

SOME ARITHMETIC IDENTITIES INVOLVING DIVISOR FUNCTIONS

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Abstract: For a positive integer n , let

$$\sigma(n) := \sum_{\substack{d \in \mathbb{N} \\ d|n}} d.$$

The explicit evaluation of such arithmetic sums as

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+4c=n}} \sigma(a)\sigma(b)\sigma(c) \quad \text{and} \quad \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b)$$

is carried out for all positive integers n .

Keywords: sum of divisors function, Eisenstein series.

1. Introduction

Let \mathbb{N} denote the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and \mathbb{Z} the set of all integers. For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define as usual

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d|n}} d^k. \tag{1.1}$$

If $n \notin \mathbb{N}$, we set $\sigma_k(n) = 0$. Further, we set $\sigma(n) := \sigma_1(n)$.

For a complex variable q satisfying $|q| < 1$, Ramanujan's Eisenstein series

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$L(q), M(q)$ and $N(q)$ are defined by

$$L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \tag{1.2}$$

$$M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \tag{1.3}$$

$$N(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n. \tag{1.4}$$

Further, for $r \in \mathbb{N}_0$ and $s, k \in \mathbb{N}$, we define

$$[r, s]_k := \sum_{n=1}^{\infty} n^r \sigma_s(n)q^{kn}. \tag{1.5}$$

Lahiri [4, p. 149] has derived the following thirteen identities from the work of Ramanujan [5], [6, Theorem 2.4, pp. 141 - 142]:

$$L^2(q) = 1 - 288[1, 1]_1 + 240[0, 3]_1, \tag{1.6}$$

$$L^3(q) = 1 - 1728[2, 1]_1 + 2160[1, 3]_1 - 504[0, 5]_1, \tag{1.7}$$

$$L^4(q) = 1 - 6912[3, 1]_1 + 10368[2, 3]_1 - 4032[1, 5]_1 + 480[0, 7]_1, \tag{1.8}$$

$$L^5(q) = 1 - 20736[4, 1]_1 + 34560[3, 3]_1 - 17280[2, 5]_1 + 3600[1, 7]_1 - 264[0, 9]_1, \tag{1.9}$$

$$M^2(q) = 1 + 480[0, 7]_1, \tag{1.10}$$

$$L(q)M(q) = 1 + 720[1, 3]_1 - 504[0, 5]_1, \tag{1.11}$$

$$L^2(q)M(q) = 1 + 1728[2, 3]_1 - 2016[1, 5]_1 + 480[0, 7]_1, \tag{1.12}$$

$$L^3(q)M(q) = 1 + 3456[3, 3]_1 - 5184[2, 5]_1 + 2160[1, 7]_1 - 264[0, 9]_1, \tag{1.13}$$

$$L(q)M^2(q) = 1 + 720[1, 7]_1 - 264[0, 9]_1, \tag{1.14}$$

$$L(q)N(q) = 1 - 1008[1, 5]_1 + 480[0, 7]_1, \tag{1.15}$$

$$L^2(q)N(q) = 1 - 1728[2, 5]_1 + 1440[1, 7]_1 - 264[0, 9]_1, \tag{1.16}$$

$$M(q)N(q) = 1 - 264[0, 9]_1, \tag{1.17}$$

$$M^2(q)N(q) = 1 - 24[0, 13]_1. \tag{1.18}$$

In this paper, we are interested in products of the form

$$L^{a_1}(q)L^{a_2}(q^2)L^{a_4}(q^4)M^{b_1}(q)M^{b_2}(q^2)M^{b_4}(q^4)N^{c_1}(q)N^{c_2}(q^2)N^{c_4}(q^4) \tag{1.19}$$

with

$$\begin{cases} a_1, a_2, a_4, b_1, b_2, b_4, c_1, c_2, c_4 \in \mathbb{N}_0, \\ \max(a_1, b_1, c_1) \in \mathbb{N}, \\ \max(a_2, b_2, c_2, a_4, b_4, c_4) \in \mathbb{N}. \end{cases} \tag{1.20}$$

We seek those products which can be expressed as a linear combination of the series $[r, s]_1, [r, s]_2$ and $[r, s]_4$ ($r \in \mathbb{N}_0, s \in \mathbb{N}$). Guided by (1.6)-(1.18), a computer search for such products was carried out in the range

$$\begin{cases} \max(a_1, a_2, a_4) = 5, \\ \max(b_1, b_2, b_4) = 2, \\ \max(c_1, c_2, c_4) = 1, \\ \operatorname{sgn}(a_1) + \operatorname{sgn}(a_2) + \operatorname{sgn}(a_4) + \operatorname{sgn}(b_1) \\ + \operatorname{sgn}(b_2) + \operatorname{sgn}(b_4) + \operatorname{sgn}(c_1) + \operatorname{sgn}(c_2) + \operatorname{sgn}(c_4) \leq 3. \end{cases} \tag{1.21}$$

The search produced seven possible identities. All seven candidates were subsequently proved and are given in Theorem 1.1. The proof of Theorem 1.1 is given in Section 2.

Theorem 1.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(i)

$$L(q)L(q^2) = 1 - 72[1, 1]_1 + 48[0, 3]_1 - 288[1, 1]_2 + 192[0, 3]_2,$$

(ii)

$$L(q)L(q^4) = 1 - 36[1, 1]_1 + 12[0, 3]_1 + 36[0, 3]_2 - 576[1, 1]_4 + 192[0, 3]_4,$$

(iii)

$$L(q)M(q^2) = 1 - 24[0, 5]_1 + 1440[1, 3]_2 - 480[0, 5]_2,$$

(iv)

$$L(q^2)M(q) = 1 + 360[1, 3]_1 - 120[0, 5]_1 - 384[0, 5]_2,$$

(v)

$$\begin{aligned} L(q)L^2(q^2) &= 1 - 144[2, 1]_1 + 144[1, 3]_1 - 24[0, 5]_1 - 2304[2, 1]_2 \\ &\quad + 2592[1, 3]_2 - 480[0, 5]_2, \end{aligned}$$

(vi)

$$\begin{aligned} L^2(q)L(q^2) &= 1 - 576[2, 1]_1 + 648[1, 3]_1 - 120[0, 5]_1 - 2304[2, 1]_2 \\ &\quad + 2304[1, 3]_2 - 384[0, 5]_2, \end{aligned}$$

(vii)

$$\begin{aligned} L(q)L(q^2)L(q^4) &= 1 - 72[2, 1]_1 + 54[1, 3]_1 - 6[0, 5]_1 \\ &\quad - 576[2, 1]_2 + 684[1, 3]_2 - 114[0, 5]_2 \\ &\quad - 4608[2, 1]_4 + 3456[1, 3]_4 - 384[0, 5]_4. \end{aligned}$$

Equating the coefficients of q^n ($n \in \mathbb{N}$) in these identities, we obtain seven arithmetic identities. These identities are given in Theorem 1.2. The proof of Theorem 1.2 is given in Section 3.

Theorem 1.2. *Let $n \in \mathbb{N}$. We have*

(i)

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} \sigma(a)\sigma(b) = \frac{1}{12}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma(n) + \frac{1}{3}\sigma_3(n/2) \\ + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/2),$$

(ii)

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+4b=n}} \sigma(a)\sigma(b) = \frac{1}{48}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma(n) + \frac{1}{16}\sigma_3(n/2) \\ + \frac{1}{3}\sigma_3(n/4) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n/4),$$

(iii)

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} \sigma(a)\sigma_3(b) = \frac{1}{240}\sigma_5(n) - \frac{1}{240}\sigma(n) + \frac{1}{12}\sigma_5(n/2) \\ + \left(\frac{1}{24} - \frac{1}{8}n\right)\sigma_3(n/2),$$

(iv)

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+b=n}} \sigma(a)\sigma_3(b) = \frac{1}{48}\sigma_5(n) + \left(\frac{1}{24} - \frac{1}{16}n\right)\sigma_3(n) + \frac{1}{15}\sigma_5(n/2) \\ - \frac{1}{240}\sigma(n/2),$$

(v)

$$\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+2c=n}} \sigma(a)\sigma(b)\sigma(c) = \frac{1}{576}\sigma_5(n) + \left(\frac{1}{144} - \frac{1}{96}n\right)\sigma_3(n) \\ + \left(\frac{1}{576} - \frac{1}{96}n + \frac{1}{96}n^2\right)\sigma(n) + \frac{5}{144}\sigma_5(n/2) \\ + \left(\frac{13}{288} - \frac{3}{32}n\right)\sigma_3(n/2) \\ + \left(\frac{1}{288} - \frac{1}{32}n + \frac{1}{24}n^2\right)\sigma(n/2),$$

(vi)

$$\begin{aligned} \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+2c=n}} \sigma(a)\sigma(b)\sigma(c) &= \frac{5}{576}\sigma_5(n) + \left(\frac{7}{288} - \frac{3}{64}n\right)\sigma_3(n) \\ &+ \left(\frac{1}{288} - \frac{1}{32}n + \frac{1}{24}n^2\right)\sigma(n) \\ &+ \frac{1}{36}\sigma_5(n/2) + \left(\frac{1}{36} - \frac{1}{12}n\right)\sigma_3(n/2) \\ &+ \left(\frac{1}{576} - \frac{1}{48}n + \frac{1}{24}n^2\right)\sigma(n/2), \end{aligned}$$

(vii)

$$\begin{aligned} \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+2b+4c=n}} \sigma(a)\sigma(b)\sigma(c) &= \frac{1}{2304}\sigma_5(n) + \left(\frac{5}{1152} - \frac{1}{256}n\right)\sigma_3(n) \\ &+ \left(\frac{1}{576} - \frac{1}{128}n + \frac{1}{192}n^2\right)\sigma(n) \\ &+ \frac{19}{2304}\sigma_5(n/2) + \left(\frac{23}{1152} - \frac{19}{768}n\right)\sigma_3(n/2) \\ &+ \left(\frac{1}{576} - \frac{5}{384}n + \frac{1}{96}n^2\right)\sigma(n/2) \\ &+ \frac{1}{36}\sigma_5(n/4) + \left(\frac{1}{36} - \frac{1}{16}n\right)\sigma_3(n/4) \\ &+ \left(\frac{1}{576} - \frac{1}{64}n + \frac{1}{48}n^2\right)\sigma(n/4). \end{aligned}$$

Parts (i), (ii), (iii) and (iv) of Theorem 1.2 are due to Huard, Ou, Spearman and Williams [3, Theorems 2, 4, 6]. The rest are new. Two further arithmetic identities are deduced from Theorem 1.1 in Section 4 (see Theorem 4.1).

2. Proof of Theorem 1.1

Let q be a complex variable satisfying $|q| < 1$. Define the theta function $\varphi(q)$ (Ramanujan’s notation [1, p. 6]) by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{2.1}$$

Further, we define the parameters $x = x(q)$ and $z = z(q)$ by

$$x := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z := \varphi^2(q), \tag{2.2}$$

(see [1, pp. 119 - 120]). Then it is known that

$$L(q) = (1 - 5x)z^2 + 12x(1 - x)zz', \tag{2.3}$$

$$L(q^2) = (1 - 2x)z^2 + 6x(1 - x)zz', \tag{2.4}$$

$$L(q^4) = \left(1 - \frac{5}{4}x\right)z^2 + 3x(1 - x)zz', \tag{2.5}$$

$$M(q) = (1 + 14x + x^2)z^4, \tag{2.6}$$

$$M(q^2) = (1 - x + x^2)z^4, \tag{2.7}$$

$$M(q^4) = \left(1 - x + \frac{1}{16}x^2\right)z^4, \tag{2.8}$$

$$N(q) = (1 - 33x - 33x^2 + x^3)z^6, \tag{2.9}$$

$$N(q^2) = \left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)z^6, \tag{2.10}$$

$$N(q^4) = \left(1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3\right)z^6, \tag{2.11}$$

(see [1, pp. 125 - 128]), where $z' = \frac{dz}{dx}$. From (1.3) and (1.5), we have

$$[0, 3]_1 = \sum_{n=1}^{\infty} \sigma_3(n)q^n = -\frac{1}{240} + \frac{1}{240}M(q). \tag{2.12}$$

From (1.4) and (1.5), we have

$$[0, 5]_1 = \sum_{n=1}^{\infty} \sigma_5(n)q^n = \frac{1}{504} - \frac{1}{504}N(q). \tag{2.13}$$

Then, from (1.6) and (2.12), we deduce

$$[1, 1]_1 = -\frac{1}{288}L^2(q) + \frac{1}{288}M(q). \tag{2.14}$$

From (1.11) and (2.13), we have

$$[1, 3]_1 = \frac{1}{720}L(q)M(q) - \frac{1}{720}N(q). \tag{2.15}$$

From (1.7), (2.13) and (2.15), we obtain

$$[2, 1]_1 = -\frac{1}{1728}L^3(q) + \frac{1}{576}L(q)M(q) - \frac{1}{864}N(q). \tag{2.16}$$

We now carry out the details of the proof of part (v) of Theorem 1.1. The remaining parts can be proved in a similar manner.

Appealing to (2.3), (2.4), (2.6), (2.7), (2.9), (2.10), (2.13), (2.15) and (2.16), we obtain

$$\begin{aligned}
 & 1 - 144[2, 1]_1 + 144[1, 3]_1 - 24[0, 5]_1 - 2304[2, 1]_2 + 2592[1, 3]_2 - 480[0, 5]_2 \\
 &= 1 - 144\left(-\frac{1}{1728}L^3(q) + \frac{1}{576}L(q)M(q) - \frac{1}{864}N(q)\right) \\
 &\quad + 144\left(\frac{1}{720}L(q)M(q) - \frac{1}{720}N(q)\right) - 24\left(\frac{1}{504} - \frac{1}{504}N(q)\right) \\
 &\quad - 2304\left(-\frac{1}{1728}L^3(q^2) + \frac{1}{576}L(q^2)M(q^2) - \frac{1}{864}N(q^2)\right) \\
 &\quad + 2592\left(\frac{1}{720}L(q^2)M(q^2) - \frac{1}{720}N(q^2)\right) - 480\left(\frac{1}{504} - \frac{1}{504}N(q^2)\right) \\
 &= \frac{1}{12}L^3(q) - \frac{1}{20}L(q)M(q) + \frac{1}{70}N(q) + \frac{4}{3}L^3(q^2) - \frac{2}{5}L(q^2)M(q^2) \\
 &\quad + \frac{2}{105}N(q^2) \\
 &= \frac{1}{12}\left((1 - 5x)z^2 + 12x(1 - x)zz'\right)^3 \\
 &\quad - \frac{1}{20}\left((1 - 5x)z^2 + 12x(1 - x)zz'\right)(1 + 14x + x^2)z^4 \\
 &\quad + \frac{1}{70}(1 - 33x - 33x^2 + x^3)z^6 + \frac{4}{3}\left((1 - 2x)z^2 + 6x(1 - x)zz'\right)^3 \\
 &\quad - \frac{2}{5}\left((1 - 2x)z^2 + 6x(1 - x)zz'\right)(1 - x + x^2)z^4 \\
 &\quad + \frac{2}{105}\left(1 - \frac{3}{2}x - \frac{3}{2}x^2 + x^3\right)z^6 \\
 &= (1 - 2x)^2(1 - 5x)z^6 + 12x(1 - x)(1 - 2x)(2 - 7x)z^5z' \\
 &\quad + 36x^2(1 - x)^2(5 - 13x)z^4z'^2 + 432x^3(1 - x)^3z^3z'^3 \\
 &= \left((1 - 5x)z^2 + 12x(1 - x)zz'\right)\left((1 - 2x)z^2 + 6x(1 - x)zz'\right)^2 \\
 &= L(q)L^2(q^2),
 \end{aligned}$$

as asserted.

We remark that in the course of the proof of part (v), we proved the new identity

$$\begin{aligned}
 L(q)L^2(q^2) &= \frac{1}{12}L^3(q) - \frac{1}{20}L(q)M(q) + \frac{1}{70}N(q) + \frac{4}{3}L^3(q^2) \\
 &\quad - \frac{2}{5}L(q^2)M(q^2) + \frac{2}{105}N(q^2).
 \end{aligned} \tag{2.17}$$

Similarly, from the proof of part (i), we have

$$L(q)L(q^2) = \frac{1}{4}L^2(q) + L^2(q^2) - \frac{1}{20}M(q) - \frac{1}{5}M(q^2), \tag{2.18}$$

from part (ii)

$$L(q)L(q^4) = \frac{1}{8}L^2(q) + 2L^2(q^4) - \frac{3}{40}M(q) + \frac{3}{20}M(q^2) - \frac{6}{5}M(q^4), \tag{2.19}$$

from part (iii)

$$L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2), \tag{2.20}$$

and from part (iv)

$$L(q^2)M(q) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2). \tag{2.21}$$

The four identities (2.18) - (2.21) are due to Cheng and Williams [2, Theorems 3.1 and 4.1(i),(ii)]. The corresponding identities for parts (vi) and (vii), namely,

$$\begin{aligned} L^2(q)L(q^2) &= \frac{1}{3}L^3(q) - \frac{1}{10}L(q)M(q) + \frac{1}{210}N(q) + \frac{4}{3}L^3(q^2) \\ &\quad - \frac{4}{5}L(q^2)M(q^2) + \frac{8}{35}N(q^2) \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} L(q)L(q^2)L(q^4) &= \frac{1}{24}L^3(q) - \frac{1}{20}L(q)M(q) + \frac{17}{840}N(q) + \frac{1}{3}L^3(q^2) \\ &\quad - \frac{1}{20}L(q^2)M(q^2) - \frac{2}{35}N(q^2) + \frac{8}{3}L^3(q^4) \\ &\quad - \frac{16}{5}L(q^4)M(q^4) + \frac{136}{105}N(q^4) \end{aligned} \tag{2.23}$$

are new.

3. Proof of Theorem 1.2

We only prove part (vi) of Theorem 1.2 as the remaining parts can be proved in a similar manner.

Using (1.2) we have

$$\begin{aligned} 13824 \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+2c=n}} \sigma(a)\sigma(b)\sigma(c) \right) q^n &= 24^3 \sum_{(a,b,c) \in \mathbb{N}^3} \sigma(a)\sigma(b)\sigma(c)q^{a+b+2c} \\ &= \left(24 \sum_{a=1}^{\infty} \sigma(a)q^a \right)^2 \left(24 \sum_{c=1}^{\infty} \sigma(c)q^{2c} \right) \\ &= (1 - L(q))^2(1 - L(q^2)) \\ &= 1 - 2L(q) + L^2(q) - L(q^2) \\ &\quad + 2L(q)L(q^2) - L^2(q)L(q^2). \end{aligned}$$

Appealing to (1.2), (1.6), (1.2) (with q replaced by q^2), Theorem 1.1(i) and Theorem 1.1(vi) for the series expansions in powers of q of $L(q)$, $L^2(q)$, $L(q^2)$, $L(q)L(q^2)$ and $L^2(q)L(q^2)$ respectively, and then equating the coefficients of q^n ($n \in \mathbb{N}$), we obtain the asserted formula on dividing by 13824. ■

4. Two Arithmetic Identities

Theorem 1.1 enables us to prove two more new identities involving the divisor function σ .

Theorem 4.1. *Let $n \in \mathbb{N}$. We have*

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b) = \frac{1}{24}n\sigma_3(n) + \left(\frac{1}{24}n - \frac{1}{12}n^2\right)\sigma(n) + \frac{1}{6}n\sigma_3(n/2) - \frac{1}{12}n^2\sigma(n/2)$$

and

$$\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma(n) + \frac{1}{12}n\sigma_3(n/2) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma(n/2).$$

Proof. Differentiating (1.2) with respect to q , we obtain

$$qL'(q) = -24 \sum_{n=1}^{\infty} n\sigma(n)q^n = -24[1, 1]_1. \tag{4.1}$$

Hence, by (1.6) and (2.12), we deduce

$$\begin{aligned} L^2(q) &= 1 - 288[1, 1]_1 + 240[0, 3]_1 \\ &= 1 + 12qL'(q) + 240\left(-\frac{1}{240} + \frac{1}{240}M(q)\right) \\ &= 12qL'(q) + M(q), \end{aligned}$$

so

$$qL'(q) = \frac{1}{12}L^2(q) - \frac{1}{12}M(q), \tag{4.2}$$

which is a well-known identity.

We now prove the first identity of the theorem. Multiplying (4.2) by $L(q^2)$, we obtain

$$qL'(q)L(q^2) = \frac{1}{12}L^2(q)L(q^2) - \frac{1}{12}L(q^2)M(q).$$

Appealing to Theorem 1.1 for the values of $L^2(q)L(q^2)$ and $L(q^2)M(q)$, we obtain

$$qL'(q)L(q^2) = -48[2, 1]_1 + 24[1, 3]_1 - 192[2, 1]_2 + 192[1, 3]_2.$$

Now, by (4.2) and (1.2), we have

$$\begin{aligned} qL'(q)L(q^2) &= \left(-24 \sum_{e=1}^{\infty} a\sigma(a)q^a \right) \left(1 - 24 \sum_{b=1}^{\infty} \sigma(b)q^{2b} \right) \\ &= -24[1, 1]_1 + 576 \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b) \right) q^n, \end{aligned}$$

so

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b) \right) q^n \\ = \frac{1}{24}[1, 1]_1 - \frac{1}{12}[2, 1]_1 + \frac{1}{24}[1, 3]_1 - \frac{1}{3}[2, 1]_2 + \frac{1}{3}[1, 3]_2. \end{aligned}$$

Equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the first identity.

We now prove the second identity. Replacing q by q^2 in (4.2) and then multiplying by $L(q)$, we obtain

$$q^2L'(q^2)L(q) = \frac{1}{12}L(q)L^2(q^2) - \frac{1}{12}L(q)M(q^2).$$

Appealing to Theorem 1.1 for the values of $L(q)L^2(q^2)$ and $L(q)M(q^2)$, we have

$$q^2L'(q^2)L(q) = 12[1, 3]_1 - 12[2, 1]_1 + 96[1, 3]_2 - 192[2, 1]_2.$$

Now

$$\begin{aligned} q^2L'(q^2)L(q) &= -24 \sum_{a=1}^{\infty} a\sigma(a)q^{2a} \left(1 - 24 \sum_{b=1}^{\infty} \sigma(b)q^b \right) \\ &= -24[1, 1]_2 + 576 \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b) \right) q^n, \end{aligned}$$

so

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b) \right) q^n \\ = \frac{1}{24}[1, 1]_2 + \frac{1}{48}[1, 3]_1 - \frac{1}{48}[2, 1]_1 + \frac{1}{6}[1, 3]_2 - \frac{1}{3}[2, 1]_2. \end{aligned}$$

Equating the coefficients of q^n ($n \in \mathbb{N}$), we obtain the second identity. ■

The second identity of Theorem 4.1 can be deduced from the first identity as follows:

$$\begin{aligned}
 \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ 2a+b=n}} a\sigma(a)\sigma(b) &= \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} b\sigma(a)\sigma(b) = \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} \left(\frac{n-a}{2}\right) \sigma(a)\sigma(b) \\
 &= \frac{n}{2} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} \sigma(a)\sigma(b) - \frac{1}{2} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a+2b=n}} a\sigma(a)\sigma(b) \\
 &= \frac{n}{2} \left(\frac{1}{12} \sigma_3(n) + \left(\frac{1}{24} - \frac{1}{8} n \right) \sigma(n) + \frac{1}{3} \sigma_3(n/2) + \left(\frac{1}{24} - \frac{1}{4} n \right) \sigma(n/2) \right) \\
 &\quad - \frac{1}{2} \left(\frac{1}{24} n \sigma_3(n) + \left(\frac{1}{24} n - \frac{1}{12} n^2 \right) \sigma(n) + \frac{1}{6} n \sigma_3(n/2) - \frac{1}{12} n^2 \sigma(n/2) \right) \\
 &= \frac{1}{48} n \sigma_3(n) - \frac{1}{48} n^2 \sigma(n) + \frac{1}{12} n \sigma_3(n/2) + \left(\frac{1}{48} n - \frac{1}{12} n^2 \right) \sigma(n/2)
 \end{aligned}$$

as claimed.

References

- [1] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, Rhode Island, 2006.
- [2] N. Cheng and K. S. Williams, *Evaluation of some convolution sums involving the sum of divisor functions*, *Yokohama Math. J.* **52** (2005), 39–57.
- [3] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, *Elementary evaluation of certain convolution sums involving divisor functions*, *Number Theory for the Millenium II*, edited by M. A. Bennett, B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand and W. Philipp, A K Peters, Natick, Massachusetts, USA, 2002, pp. 229–274.
- [4] D. B. Lahiri, *On Ramanujan's function $\tau(n)$ and the divisor function $\sigma_k(n)$ -I*, *Bull. Calcutta Math. Soc.* **38** (1946), 193–206.
- [5] S. Ramanujan, *On certain arithmetical functions*, *Trans. Cambridge Philos. Soc.* **22** (1916), 159–184.
- [6] S. Ramanujan, *Collected Papers*, AMS Chelsea Publishing, Providence, Rhode Island, USA, 2000.

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