i(x,y) = |x - b_i y|^2 - R_i^2|y|^2 = (1, b_i, |b_i|^2 - R_i^2)$, so that $\Phi_i(z,1) + t^2 = R_i^2$ is an equation of the hemisphere S_i in H^3. Then the corresponding Hermitian forms Φ_i are integral (see e.g. [31], p. 29) and the values of $r(\Phi_i) = R_i^2|D| = 60, 40, 24, 15, 10, 6$ for S_i, $i = 2, 3, ..., 7$ respectively. Notice that

$$\Phi_i(x,y) = \Phi_{i+3}(x - y, 2y), \ i = 2, 3, \quad \Phi_4(x, -y) = \Phi_7(x - (1+\omega)y, 2y).$$

The hemisphere S_7 is anisotropic, that is the only solution of $\Phi_7(x,y) = 0$ in $x, y \in \mathcal{O}_{30}$ is $x = y = 0$, and the lowest face of $\mathcal{D}$ lies in S_7. Since the diameter of

S is $1/\sqrt{5}$, the highest Hermitian point of $\mathcal{M}(B_{30})$ is

$$H_{30} = \sqrt{5}$$

(cf. [34], Example 5.1). Denote by h_{ij} the height of the edge $L_{ij} = S_i \cap S_j$ of $\mathcal{D}$. Then

$$\begin{aligned} &h_{18} = \frac{\sqrt{3}}{2}, && h_{28} = \frac{1}{2}, && h_{23} = \frac{1}{\sqrt{5}}, && h_{14} = \frac{1}{\sqrt{6}}, \\ &h_{34} = \frac{1}{\sqrt{8}}, && h_{35} = \frac{1}{\sqrt{11}}, && h_{38} = \frac{1}{\sqrt{12}}, && h_{16} = h_{45} = \frac{1}{\sqrt{13}}, \\ &h_{46} = \frac{1}{\sqrt{17}}, && h_{56} = \frac{1}{\sqrt{20}}, && h_{27} = \frac{1}{\sqrt{22}}, && h_{37} = \frac{1}{\sqrt{23}}. \end{aligned}$$

Since the height of any edge of $\mathcal{D} \geqslant 1/\sqrt{23}$, there is no extremal geodesic L in H^3, whose height is less than $1/\sqrt{23}$ (see [28], Theorem 1).

The relations can be given in terms of τ_i, where τ_i is the reflection in the plane S_i in H^3.

The group $G = \mathrm{Stab}(S_7, B_{30}) = \langle \tau_2, \tau_3, \tau_8 \rangle$. Since $(\tau_2\tau_3)^2 = (\tau_2\tau_8)^4 = (\tau_3\tau_8)^6 = 1$, the group G contains the $(2,4,6)$-triangle group as a subgroup of index two. The triangular face D_S of $\mathcal{D}$, which lies in S_7, with vertices at

$$v_1 = \left(\frac{1}{2} + \frac{5\omega}{12}, \frac{1}{\sqrt{24}}\right), \qquad v_2 = \left(\frac{1}{2} + \frac{3\omega}{8}, \frac{1}{\sqrt{32}}\right), \qquad v_3 = \left(\frac{2}{5} + \frac{2\omega}{5}, \frac{1}{5}\right)$$

is a fundamental domain of G, $S_7 \cap K(\infty)$ consists of two copies of D_S. and

$$\begin{aligned} \mathrm{Stab}(v_1, B_{30}) &= \{\tau_2, \tau_7, \tau_8 : (\tau_2\tau_7)^2 = (\tau_7\tau_8)^2 = (\tau_8\tau_2)^4 = 1\}, \\ \mathrm{Stab}(v_2, B_{30}) &= \{\tau_3, \tau_7, \tau_8 : (\tau_3\tau_7)^2 = (\tau_7\tau_8)^2 = (\tau_8\tau_3)^6 = 1\}, \\ \mathrm{Stab}(v_3, B_{30}) &= \{\tau_2, \tau_3, \tau_7 : (\tau_2\tau_3)^2 = (\tau_3\tau_7)^2 = (\tau_7\tau_2)^2 = 1\}. \end{aligned}$$

The geodesic L_{37} is perpendicular to S_2 and the axis L_{38} of $\tau_8\tau_3$. Denote $U = (\tau_8\tau_3)^3$. Then $\Gamma_{37} = \mathrm{Stab}(L_{37}, B_{30}) = \langle \tau_2, U \rangle$. Let $t_1 = L_{37} \cap S_2$ and $t_2 = L_{37} \cap L_{38}$. Then the arc $[t_1, t_2] = L_{37} \cap K(\infty)$ is a fundamental domain of Γ_{37} on L_{37} and, by Corollary 24, [30], L_{37} is extremal. Since the height of L_{37} is $1/\sqrt{23}$ and, as shown above, the height of any extremal geodesic is at least $1/\sqrt{23}$, the Hurwitz constant of the field $Q\left(\sqrt{-30}\right)$ is

$$C_{30} = \sqrt{23}/2$$

(cf. [35], Table 1).

3.1.1. The v-cells $N(v_1)$, $N(v_2)$, and $N(v_3)$.

The hemisphere S_7 contains four vertices of $\mathcal{D}$, v_1, v_2, v_3 and $v_3' = \tau_8(v_3)$. The v-cell $N(v_3)$ is a rectangular parallelepiped. The v-cells $N(v_1)$ and $N(v_2)$ are square and hexagonal prisms respectively.

The vertices of $N(v_3)$ are the points $B = \omega/2$, $C = \omega/3$, $D = 2\omega/5$, $E = 1/2 + 2\omega/5$, $F = 2/5 + 2\omega/5$, $J = (10 + 9\omega)/23$, $K = (10 + 9\omega)/22$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_3)$ from infinity into $\mathbf{C}$ is the triangle with vertices at B, C and E. The Farey constant of the congruent faces $ABDC$ and $KEJF$ is $2h_{23} = 2/\sqrt{5}$, the Farey constant of the congruent faces $ABKE$ and $CDFJ$ is $2h_{27} = 2/\sqrt{22}$ and the Farey constant of the congruent faces $ACJE$ and $BDFK$ is $2h_{37} = 2/\sqrt{23}$. Since the v-cells $N(v_3)$ and $N(v_3')$ are symmetrical with respect to the vertical plane in H^3 through the line $\operatorname{Re} z = 1/2$ in $\mathbf{C}$, if X is a vertex of $N(v_3)$, then $X' = 1 - \overline{X}$ is the corresponding vertex of $N(v_3')$.

The vertices of $N(v_1)$ are the points B, $B' = 1 + \omega/2$, $B_1 = 1/2 + \omega/2$, E, K, $K' = (12 + 9\omega)/22$, $L = 1/2 + 5\omega/12$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_1)$ from infinity into $\mathbf{C}$ is the triangle with vertices at B, B' and E. There are two congruent square faces ABB_1B' and $EKLK'$, whose Farey constant is $2h_{28} = 1$, and four congruent rectangular faces $ABKE$, BB_1LK, $AB'K'E$, $B'B_1LK'$. (We call these faces squares and rectangles only because of their groups of symmetry). The axis of $\tau_8\tau_2$ is the axis of order four in $N(v_1)$.

The vertices of $N(v_2)$ are the points C, $C' = 1 + \omega/3$, $C_1 = (1 + \omega)/3$, $C_2 = 1/2 + \omega/3$, $C_1' = (2 + \omega)/3$, E, J, $J' = (13 + 9\omega)/23$, $M = 1/2 + 3\omega/8$, $N = (13 + 11\omega)/29$, $N' = (16 + 11\omega)/29$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_2)$ from infinity into $\mathbf{C}$ is the triangle with vertices at C, C' and E. There are two congruent hexagonal faces $ACC_1C_2C_1'C'$ and $EJNMN'J'$, whose Farey constant is $2h_{38} = 1/\sqrt{3}$, and six congruent rectangular faces $ACJE$, CC_1NJ, C_1NMC_2, $AC'J'E$, $C'C_1'N'J'$, $C_1'N'MC_2'$, whose Farey constant is $2h_{37} = 2/\sqrt{23}$. The axis of $\tau_8\tau_3$ is the axis of order six in $N(v_2)$.

Let $2/\sqrt{23} \leqslant k < 1/\sqrt{5}$. Then $N(v_i, k)$ has a geodesic face ϕ if and only if ϕ lies in a rectangular face of $N(v_i)$, which is congruent to $ABKE$ or $ACJE$, since only for such a face the Farey constant is less than $1/\sqrt{5}$. But, the centers of all such faces lie in S_7. Hence, if the height of an extremal geodesic L is less than $1/\sqrt{20}$, then $g(L) \subset S_7$ for some translation $g \in B_{30}$. Indeed, an extremal geodesic L, which cuts $N(v, k)$, must enter through one of its geodesic faces and exit through another. Since the limit points of the sequence of v-cells cut by L are the endpoints of L and they lie in S_7, L itself lies in S_7.

Lemma 3.1. *If the height of an extremal geodesic L in H^3 is less than $1/\sqrt{20}$, then $L \subset gS$, $g \in G_\infty$. Thus, $\mathcal{M}(B_{30}) \cap (\sqrt{5}, \sqrt{23}/2] = \mathcal{M}_S \cap (\sqrt{5}, \sqrt{23}/2]$.*

3.2. A group with signature (0; 2, 4, 6)

The reflections τ_2, τ_3 and τ_8 are represented in G by $\tau_2\tau_7$, $\tau_3\tau_7$ and $\tau_8\tau_7$. respectively. By Theorem 2.2 from [31], if a geodesic $L \subset S_7$ is extremal with respect to G_S, then L is extremal with respect to B_{30}, and therefore $\mathcal{M}_S \subset \mathcal{M}(B_{30})$.

Let $\rho = \begin{pmatrix} R_7 & b_7 \\ 0 & 1 \end{pmatrix}$. Let C_1 be the circle $|z - b_7| = 1/\sqrt{20}$ in the complex plane $\mathbf{C}$. Then $G_S = \operatorname{Stab}(C_1, B_{30})$ and $C_1 = \rho(C)$, where C is the unit circle $|z| = 1$. The Klein model D^2 of the hyperbolic plane, which is used in [32], is obtained as the projection of the unit upper hemisphere model of the hyperbolic plane in H^3 from ∞ into $\mathbf{C}$, so that C is the boundary of D^2 (cf. [19], p. 68).

The group $\Gamma = \rho^{-1}G_S\,\rho$ is a discrete cocompact subgroup of the group of isometries G_C of D^2. Below, we shall denote the fixed point of $F \in G_C$ by the corresponding lower case letter. Thus, the fixed point of F_1 is f_1. The fixed point of $F = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G_C$ in $\mathbf{C}$ is $f = ib/\operatorname{Im} a$. The fixed points of F and F' in G_C are said to be Γ-*equivalent* if there is $g \in \Gamma$ such that $F' = gFg^{-1}$.

Let $c = \sqrt{5} + i\sqrt{6}$ and $c_1 = -3\sqrt{5} + 2i\sqrt{6}$. The group Γ is generated by reflections

$$\sigma = \begin{bmatrix} -1 & -\bar{c} \\ c & 1 \end{bmatrix}, \qquad \sigma_0 = \begin{bmatrix} 3 & -\overline{c_1} \\ c_1 & -3 \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

across the sides of the triangle with vertices $a = -i\sqrt{6}/4$, $b = i\sqrt{6}/6$, and $s = -1/\sqrt{5}$, which are the fixed points of $A = \sigma_1\sigma_0$, $B = \sigma_1\sigma$, $S = \sigma\sigma_0$. Here $\sigma = \rho^{-1}\tau_2\tau_7\rho$, $\sigma_0 = \rho^{-1}\tau_3\tau_7\rho$ and $\sigma_1 = \rho^{-1}\tau_8\tau_7\rho$. One has $S^2 = A^6 = B^4 = \mathrm{id}$, and $\Gamma = \left\langle \sigma, \sigma_0, \sigma_1 : (\sigma_0\sigma)^2 = (\sigma\sigma_1)^4 = (\sigma_1\sigma_0)^6 = 1 \right\rangle$.

Denote $\sigma_{i+1} = A\sigma_i$, $i = 0, ..., 5$, and $\sigma_{i+6} = \sigma_i$, $S_1 = ASA^{-1} = \sigma_1 S\sigma_1$, $S_2 = A^2SA^{-2} = \sigma_0' S\sigma_0'$, where $\sigma_0' = \sigma_2 = \sigma_1\sigma_0\sigma_1$, $U = A^3$, $\phi = S\sigma = \sigma_3 S$, $\phi_1 = \sigma' S = S\sigma_0$, $\phi_2 = \sigma_0' S = S_2\sigma_0'$, where $\sigma' = \sigma_1\sigma\sigma_1$.

It is shown above that the axis L_{37} of $\rho\phi\rho^{-1}$ is extremal with respect to B_{30}. Similarly, the axes of $\rho\phi_1\rho^{-1}$ and $\rho\phi_2\rho^{-1}$ are also extremal with respect to B_{30}.

Lemma 3.2. *For the fixed points f, f_1, f_2 of ϕ, ϕ_1 and ϕ_2 respectively, we have*

$$\begin{aligned}
(f, \sigma w) = (f, \sigma_3 w) = -(f, \sigma_0 w) = -(f, Sw) = -(f, Uw) = 1, \\
(f_1, \sigma' w) = (f_1, \sigma_0 w) = -(f_1, \sigma w) = -(f_1, Sw) = -(f_1, B^2 w) = 1, \\
(f_2, \sigma_0' w) = (f_2, S\sigma_0' Sw) = -(f_2, \varphi_2 w) = -(f_2, S_2 w) = -(f_2, Sw) = 1.
\end{aligned}$$

Proof. Since f is the fixed point of the reflection σ_0, $(f, \sigma_0 w) = -1$. Since f is the fixed point of $\phi = S\sigma = \sigma_3 S$, $(f, \sigma w) = (f, \sigma_3 w) = -(f, Sw) = 1$. Since $U = \sigma_0\sigma_3$, $(f, Uw) = -1$.

Similarly, since f_1 is the fixed point of the reflection σ, $(f_1, \sigma w) = -1$, and since f_1 is the fixed point of $\phi_1 = \sigma' S = S\sigma_0$, $(f_1, \sigma' w) = (f, \sigma_0 w) = -(f, Sw) = 1$. Since $B^2 = \sigma\sigma'$, $(f, B^2 w) = -1$.

The last statement is true because f_2 is the fixed point of $\phi_2 = \sigma_0' S$. ■

3.3. Uniqueness

Denote by D_T the disc $|z| < 11/\sqrt{6}$ and $D_T' = \{z \in D_T : \operatorname{Re} z \leqslant 0\}$. Since $\Gamma_w = \langle\sigma_1\rangle$, we can assume that $z \in D_T'$. Below, we assume that $z \in D_T'$ is an extremal point.

Definition 3.1. *For any $g \in G_C$, denote by $P(g)$ and $N(g)$ the conditions $(z, gw) \geqslant 1$ and $(z, gw) \leqslant -1$ respectively.*

If $P(\sigma)$ and $P(\sigma_0)$ hold, then $z \in \mathcal{D}_\Gamma$, a fundamental domain of Γ. If $N(\sigma')$ holds, then $|z| \geqslant 11/\sqrt{6}$. If $N(\sigma_0')$ holds, then $|z| \geqslant 23/(2\sqrt{6})$. In both cases, $z \notin D_T'$. Thus, for $z \in D_T'$, $P(\sigma')$ and $P(\sigma_0')$ both hold. If $P(S)$ holds, then an indefinite $z \in D_T'$ belongs to $p(U)$. Hence any indefinite $z \in D_T'$ belongs to D_T'', the part of D_T', where $P(\sigma')$, $P(\sigma_0')$ and $N(S)$ hold.

Assume that $z \in D_T''$. If $P(\sigma_0)$ holds, then either, by Lemma 3.2, $z = f_1 = -\sqrt{5} + i\sqrt{6}$ or $N(\sigma_0)$ holds. If $N(\sigma_0)$ and $N(B^2)$ hold, then $|z|^2 \geqslant 33$ and an extremal $z \notin D_T''$. Hence, $P(B^2)$ holds. If $N(\sigma)$ and $P(B^2)$ hold, then $|z| \geqslant |-2\sqrt{5}+i/\sqrt{6}| = 11/\sqrt{6}$ and $z \notin D_T''$. Thus, $P(\sigma)$ holds. If $P(\sigma)$ and $P(\sigma_3)$ hold, then either, by Lemma 3.2, $z = f = -\sqrt{5} - i\frac{2}{3}\sqrt{6}$ or $N(\sigma_3)$ holds. If $N(\sigma_3)$ and $P(\phi_2)$ hold, then $|z| \geqslant \left|-\frac{13}{9}\sqrt{5} - i\frac{38}{27}\sqrt{6}\right|$ and $z \notin D_T''$. Hence, $N(\phi_2)$ holds. If $P(S_2)$ and $N(\phi_2)$ hold, then $|z|^2 \geqslant 97$ and $z \notin D_T''$. Hence, $N(S_2)$ holds.

If $N(S_2)$ and $N(S\sigma_0'S)$ hold, then $|z|^2 \geqslant \left|-\frac{5}{4}\sqrt{5} - i\frac{41}{24}\sqrt{6}\right|^2$ and $z \notin D_T''$. Thus, $P(S\sigma_0'S)$ holds and, by Lemma 3.2, $z = f_2 = -\sqrt{5} - i\frac{3}{2}\sqrt{6}$.

We have proved the following.

Lemma 3.3. *Let an extremal* $z \in D_T'$. *Then* $z = f = -\sqrt{5} - i\frac{2}{3}\sqrt{6}$, *or* $z = f_1 = -\sqrt{5} + i\sqrt{6}$, *or* $z = f_2 = -\sqrt{5} - i\frac{3}{2}\sqrt{6}$, *or* $|z| \geqslant 11/\sqrt{6}$.

Let $S = S(b, R)$ and let L be a geodesic in S of height h. Let $\rho(S) = S(0, 1)$. Let f be the pole of the projection of $\rho(L)$ from ∞ into $\mathbf{C}$ in H^3. Then

$$h = R\sqrt{1 - |f|^{-2}}. \tag{3.1}$$

Thus, by (3.1) and Lemma 3.3, the first three points of $\mathcal{M}(B_{30})$ are $\sqrt{23}/2$, $\sqrt{22}/2$ and $\sqrt{37/7}$. We have proved the following.

Theorem 3.1. $\mathcal{M}(B_{30}) \cap (2.2936, \infty) = \left\{\sqrt{37/7},\ \sqrt{22}/2,\ \sqrt{23}/2\right\}$.

Let L_0, L_1 *and* L_2 *be the axes of* $\rho\phi\rho^{-1}$, $\rho\phi_1\rho^{-1}$ *and* $\rho\phi_2\rho^{-1}$ *respectively.*
If $\nu(L) = \sqrt{23}/2$ *for a geodesic* L *in* H^3, *then* $L = g(L_0)$ *for some* $g \in B_{30}$.
If $\nu(L) = \sqrt{22}/2$ *for a geodesic* L *in* H^3, *then* $L = g(L_1)$ *for some* $g \in B_{30}$.
If $\nu(L) = \sqrt{37/7}$ *for a geodesic* L *in* H^3, *then* $L = g(L_2)$ *for some* $g \in B_{30}$.

As mentioned above, $\mathcal{M}(B_{30})$ coincides with the Markov spectrum of the field $\mathbf{Q}(\sqrt{-30})$, which, as follows from Theorem 3.1, equals to $\mathcal{L}_{30}$ in the interval $(2.2936, \infty)$ (see e.g. [31], p. 41).

4. Diophantine approximation in $\mathbf{Q}(\sqrt{-33})$

4.1. Reduction

Let $d = 33$ and $\omega = \sqrt{-33}$. Then $\{1, \omega\}$ is the standard basis of the ring of integers $\mathcal{O}_{33}$ of the field $\mathbf{Q}(\sqrt{-33})$. The group $G_\infty = \mathrm{Stab}(\infty, B_{33})$ is generated by reflections in the vertical plains in H^3 through the lines $\operatorname{Re} z = 0$, $\operatorname{Re} z = 1/2$, $\operatorname{Im} z = 0$ and $\operatorname{Im} z = \sqrt{33}/2$ in $\mathbf{C}$, which will be denoted by S_7, S_8, S_9, and S_{10} respectively. It is shown in [13] that the group B_{33} is reflective. But B_{33} itself is not generated by reflections in the faces of their isometric fundamental domain $\mathcal{D}$. The floor of $\mathcal{D}$ lies in six hemispheres

$$S_1 = S(0,1), \qquad S_2 = S\left(\frac{1+\omega}{2}, \frac{1}{\sqrt{2}}\right), \qquad S_3 = S\left(\frac{\omega}{3}, \frac{1}{\sqrt{3}}\right),$$
$$S_4 = S\left(\frac{3+\omega}{6}, \frac{1}{\sqrt{6}}\right), \qquad S_5 = S\left(\frac{1+\omega}{4}, \frac{1}{2\sqrt{2}}\right), \qquad S_6 = S\left(\frac{2\omega}{11}, \frac{1}{\sqrt{11}}\right).$$

The reflections in the hemispheres S_i, $i \neq 3$, belong to B_{33}, but the reflection in S_3 does not. The axis of $\rho_0 = \begin{pmatrix} \omega & 12 \\ 3 & -\omega \end{pmatrix}_{-3} \in B_{33}$ with endpoints $\omega/3 \pm 1/\sqrt{3}$ in $\mathbf{C}$ belongs to S_3, $\rho_0(\infty) = \omega/3$ and $\rho_0(S_3) = S_3$.

We have $\Phi_2 = (1,(1+\omega)/2,8)$, $\Phi_5 = (1,(1+\omega)/4,2)$, and $\rho_0^*\Phi_2\rho_0 = \Phi_5$, where $\rho_0^* = (\overline{\rho_0})^T$. Hence $\rho(S_2) = S_5$. Thus, B_{33} is generated by ρ and the reflections in the hemispheres S_i, $1 \leqslant i \leqslant 10$, $i \neq 3$.

The hemisphere S_6 is anisotropic, and the lowest face of $\mathcal{D}$ lies in S_6. Since the diameter of S_6 is $2/\sqrt{11}$, the highest Hermitian point of $\mathcal{M}(B_{33})$ is

$$H_{33} = \sqrt{11}/2$$

(cf. [34], Example 5.3). Denote by h_{ij} the height of the edge $L_{ij} = S_i \cap S_j$ of $\mathcal{D}$. Then

$$h_{18} = \frac{\sqrt{3}}{2}, \qquad h_{27} = \frac{1}{2}, \qquad h_{23} = \frac{1}{\sqrt{5}}, \qquad h_{48} = \frac{1}{\sqrt{6}},$$
$$h_{14} = \frac{1}{\sqrt{7}}, \qquad h_{35} = h_{23} = \sqrt{\frac{5}{42}}, \qquad h_{38} = h_{16} = \frac{1}{\sqrt{12}}, \qquad h_{45} = \frac{1}{\sqrt{14}},$$
$$h_{57} = h_{58} = \frac{1}{4}, \qquad h_{46} = \frac{1}{\sqrt{17}}, \qquad h_{56} = \frac{1}{\sqrt{19}}.$$

Since the height of any edge of $\mathcal{D} \geqslant 1/\sqrt{19}$, there is no extremal geodesic L in H^3, whose height is less than $1/\sqrt{19}$ (see [28], Theorem 1).

Denote by τ_i the reflection in the plane S_i, $i \neq 3$, in H^3.

The group $\Gamma_S = \mathrm{Stab}(S_6, B_{33}) = \langle \tau_1, \tau_4, \tau_5, \tau_7 \rangle$. Since $(\tau_1\tau_7)^2 = (\tau_1\tau_4)^2 = (\tau_4\tau_5)^2 = (\tau_5\tau_7)^4 = 1$, the group Γ_S contains a group with signature $(0; 2,2,2,4)$ as a subgroup of index two. The quadrangular face D_S of $\mathcal{D}$, which lies in S_6, with vertices

$$v_1 = \left(\frac{1+\omega}{6}, \frac{1}{\sqrt{18}}\right), \qquad v_2 = \left(\frac{1+\omega}{5}, \frac{1}{5}\right), \qquad v_3 = \left(\frac{2\omega}{9}, \frac{1}{\sqrt{27}}\right).$$

and $v_4 = (\omega/6, 1/\sqrt{12}) = S_1 \cap S_6 \cap S_7$ is a fundamental domain of Γ_S, and $S_6 \cap K(\infty) = D_S \cup \tau_7(D_S)$ is the pentagon with vertices at v_1, v_2, v_3, $\tau_7(v_1)$ and $\tau_7(v_2)$. Thus, by Theorem 2.2 from [31], if a geodesic $L \subset S_6$ is extremal with respect to Γ_S, then L is extremal with respect to B_{33}, and therefore

$\mathcal{M}_S \subset \mathcal{M}(B_{33})$. We have

$$\begin{aligned}
\mathrm{Stab}(v_1, B_{33}) &= \{\tau_1, \tau_4, \tau_5 : (\tau_1\tau_4)^2 = (\tau_4\tau_6)^2 = (\tau_6\tau_1)^2 = 1\},\\
\mathrm{Stab}(v_2, B_{33}) &= \{\tau_4, \tau_5, \tau_6 : (\tau_4\tau_5)^2 = (\tau_5\tau_6)^2 = (\tau_6\tau_4)^2 = 1\},\\
\mathrm{Stab}(v_3, B_{33}) &= \{\tau_5, \tau_6, \tau_7 : (\tau_7\tau_6)^2 = (\tau_6\tau_5)^2 = (\tau_5\tau_7)^4 = 1\}.
\end{aligned}$$

The geodesic L_{56} is perpendicular to the planes S_4 and $S_5' = \tau_7(S_5)$. Denote $\tau_5' = \tau_7\tau_5\tau_7$. Then $\Gamma_{56} = \mathrm{Stab}(L_{56}, B_{33}) = \langle \tau_4, \tau_5' \rangle$. Let $t_1 = L_{56} \cap S_4$ and $t_2 = L_{56} \cap S_5'$. Then the arc $[t_1, t_2] = L_{56} \cap K(\infty)$ is a fundamental domain of Γ_{56} on L_{56} and, by Corollary 24, [30], L_{56} is extremal. Since the height of L_{56} is $1/\sqrt{19}$ and, as shown above, the height of any extremal geodesic is at least $1/\sqrt{19}$, the Hurwitz constant of the field $\mathbf{Q}\left(\sqrt{-33}\right)$ is

$$C_{33} = \sqrt{19}/2$$

(cf. [35], Table 3).

4.2. The v-cells $N(v_1)$, $N(v_2)$ and $N(v_3)$

The v-cells $N(v_1)$ and $N(v_2)$ are rectangular parallelepipeds and $N(v_3)$ is a square prism.

The vertices of $N(v_1)$ are the points $B = 0$, $C = \omega/6$, $D = 2\omega/11$, $E = 6/(1-\omega)$, $F = (1+\omega)/6$, $J = 6/(3-\omega)$, $K = (3+\omega)/6$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_1)$ from infinity into $\mathbf{C}$ is the triangle with vertices at B, D and K. The Farey constant of the congruent faces $ABDC$ and $KEFJ$ is $2h_{16} = 1/\sqrt{3}$, the Farey constant of the congruent faces $ABJK$ and $CDEF$ is $2h_{14} = 2/\sqrt{7}$ and the Farey constant of the congruent faces $ADEK$ and $BCFJ$ is $2h_{46} = 2/\sqrt{17}$.

The vertices of $N(v_2)$ are the points D, E, K, $L = (1+\omega)/5$, $M = (1+\omega)/4$, $N = (2+4\omega)/19$, $P = (5+3\omega)/14$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_2)$ from infinity into $\mathbf{C}$ is the triangle with vertices at M, D and K. The Farey constant of the congruent faces $ADNM$ and $KELP$ is $2h_{56} = 2/\sqrt{19}$, the Farey constant of the congruent faces $AKPM$ and $DELN$ is $2h_{45} = 2/\sqrt{14}$ and the Farey constant of the congruent faces $ADEK$ and $LNMP$ is $2h_{46} = 2/\sqrt{17}$.

Since the v-cells $N(v_k)$ and $N(v_k')$, $k = 1, 2$, are symmetrical with respect to the vertical plane in H^3 through the line $\mathrm{Re}\, z = 1/2$ in $\mathbf{C}$, if X is a vertex of $N(v_k)$, then $X' = 1 - \overline{X}$ is the corresponding vertex of $N(v_k')$.

The vertices of $N(v_3)$ are the points $A_1 = \omega/4$, D, $D_1 = 2\omega/9$, M, $M' = (-1+\omega)/4$, N, $N' = (-2+4\omega)/19$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_1)$ from infinity into $\mathbf{C}$ is the triangle with vertices at D, M and M'. There are two congruent square faces AMA_1M' and DND_1N', whose Farey constant is $2h_{28} = 1/2$, and four congruent rectangular faces $ADNM$, A_1D_1NM, $ADN'M'$, $A_1D_1N'M'$. (We call these faces squares and rectangles only because of their groups of symmetry). The axis of $\tau_5\tau_7$ is the axis of order four in $N(v_3)$.

Let $2/\sqrt{19} \leqslant k < 2/\sqrt{14}$. If a geodesic L cuts a square face, which is congruent to AMA_1M', we can assume that L cuts AMA_1M'. If L is extremal, then L cuts the geodesic face ϕ in AMA_1M', which exists for $1/2 < k < 2/\sqrt{14}$. For these values of k, $\phi \subset Q(\omega/3, k)$, the horoball with equation $|z-\omega/3|^2+(t-1/(3k))^2 < 1/(3k)^2$ in H^3. Hence, there is no extremal geodesic, which cuts a square face of $N(v_3)$. Thus, an extremal geodesic can cut only the rectangular faces of $N(v_i)$. Assume that an extremal geodesic L cuts $N(v_i, k)$. Then L cuts the geodesic faces of $N(v_i, k)$, which lie in the rectangular faces of $N(v_i)$ congruent to $ADNM$ or $ADEK$, whose Farey constants are less than $2/\sqrt{14}$. The centers of all such rectangular faces lie in S_6. Hence, if the height of an extremal geodesic L is less than $1/\sqrt{14}$, then $g(L) \subset S_6$ for some translation $g \in B_{33}$. Indeed, an extremal geodesic L, which cuts $N(v_i, k)$, must enter through one of its geodesic faces and exit through another. Since the limit points of the sequence of v-cells cut by L are the endpoints of L and they lie in S_6, L itself lies in S_6.

The reflection τ_{11} with respect to the hemisphere

$$S_{11} = S\left(\frac{1+\omega}{3}, \frac{1}{3}\right)$$

belongs to B_{33} and $S_{11}\cap S_8 = S_3\cap S_8$. Let $v_4 = S_{11}\cap S_8\cap S_5$ and $v_5 = S_4\cap S_8\cap S_5$. Then

$$v_4 = \left(\frac{1}{2}+\frac{19\omega}{66}, \frac{1}{\sqrt{66}}\right), \qquad v_5 = \left(\frac{1}{2}+\frac{5\omega}{22}, \frac{1}{\sqrt{22}}\right),$$

and

$$\begin{aligned}\mathrm{Stab}(v_4, B_{33}) &= \{\tau_5, \tau_8, \tau_{11} : (\tau_{11}\tau_5)^2 = (\tau_5\tau_8)^4 = (\tau_8\tau_{11})^3 = 1\},\\ \mathrm{Stab}(v_5, B_{33}) &= \{\tau_4, \tau_5, \tau_8 : (\tau_4\tau_5)^2 = (\tau_5\tau_8)^4 = (\tau_8\tau_4)^2 = 1\}.\end{aligned}$$

4.3. The v-cells $N(v_4)$ and $N(v_5)$

The geodesic $L_{58} = S_8 \cap S_5$ is a common axis of order four of $N(v_4)$ and $N(v_5)$ both. $N(v_4)$ is the same v-cell, which appears in the case of $d = 6$ (see [31], Sec. 5.1). It is a cube whose vertices and edges are replaced by triangular and rectangular faces respectively. $N(v_5)$ is a square prism.

We describe the v-cell $N(v_4)$. Denote

$$\rho_1 = \tau_8\tau_{11} = \begin{pmatrix} 1+\omega & 10-\omega \\ 3 & -2-\omega \end{pmatrix}, \qquad \tau = \tau_{11}\tau_5 = \begin{pmatrix} -10 & 3+3\omega \\ -1+\omega & 10 \end{pmatrix},$$

The group $S_4 = \langle \tau, \rho_1 \rangle$ is the subgroup of the orientation-preserving isomorphisms in $\mathrm{Stab}(v, B_{33})$. One has $\tau^2 = \rho_1^3 = (\tau\rho_1)^4 = id$. The order of S_4 is 24. The vertical plane S_8 in H^3 is the plane of symmetry of the v-cell $N(v_4)$. Hence, if X is a vertex of $N(v)$ in $\mathbf{C}$, then $X' = 1-\overline{X}$ is also a vertex. Thus it is enough to enumerate the vertices of $N(v_4)$ on the line $\mathrm{Re}\, z = 1/2$ and to the left of this line. At any vertex of $N(v_4)$, one triangular, one square, and two rectangular faces of $N(v_4)$ meet. The group S_4 acts transitively on the vertices of $N(v_4)$.

Thus $N(v_4)$ has 24 vertices: $B = (1+\omega)/4$, $C = (1+\omega)/3$, $D = 5(1+\omega)/17$, $E = (13+9\omega)/29$, $F = (15+11\omega)/37$, $H = (27+17\omega)/58$, $K = (3+2\omega)/7$, $L = (27+16\omega)/57$, $M = (11+8\omega)/29$, $N = (17+10\omega)/37$ and their reflections across the line $\operatorname{Re} z = 1/2$ in $\mathbf{C}$, the points $B_1 = 1/2+\omega/4$, $G = 1/2+3\omega/10$ and $J = 1/2+19\omega/66$ on this line, and $A = \infty$. The projection of $N(v_4)$ from infinity into $\mathbf{C}$ is the trapezoid with vertices at B, C, C', B'.

The v-cell $N(v_4)$ has 6 congruent square faces: ABB_1B', $CDFE$, $C'D'F'E'$, $GHJH'$, $KLNM$, $K'L'N'M'$, whose Farey constant equals $2h_{58} = 1/2$; 12 congruent rectangular faces: $ABDC$, $AB'D'C'$, BB_1NM, $B'B_1N'M'$, $DFKM$, $D'F'K'M'$, $JLKH$, $JL'K'H'$, $EFHG$, $E'F'H'G$, $LNN'L'$ and $CEE'C'$, whose Farey constant equal $2h_{5,11} = 2/\sqrt{17}$; and 8 congruent triangular faces: ACC', GEE', JLL', B_1NN', BDM, $B'D'M'$, FHK and $F'H'K'$, whose Farey constant equals $2h_{38} = 1/\sqrt{3}$.

The vertices of $N(v_5)$ are the points B, B_1, $B' = (3+\omega)/4$, $P = 1/2+\omega/6$, $R = (5+3\omega)/14$, $R' = (9+3\omega)/14$, $T = 1/2+5\omega/22$ in $\mathbf{C}$ and $A = \infty$. The projection of $N(v_5)$ from infinity into $\mathbf{C}$ is the triangle with vertices at B, B' and P. There are two congruent square faces ABB_1B' and $PRTR'$, whose Farey constant is $2h_{58} = 1/2$, and four congruent rectangular faces $ABRP$, BB_1TR, $AB'R'P$, $B'B_1TR'$, whose Farey constant is $2h_{45} = 2/\sqrt{14}$. The axis of $\tau_8\tau_5$ is the axis of order four in $N(v_5)$.

Let $2/\sqrt{19} \leqslant k < 2/\sqrt{14}$. If a geodesic L cuts a rectangular face, which is congruent to $ABDC$, we can assume that L cuts $ABDC$. If L is extremal, then L cuts the geodesic face ϕ in $ABDC$, which exists for $2/\sqrt{17} < k < 2/\sqrt{14}$. For these values of k, $\phi \subset Q(\omega/3, k)$. Hence, there is no extremal geodesic, which cuts a triangular or rectangular face of $N(v_4)$. Thus, an extremal geodesic L, which cuts $N(v_4)$ or $N(v_5)$, can cut only the square faces of these v-cells. Up to a symmetry, L cuts either (1) the opposite or (2) adjacent square faces of $N(v_4)$. If L cuts only the opposite faces of cubes, then $L = L_{58}$, whose endpoints are $1/2+\omega/4 \pm i/4$. Since the arc $[v_4, v_5] = L \cap K(\infty)$ is a fundamental domain of $\operatorname{Stab}(L, B_{33})$ on L, L is an extremal geodesic and $\nu(L) = 1/(2h_{58}) = 2$.

Let $T_0 = ((1+\omega)/4+\sqrt{1/8-k^2/4}, k/2)$ and $T_1 = ((3+\omega)/4-\sqrt{1/8-k^2/4}, k/2)$. The geodesic face ψ of $N(v_4, k)$ that lies in the common vertical square face ABB_1B' of $N(v_4)$ and $N(v_5)$ has one side T_0T_1, which lies on the line $t = k/2$ or, more precisely, on $Q(A, k) \cap ABB_1B'$. The opposite side T_2T_3 of ψ lies on the circle $Q(B_1, k) \cap ABB_1B'$. The diagonals T_0T_3 and T_1T_2 of ψ lie on $S_5 \cap ABB_1B'$ and $S_5' \cap ABB_1B'$ respectively. Here $S_5' = S((3+\omega)/4, 1/\sqrt{8})$. The point of intersection of these diagonals is $C_s = (1/2+\omega/4, 1/4)$. It is the center of symmetry of both ABB_1B' and ψ. Notice that when $k = 1/\sqrt{2}$, all four vertices of ψ lie on the boundary of ABB_1B'.

Assume that an extremal geodesic L cuts two adjacent square faces of $N(v_5)$. We can assume that L cuts ψ and $\rho(\psi)$, the geodesic face in $\rho(ABB_1B') = CDFE$. Then L also cuts $\tau_4(\psi)$, the geodesic face in the square face $\tau_4(ABB_1B')$ of $N(v_4)$ with center $(1/2+3\omega/14, 1/154)$. But, any geodesic which cuts $\rho(\psi)$ and $\tau_4(\psi)$ does not meet ψ. Hence there is no extremal geodesic L, which cuts two adjacent square faces of $N(v_5)$.

Geodesic L_{45} is perpendicular to the hemisphere S_6. Since the arc $[v_2, v_5] = L_{45} \cap K(\infty)$ is a fundamental domain of $\mathrm{Stab}(L_{45}, B_{33})$ on L_{45}, L_{45} is an extremal geodesic and $\nu(L_{45}) = 1/(2h_{45}) = 2/\sqrt{14}$.

We have proved the following.

Lemma 4.1. *Let the height of a geodesic L in H^3 be less than $1/\sqrt{14}$. If L is extremal, then L is equivalent to L_{58} or $L \subset gS_6$, $g \in G_\infty$. Thus, $\mathcal{M}(B_{33}) \cap (\sqrt{14}/2, \sqrt{19}/2] = \mathcal{M}_S \cap (\sqrt{14}/2, \sqrt{19}/2] \cup \{2\}$.*

The geodesic L_{45} is extremal with respect to B_{33} and $\nu(L_{45}) = 2/\sqrt{14}$.

Remark. The hemisphere S_4 is anisotropic and $r(\Phi_4) = 22$. The group $\mathrm{Stab}(S_4, B_{33})$ is generated by reflections in L_{48}, L_{14}, L_{45}, and L_{46} with heights $1/\sqrt{6}$, $1/\sqrt{7}$, $1/\sqrt{14}$, and $1/\sqrt{17}$ respectively. It contains a subgroup with signature $(0; 2, 2, 3, 4)$ as a subgroup of index 2.

4.4. A group with signature $(0; 2, 2, 2, 4)$

Now let $\rho = \begin{pmatrix} i\sqrt{3} & -6 \\ 0 & \omega \end{pmatrix}$. Let C_1 be the circle $|z - 2\omega/11| = 1/\sqrt{11}$ in the complex plane $\mathbf{C}$. Then $G_S = \mathrm{Stab}(C_1, B_{33})$ and $C_1 = \rho(C)$, where C is the unit circle $|z| = 1$. The group $\Gamma = \rho^{-1}\Gamma_S\,\rho$ is a discrete cocompact subgroup of the group of isometries G_S of D^2. It is generated by reflections

$$\sigma = \begin{pmatrix} 1 & 2i\sqrt{3} \\ 2i\sqrt{3} & -1 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} -4 & \sqrt{11} + 3i\sqrt{3} \\ -\sqrt{11} + 3i\sqrt{3} & 4 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} -2i\sqrt{3} & 1 + \omega \\ 1 - \omega & 2i\sqrt{3} \end{pmatrix}$$

across the sides of the quadrilateral with vertices $s_0 = i(4/9)\sqrt{3}$, $v = -i\sqrt{3}/6$, $s_1 = \left(\sqrt{11} - i\sqrt{3}\right)/6$, and $u = \left(\sqrt{11} + i\sqrt{3}\right)/5$, which are the fixed points of

$$A = \sigma_0\sigma_1, \qquad V = \sigma_1\sigma, \qquad S_1 = \sigma\sigma_2, \qquad U = \sigma_2\sigma_0$$

respectively. One has $A^4 = V^2 = S_1^2 = U^2 = ABS_1U = \mathrm{id}$, and $\Gamma =< \sigma, \sigma_0, \sigma_1, \sigma_2 : (\sigma_0\sigma_1)^4 = (\sigma_1\sigma)^2 = (\sigma\sigma_2)^2 = (\sigma_2\sigma_0)^2 = 1 >$. Here $\sigma = \rho^{-1}\tau_6\tau_1\rho$, $\sigma_0 = \rho^{-1}\tau_6\tau_5\rho$, $\sigma_1 = \rho^{-1}\tau_6\tau_7\rho$ and $\sigma_2 = \rho^{-1}\tau_6\tau_4\rho$.

For $g \in G_S$, denote $g' = \sigma_1 g \sigma_1$. If z is the fixed point of g, then the fixed points of g' is $-\overline{z}$. Denote

$$S_0 = A^2, \qquad \varphi = U'\sigma_0, \qquad \psi = \sigma_0 S_1,$$

$$F_0 = U'S_0 = \varphi\sigma_0', \qquad H_0 = US_1 = \sigma_2\psi,$$

$$H = \varphi\psi^{-1} = U'S_2, \qquad H'' = \sigma_0 H \sigma_0 = \varphi^{-1}H^{-1}\varphi = \psi^{-1}H^{-1}\psi = \varphi^{-1}\psi.$$

Then

$$f_0 = -\frac{1}{4}\sqrt{11} + i\frac{3}{4}\sqrt{3}, \qquad h_0 = \frac{1}{2}\sqrt{11} - i\frac{1}{6}\sqrt{3},$$
$$h = -\frac{1}{8}\sqrt{11} + i\frac{29}{24}\sqrt{3}, \qquad h'' = \frac{5}{8}\sqrt{11} + i\frac{7}{24}\sqrt{3}$$

and f_0 is the common fixed point of σ_0', φ and F_0, h_0 is the common fixed points of σ_2, ψ and H_0. We have $F_0 = U'S_0 = \sigma_2'\sigma_0$ and $H_0 = US_1 = \sigma_0\sigma$.

Lemma 4.2. *For the fixed points of F_0, H_0, H and H'',*

$$\begin{aligned}
(f_0, \sigma_0 w) = (f_0, \sigma_2' w) = -(f_0, \sigma_0' w) = -(f_0, S_0 w) = -(f_0, U'w) = 1,\\
(h_0, \sigma w) = (h_0, \sigma_0 w) = -(h_0, \sigma_2 w) = -(h_0, S_1 w) = -(h_0, Uw) = 1,\\
(h, \varphi w) = (h, \psi w) = -(h, \sigma_0 w) = -(h, S_2 w) = -(h, U'w) = 1,\\
(h'', \varphi^{-1} w) = (h'', \psi^{-1} w) = -(h'', \sigma_0 w) = -(h'', S_1 w) = 1.
\end{aligned}$$

Proof. Since $h'' = -\sigma_0 h$, we have $(h, U'w) = -(\sigma_0 h'', U'w) = -(h'', \sigma_0 U'w) = -(h'', \varphi^{-1} w) = -1$ and $(h, \psi w) = (h, H\psi w) = (h, \varphi w) = 1$. Similarly,

$$(h'', S_1 w) = -(\sigma_0 h, S_1 w) = -(h, \sigma_0 S_1 w) = -(h, \psi w) = -1$$

and

$$(h'', \varphi^{-1} w) = (h'', (H'')^{-1}\varphi^{-1} w) = (h'', \psi^{-1} w) = 1,$$

since $H'' = \psi^{-1} H^{-1} \psi = \psi^{-1}\psi\varphi^{-1}\psi = \varphi^{-1}\psi$. ■

4.5. Uniqueness

Denote by D_T the disc $|z|^2 < 14/3$ and $D_T' = \{z \in D_T : \operatorname{Re} z \leqslant 0\}$. Since $\Gamma_w = \langle \sigma_1 \rangle$, we can assume that $z \in D_T'$.

If $N(\sigma)$ holds, then $|z|^2 \geqslant |-i2\sqrt{3}|^2 = 12$. Thus, for an extremal $z \in D_T'$, $P(\sigma)$ holds. If $P(\sigma_0)$ and $P(\sigma_2)$ hold, then $z \in \mathcal{D}_\Gamma$, a fundamental domain of Γ.

If $N(\sigma_2)$ and $P(\sigma_0)$ hold, then $z = h_0 = \sqrt{11}/2 - i\sqrt{3}/6$, $|h_0|^2 = 17/6 = 2.8333$ (see Lemma 4.2).

Let us assume that $P(\sigma_2)$ and $N(\sigma_0)$ hold. If $P(\sigma_0')$ holds, then $z = f_0' = \left(\sqrt{11} + i3\sqrt{3}\right)/4$, $|f_0|^2 = 19/8 = 2.375$ (see Lemma 4.2).

If $P(U)$ holds, then $z \in \mathcal{D}_\Gamma$. Hence we can assume that $N(U)$ holds. If $N(\sigma_0')$ holds, then $|z|^2 \geqslant |h'|^2 = |\sqrt{11}/8 + i29\sqrt{3}/24|^2 = 427/96 = 4.55208333$.

If $N(\sigma_2)$ and $N(\sigma_0)$ hold, then $N(S_1)$ and $N(\sigma_0)$ hold, in which case $|z|^2 \geqslant |h''|^2 = |5\sqrt{11}/8 + i7\sqrt{3}/24|^2 = 427/96 = 4.55208333$.

We have proved the following.

Lemma 4.3. *Let an extremal indefinite $z \in D'_T$. Then $z = f'_0$ or $z = h_0$ or*

1) *$N(\sigma'_0)$ and $N(U)$ hold, or*
2) *$N(S_1)$ and $N(\sigma_0)$ hold.*

1) Assume that $N(\sigma_0)$ and $N(U')$ hold. If $P(\varphi)$ and $P(\psi)$ hold, then $z = h = -\sqrt{11}/8 + i29\sqrt{3}/24$ (see Lemma 4.2). If $N(\varphi)$ and $N(\sigma_0)$ hold, then $|z|^2 > 5.27348$. If $N(\psi)$ and $N(U')$ hold, then $|z|^2 > 6.24218$. If $N(\varphi)$ and $N(\psi)$ hold, then $|z|^2 > 8.3383$. Thus, if $N(\sigma_0)$ and $N(U')$ hold, then either $z = h$ or $|z|^2 > 5.27348$.

2) Let $N(\sigma_0)$ and $N(S_1)$ hold. If $P(\varphi^{-1})$ and $P(\psi^{-1})$ hold, then $z = h'' = 5\sqrt{11}/8+i7\sqrt{3}/24$ (see Lemma 4.2). If $N(\varphi^{-1})$ and $N(S_1)$ hold, then $|z|^2 > 5.2324$. If $N(\varphi^{-1})$ and $N(\psi^{-1})$ hold, then $|z|^2 > 8.3383$. If $N(\psi^{-1})$ and $N(\sigma_0)$ hold, then $|z|^2 > 5.9738$. Thus, if $N(\sigma_0)$ and $N(S_1)$ hold, then either $z = h''$ or $|z|^2 > 5.2324$.

We have proved the following.

Lemma 4.4. *Let an extremal indefinite $z \in D'_T$. Then $z = f'_0$ or $z = h_0$ or $z = h$ or $z = h''$.*

By (3.1), Lemmas 4.1 and 4.4 imply the following.

Theorem 4.1. $\mathcal{M}(B_{33}) \cap [\sqrt{14}/2, \infty) = \{\sqrt{14}/2, \sqrt{437/124}, 2, \sqrt{17}/2, \sqrt{19}/2\}$.

Let L_0, L_1 and L_2 be the axes of $\rho F_0 \rho^{-1}$, $\rho H_0 \rho^{-1}$ and $\rho H \rho^{-1}$ respectively.

If $\nu(L) = \sqrt{19}/2$ for a geodesic L in H^3, then $L = g(L_0)$ for some $g \in B_{33}$.

If $\nu(L) = \sqrt{17}/2$ for a geodesic L in H^3, then $L = g(L_1)$ for some $g \in B_{33}$.

If $\nu(L) = 2$ for a geodesic L in H^3, then $L = g(L_{58})$ for some $g \in B_{33}$.

If $\nu(L) = \sqrt{437/124}$ for a geodesic L in H^3, then $L = g(L_2)$ for some $g \in B_{33}$.

The geodesic L_{45} is extremal with respect to B_{33} and $\nu(L_{45}) = 2/\sqrt{14}$.

As mentioned above, $\mathcal{M}(B_{33})$ coincides with the Markov spectrum of the field $\mathbf{Q}(\sqrt{-33})$, which, as follows from Theorem 4.1, equals to $\mathcal{L}_{33}$ in the interval $[\sqrt{14}/2, \infty)$ (see e.g. [31], p. 41).

5. Diophantine approximation in $\mathbf{Q}(\sqrt{-57})$

5.1. Reduction

Let $d = 57$ and $\omega = i\sqrt{57}$. The group $G_\infty = \mathrm{Stab}(\infty, B_{57})$ is generated by reflections in the vertical plains in H^3 through the lines $\mathrm{Re}\, z = 0$, $\mathrm{Re}\, z = 1/2$, $\mathrm{Im}\, z = 0$ and $\mathrm{Im}\, z = \sqrt{57}/2$ in $\mathbf{C}$, which will be denoted by S_{15}, S_{16}, S_{17}, and S_{18}

respectively. The floor of the isometric fundamental domain $\mathcal{D}$ lies in hemispheres

$$S_1 = S(0,1), \qquad S_2 = S\left(\frac{1+\omega}{2}, \frac{1}{\sqrt{2}}\right), \quad S_3 = S\left(\frac{\omega}{3}, \frac{1}{\sqrt{3}}\right),$$
$$S_4 = S\left(\frac{3+\omega}{6}, \frac{1}{\sqrt{6}}\right), \qquad S_5 = S\left(\frac{1+\omega}{4}, \frac{1}{\sqrt{8}}\right), \quad S_6 = S\left(\frac{\omega}{6}, \frac{1}{\sqrt{12}}\right),$$
$$S_7 = S\left(\frac{-24}{\omega}, \frac{1}{\sqrt{19}}\right), \qquad S_8 = S\left(\frac{-12}{\omega}, \frac{1}{\sqrt{19}}\right), \quad S_9 = S\left(\frac{24+\omega}{3-\omega}, \frac{1}{\sqrt{22}}\right),$$
$$S_{10} = S\left(\frac{9+\omega}{9-\omega}, \frac{1}{\sqrt{23}}\right), \quad S_{11} = S\left(\frac{-18+4\omega}{9+\omega}, \frac{1}{\sqrt{23}}\right),$$
$$S_{12} = S\left(\frac{2(1+\omega)}{5}, \frac{1}{5}\right), \quad S_{13} = S\left(\frac{1+\omega}{5}, \frac{1}{5}\right), \quad S_{14} = S\left(\frac{-33+\omega}{2\omega}, \frac{1}{\sqrt{38}}\right).$$

The forms $\Phi_5 = (2,(1+\omega)/2,7)$ and $\Phi_6 = (2,\omega/3,3)$ are reflection forms since $\Delta(\Phi_5) = 1/2$, $\Delta(\Phi_6) = 1/3$ and $6|D$ (see [26], Lemma 4). Reflections in hemispheres S_i, $i = 1,2,3,5,6,14,15,16,17,18$, belong to B_{57}. Hemispheres S_3, S_5 and S_{14} are mutually perpendicular. Also, S_{14} is a boundary hemisphere with $r(\Phi_{14}) = 6$. Denote by τ_i reflection in the hemisphere S_i and

$$H_1 = \begin{bmatrix} -24 & 5\omega \\ \omega & 12 \end{bmatrix}_{-3}, \qquad \tau = \begin{bmatrix} 24+\omega & 18-10\omega \\ 3-\omega & -24-\omega \end{bmatrix}_{-3},$$

$$\tau' = \begin{bmatrix} 3+\omega & 9-\omega \\ 6 & -3-\omega \end{bmatrix}_{-6},$$

$H_2 = \tau H_1$ and $H_3 = \tau'(\overline{H_2})^{-1}$. The axis of H_1 lies in the plane $\operatorname{Re} z = 0$, the axis of τ' is perpendicular to the plane $\operatorname{Re} z = 1/2$ and the axis of τ is perpendicular to the hemispheres S_2 and S_3.

Since the isometric fundamental domain of $\langle H_1\rangle$ is the exterior of the hemispheres S_7 and S_8 in H^3, the isometric fundamental domain of $\langle H_2\rangle$ is the exterior of the hemispheres S_{12} and $\tau_{15}(S_{13})$ in H^3, the isometric fundamental domain of $\langle H_3\rangle$ is the exterior of the hemispheres S_{10} and $\tau_{15}(S_{11})$ in H^3, $\tau(S_9) = S_9$ and $\tau'(S_4) = S_4$, we have proved the following.

Theorem 5.1. *The extended Bianchi group B_{57} is generated by τ, τ', H_1 and by reflections in S_i, $i = 1,2,3,5,6,14,15,16,17,18$.*

The hemisphere S_{14} is anisotropic, and the lowest face of $\mathcal{D}$ lies in S_{14}. Since the diameter of S_{14} is $2/\sqrt{38}$, the highest Hermitian point of $\mathcal{M}(B_{57})$ is

$$H_{57} = \sqrt{38}/2.$$

(cf. [34], Example 5.1). Denote by h_{ij} the height of the geodesic $L_{ij} = S_i \cap S_j$.

Then

$$h_{2,15}=\frac{1}{2},\qquad h_{35}=\frac{1}{\sqrt{11}},\qquad h_{3,16}=\frac{1}{\sqrt{12}},\qquad h_{5,15}=h_{5,16}=\frac{1}{4},$$

$$h_{46}=\frac{1}{\sqrt{18}},\qquad h_{14,7}=h_{14,8}=\frac{1}{\sqrt{19}},\qquad h_{10,1}=h_{11,16}=\frac{\sqrt{91}}{46},$$

$$h_{58}=h_{27}=\sqrt{\frac{37}{874}},\qquad h_{29}=\frac{1}{\sqrt{24}},\qquad h_{39}=\frac{1}{5},$$

$$h_{10,4}=h_{4,13}=h_{12,11}=\frac{1}{5}\sqrt{\frac{67}{69}},\qquad h_{37}=h_{68}=\frac{1}{10}\sqrt{\frac{73}{19}},$$

$$h_{13,8}=h_{9,12}=h_{97}=\frac{1}{5}\sqrt{\frac{177}{209}},\qquad h_{5,13}=h_{2,12}=\frac{1}{\sqrt{29}}\sqrt{\frac{49}{50}},$$

$$h_{10,6}=h_{11,3}=\frac{1}{2}\sqrt{\frac{89}{667}},\qquad h_{6,13}=h_{3,12}=\frac{1}{\sqrt{31}}\sqrt{\frac{97}{100}},$$

$$h_{45}=h_{1,4}=\frac{1}{\sqrt{33}},\qquad h_{3,14}=\frac{1}{\sqrt{41}},\qquad h_{5,14}=\frac{1}{\sqrt{46}}.$$

If the center of S_i is located in S_{15} (or S_{16}) then $h_{i,15}=R_i$ (or $h_{i,16}=R_i$).

The hemispheres S_1, S_6, S_{10} and S_{15} meet at a vertex of $\mathcal{D}$, and the hemispheres S_2, S_{11}, S_{12} and S_{16} meet at another vertex of $\mathcal{D}$.

The group $G_S=\mathrm{Stab}(S_{14},B_{57})=\langle\gamma,\gamma_0,\gamma_1\rangle$, where $\gamma=\tau_5\tau_{14}$, $\gamma_0=\tau_3\tau_{14}$, $\gamma_1=\tau_{16}\tau_{14}$.

The heights of all the edges in the floor of $\mathcal{D}$, which do not lie in S_{14}, are greater than $1/\sqrt{38}$. Thus, the reduction for $d=57$ is similar to that for $d=30$.

Let $S=\gamma\gamma_0$, $A=\gamma_1\gamma_0$, $B=\gamma_1\gamma$. Then $S^2=A^6=B^4=id$. Thus the group G_S contains the $(2,4,6)$-triangle group as a subgroup of index two. The triangular face D_S of $\mathcal{D}$, which lies in S_{14}, with vertices at

$$v_1=\left(\frac{9+5\omega}{18},\frac{1}{\sqrt{54}}\right),v_2=\left(\frac{5+3\omega}{10},\frac{1}{\sqrt{50}}\right),v_3=\left(\frac{3+2\omega}{7},\frac{1}{7}\right),$$

is a fundamental domain of G_S, $S_{14}\cap K(\infty)$ consists of two copies of D_S. and

$$\mathrm{Stab}(v_1,B_{57})=\{\tau_5,\tau_{14},\tau_{16}:(\tau_5\tau_{14})^2=(\tau_{14}\tau_{16})^2=(\tau_{16}\tau_5)^4=1\},$$
$$\mathrm{Stab}(v_2,B_{57})=\{\tau_3,\tau_{14},\tau_{16}:(\tau_3\tau_{14})^2=(\tau_{14}\tau_{16})^2=(\tau_{16}\tau_3)^6=1\},$$
$$\mathrm{Stab}(v_3,B_{57})=\{\tau_3,\tau_5,\tau_{14}:(\tau_3\tau_5)^2=(\tau_5\tau_{14})^2=(\tau_{14}\tau_3)^2=1\}.$$

The geodesic $L_{5,14}$ is perpendicular to S_3 and to the axis $L_{5,16}$ of $\tau_{16}\tau_5$. Denote $U=(\tau_{16}\tau_5)^2$. Then $\Gamma_{5,14}=\mathrm{Stab}(L_{5,14},B_{57})=\langle\tau_3,U\rangle$. Let $t_1=L_{5,14}\cap S_3$ and $t_2=L_{5,14}\cap L_{5,16}$. Then the arc $[t_1,t_2]=L_{5,14}\cap K(\infty)$ is a fundamental domain of$\Gamma_{5,14}$ on $L_{5,14}$ and, by Corollary 24, [30], $L_{5,14}$ is extremal. Since the height of

$L_{5,14}$ is $1/\sqrt{46}$ and, as shown above, the height of any extremal geodesic is at least $1/\sqrt{46}$, the Hurwitz constant of the field $\mathbf{Q}\left(\sqrt{-57}\right)$ is

$$C_{57} = \sqrt{46}/2.$$

(cf. [35], Table 1).

The hemisphere S_{14} contains four vertices of $\mathcal{D}$, v_1, v_2, v_3 and $v_3' = \tau_{16}(v_3)$.

The v-cells $N(v_1)$, $N(v_2)$, and $N(v_3)$ are of the same type as in the case of $d = 30$ (see Section 3). The v-cell $N(v_3)$ is a rectangular parallelepiped. The v-cells $N(v_1)$ and $N(v_2)$ are square and hexagonal prisms respectively. The geodesic $L_{5,16}$ is the axis of symmetry of order 4 in $N(v_1)$ and $L_{3,16}$ is the axis of symmetry of order 6 in $N(v_2)$.

The common rectangular face ϕ_{23} of $N(v_3)$ and $N(v_2)$ has vertices at $\omega/3$, $(19+12\omega)/41$, $(-33+\omega)/(2\omega)$ and ∞. Its Farey constant is $2h_{3,14} = 2/\sqrt{41}$. All the rectangular faces of $N(v_2)$ are congruent to this face.

The common rectangular face ϕ_{13} of $N(v_3)$ and $N(v_1)$ has vertices at $(1+\omega)/4$, $(21+13\omega)/46$, $(-33+\omega)/(2\omega)$ and ∞. Its Farey constant is $2h_{5,14} = 2/\sqrt{46}$. All the rectangular faces of $N(v_1)$ are congruent to this face.

Let $2/\sqrt{46} \leqslant k < 2/\sqrt{38}$. Then $N(v_i, k)$ has a geodesic face ϕ if and only if ϕ lies in a rectangular face of $N(v_i)$, which is congruent to ϕ_{23} or ϕ_{13}, since only for such a face the Farey constant is less than $2/\sqrt{38}$. But, the centers of all such faces lie in S_{14}. Hence, if the height of an extremal geodesic L is less than $1/\sqrt{38}$, then $g(L) \subset S_{14}$ for some translation $g \in B_{57}$. Indeed, an extremal geodesic L, which cuts $N(v, k)$, must enter through one of its geodesic faces and exit through another. Since the limit points of the sequence of v-cells cut by L are the endpoints of L and they lie in S_{14}, L itself lies in S_{14}.

Lemma 5.1. *If the height of an extremal geodesic L in H^3 is less than $1/\sqrt{38}$, then $L \subset gS$, $g \in G_\infty$. Thus,*

$$\mathcal{M}(B_{57}) \cap [\sqrt{38}/2, \sqrt{46}/2] = \mathcal{M}_S \cap [\sqrt{38}/2, \sqrt{46}/2].$$

5.2. A group with signature (0; 2, 4, 6)

Let $S_{14} = S(b_{14}, R_{14})$ and $\rho = \begin{pmatrix} R_{14} & b_{14} \\ 0 & 1 \end{pmatrix}$. Let C_1 be the circle $|z - b_{14}| = 1/\sqrt{38}$ in the complex plane $\mathbf{C}$. Then $G_S = \mathrm{Stab}(C_1, B_{57})$ and $C_1 = \rho(C)$, where C is the unit circle $|z| = 1$. The group $\Gamma = \rho^{-1} G_S\, \rho$ is a discrete cocompact subgroup of the group of isometries G_C of D^2. It is generated by reflections

$$\sigma = \begin{bmatrix} i\sqrt{6} & 5+\omega \\ 5-\omega & -i\sqrt{6} \end{bmatrix}, \qquad \sigma_0 = \begin{bmatrix} -2i\sqrt{6} & 9-\omega \\ 9+\omega & 2i\sqrt{6} \end{bmatrix}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

across the sides of the triangle with vertices $a = i\sqrt{6}/5$, $b = -2i\sqrt{6}/9$, and $s = -(\sqrt{38} + i\sqrt{6})/14$, which are the fixed points of $A = \sigma_1\sigma_0$, $B = \sigma_1\sigma$ and $S = \sigma\sigma_0$ respectively. One has $S^2 = A^6 = B^4 = \mathrm{id}$. Denote $\sigma_k = \sigma_0 A^k$, $k = 1, ..., 5$. For $g \in \Gamma$, denote $g' = \sigma_1 g \sigma_1$. Let $\phi = \sigma_3 S$ and $\phi_1 = \sigma' S$.

Lemma 5.2. *For the fixed points f, f_1 of ϕ, and ϕ_1 respectively, we have*

$$(f, \sigma w) = (f, \sigma_3 w) = -(f, \sigma_0 w) = -(f, Sw) = -(f, Uw) = 1,$$
$$(f_1, \sigma' w) = (f_1, \sigma_0 w) = -(f_1, \sigma w) = -(f_1, Sw) = -(f_1, B^2 w) = 1.$$

5.3. Uniqueness

Denote by D_T the disc $|z| < 4.2$ and $D'_T = \{z \in D_T : \operatorname{Re} z \leqslant 0\}$. Since $\Gamma_w = \langle\sigma_1\rangle$, we can assume that $z \in D'_T$.

If $P(\sigma)$ and $P(\sigma_0)$ hold, then $z \in \mathcal{D}_\Gamma$, a fundamental domain of Γ. If $N(\sigma)$ and $N(\sigma_0)$ hold, then $|z| \geqslant |-57\sqrt{38} + 13i\sqrt{6}|/84 > 4.2$.

If $N(S')$ holds, then $|z| \geqslant |i7\sqrt{6}/3| = 5.7$. Hence $P(S')$ holds. If $N(\sigma')$ holds, then $|z| \geqslant |-15\sqrt{38} - 139i\sqrt{6}|/84 > 4.2$. Hence $P(\sigma')$ holds. If $N(\sigma)$ and $P(\sigma_0)$ hold, then $z = f_1 = -(\sqrt{38} + 3i\sqrt{6})/4$.

Assume that $N(\sigma_0)$ holds. If $P(S)$ holds, then $|z| \geqslant |-27\sqrt{38} + 127i\sqrt{6}|/84 > 4.2$. Hence $N(S)$ holds. If $N(\sigma_3)$ hold, then $z \notin D'_T$. Hence $P(\sigma_3)$ holds. If $P(\sigma)$ hold, then $z = f = -\sqrt{38}/2 + i5/\sqrt{6}$.

Lemma 5.3. *Let $z \in D'_T$. Then $z = f = -\sqrt{38}/2 + i5/\sqrt{6}$, or $z = f_1 = -(\sqrt{38} + 3i\sqrt{6})/4$, or $|z| \geqslant 4.2$.*

Thus, by (3.1) and Lemma 5.3, the first two points of $\mathcal{M}(B_{57})$ are $\nu(f) = \sqrt{46}/2$ and $\nu(f_1) = \sqrt{41}/2$. We have proved the following.

Theorem 5.2. $\mathcal{M}(B_{57}) \cap (3.1735, \infty) = \{\sqrt{41}/2,\ \sqrt{46}/2\}$.

If $\nu(L) = \sqrt{46}/2$ for a geodesic L in H^3, then $L = g(L_{5,14})$ for some $g \in B_{57}$.

If $\nu(L) = \sqrt{41}/2$ for a geodesic L in H^3, then $L = g(L_{3,14})$ for some $g \in B_{57}$.

As mentioned above, $\mathcal{M}(B_{57})$ coincides with the Markov spectrum of the field $\mathbf{Q}(\sqrt{-57})$, which, as follows from Theorem 5.2, equals to $\mathcal{L}_{57}$ in the interval $(3.1735, \infty)$ (see e.g. [31], p. 41).

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