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A COMBINATORIAL-GEOMETRIC VIEWPOINT OF KNOPP'S FORMULA FOR DEDEKIND SUMS

Kazuhito Kozuka

Abstract: In this paper, by means of a combinatorial-geometric method, we give a new proof of Knopp's formula for Dedekind sums and its generalizations to multiple Dedekind sums attached to Dirichlet characters. The combinatorial-geometric method for studying Dedekind sums were introduced by Beck, who proved the well-known reciprocity formula for Dedekind sums and some of its generalizations by the method. The motive of this paper is to find a similar approch to Knopp's formula .

Keywords: Dedekind sums, Knopp's formula.

1. Introduction

For $h \in \mathbf{Z}$ and $k \in \mathbf{N}$, the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\alpha \bmod k} \Big(\Big(\frac{\alpha}{k}\Big)\Big)\Big(\Big(\frac{h\alpha}{k}\Big)\Big),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbf{Z} \\ 0 & \text{if } x \in \mathbf{Z}. \end{cases}$$

Among many formulas for this sum, the following ones are well known:

(I) Reciprocity formula (Dedekind [5])

$$12hk\{s(h,k) + s(k,h)\} = h^2 - 3hk + k^2 + 1 \tag{1}$$

for $h, k \in \mathbf{N}$ with $(h,k) = 1$.

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(II) Knopp's formula (Knopp [6])

$$\sum_{\substack{ad=N \\ d>0}} \sum_{b=0}^{d-1} s(ah+bk, dk) = \sigma(N)s(h,k) \tag{2}$$

for $N \in \mathbf{N}$, where $\sigma(N) = \sum_{\delta|N} \delta$. Note that in the case that N is a prime number, the formula (2) was already known to Dedekind ([5]).

Generalizations of Dedekind sums and formulas (1) and (2) have been studied extensively with many methods. Recently, based on the works of Carlitz in [4], Beck gave geometric proofs of (1) and some of its generalizations including multivariable cases. ([1], [2], [8]). This method is deeply connected with the theory of lattice points in polytopes (cf. [3]). The basic idea for the proof of (1) is to decompose the lattice points of the first quadrant in the plane $\mathbf{R}^2$ by a certain ray. Let us sketch the method:

Suppose that $h, k \in \mathbf{N}$ and put

$$K_1 = \{(x,y) \in \mathbf{R}^2 | y \geqslant \frac{h}{k}x \geqslant 0\} \quad \text{and} \quad K_2 = \{(x,y) \in \mathbf{R}^2 | 0 \leqslant y < \frac{h}{k}x\}.$$

Then, we have the following identity of formal power series:

$$\sum_{(l,m)\in K_1 \cap \mathbf{Z}^2} u^l v^m + \sum_{(l,m)\in K_2 \cap \mathbf{Z}^2} u^l v^m = \sum_{l,m \geqslant 0} u^l v^m.$$

Both sides of this equation can be expressed by rational functions of u and v, from which the formula (1) is deduced by some calculations.

The motive of this paper is to find a similar approch to Knopp's formula (2) and its generalizations. In [7], we have already obtained a generalization of (2) by defining higher-order multiple Dedekind sums attached to Dirichlet characters ((7) of Theorem 4.1 in [7]). In this paper, we give a new proof of it by means of the combinatorial-geometric method. Let us give a description of each section.

In Section 2, we recall some definitions and state the main result.

In Section 3, for the purpose of providing a good overview, we prove the main result for the special case of non-multiple Dedekind sums without Dirichlet characters.

In Section 4, extending the idea in the previous section, we give a complete proof in the general case.

2. Definitions and the main result

Let B_p and $B_p(X)$ be the pth Bernoulli number and polynomial, respectively, defined by

$$\frac{t}{e^t - 1} = \sum_{p=0}^{\infty} B_p \frac{t^p}{p!} \quad \text{and} \quad \frac{te^{tX}}{e^t - 1} = \sum_{p=0}^{\infty} B_p(X) \frac{t^p}{p!}.$$

For any $x \in \mathbf{Q}$, we put $\{x\} = x - [x]$ and define $\tilde{B}_p(x) = B_p(\{x\})$, which is periodic of period 1.

For any primitive Dirichlet character χ, we denote by f_χ the conductor of χ. For any $x \in \mathbf{Q}$ with denominator relatively prime to f_χ, we can define the value $\chi(x)$ by multiplicativity. As in [9], we define the twisted Bernoulli function $\tilde{B}_{p,\chi}(x)$ by

$$\sum_{j=0}^{f_\chi - 1} \frac{\chi(\{x\}+j)te^{(\{x\}+j)t}}{e^{f_\chi t}-1} = \sum_{p=0}^{\infty} \tilde{B}_{p,\chi}(x)\frac{t^p}{p!},$$

or equivalently

$$\tilde{B}_{p,\chi}(x) = f_\chi^{p-1} \sum_{j \bmod f_\chi} \chi(x+j)\tilde{B}_p\left(\frac{x+j}{f_\chi}\right)$$

(cf. pp.301 of [9]). Note that $\tilde{B}_{p,\chi}(x)$ is also periodic of period 1.

In what follows, for integers $l_1, \cdots, l_n \in \mathbf{Z}$, we denote by $\gcd\{l_1, \cdots, l_n\}$ the greatest common divisor of $l_1, \cdots, l_n$. We put $\bar{\mathbf{N}} = \mathbf{N} \cup \{0\}$.

Let $P = (p_1, \cdots, p_n, q) \in \bar{\mathbf{N}}^{n+1}$, $H = (h_1, \cdots, h_n) \in \mathbf{Z}^n$ and $k \in \mathbf{N}$. Let $\Psi = (\chi_1, \cdots, \chi_n, \psi)$ be an $(n+1)$-tuple of primitive Dirichlet characters, put $f_\Psi = (\prod_{i=1}^n f_{\chi_i})f_\psi$ and assume that $\gcd\{k, f_\Psi\} = 1$. As in [7], we define the multiple Dedekind sums $S(P, H, k, \Psi)$ by

$$S(P,H,k,\Psi) = \sum_{\alpha_1,\cdots,\alpha_n \bmod k} \left(\prod_{i=1}^n \tilde{B}_{p_i,\chi_i}\left(\frac{\alpha_i}{k}\right)\right) \tilde{B}_{q,\psi}\left(\frac{h_1\alpha_1 + \cdots + h_n\alpha_n}{k}\right).$$

For any $d \in \mathbf{N}$, we put $I_d = \{(b_1, \cdots, b_n) \in \bar{\mathbf{N}}^n | 0 \leqslant b_1, \cdots, b_n \leqslant d-1\}$. For any $m, N \in \mathbf{N}$, we put $\sigma_{m,\Psi}(N) = \sum_{\delta | N} \delta^m (\chi_1 \cdots \chi_n \psi)(\delta)$. In addition, we put $s(P) = p_1 + \cdots + p_n + q - n$. Then the main result of this paper is the following.

Theorem. *Let $N \in \mathbf{N}$. Then we have*

$$N^{s(P)-q}(\chi_1 \cdots \chi_n)(N) \sum_{\substack{ad=N \\ d>0}} \sum_{B \in I_d} d^{q-n}\psi(d)S(P, aH+kB, dk, \Psi)$$
$$= \sigma_{s(P),\Psi}(N)S(P,H,k,\Psi),$$

where we put $aH + kB = (ah_1 + kb_1, \cdots, ah_n + kb_n)$ for $B = (b_1, \cdots, b_n)$.

3. Proof of the Theorem in a special case

In this section, we deal with the following sum:

$$s_{p,q}(h,k) = \sum_{\alpha \bmod k} \tilde{B}_p\left(\frac{\alpha}{k}\right)\tilde{B}_q\left(\frac{h\alpha}{k}\right).$$

for $p,q\in\bar{\mathbf{N}}, h\in\mathbf{Z}, k\in\mathbf{N}$. For this sum, our main Theorem reduces to the following formula:

$$N^{p-1}\sum_{\substack{ad=N\\ d>0}} d^{q-1}\sum_{b=0}^{d-1} s_{p,q}(ah+kb,dk)=\sum_{\delta|N}\delta^{p+q-1}s_{p,q}(h,k). \tag{3}$$

The purpose of this section is to prove (3).

We put

$$F(h,k:s,t)=\sum_{\alpha=0}^{k-1}\frac{e^{\frac{\alpha}{k}s+\{\frac{h\alpha}{k}\}t}}{(e^s-1)(e^t-1)},$$

which is expanded at $(s,t)=(0,0)$ as

$$F(h,k:s,t)=\sum_{p,q\in\bar{\mathbf{N}}} s_{p,q}(h,k)\frac{s^{p-1}t^{q-1}}{p!q!}. \tag{4}$$

By the periodicity of $\tilde{B}_q(x)$, we have

$$s_{p,q}(h+mk,k)=s_{p,q}(h,k)$$

for all $m\in\mathbf{Z}$. By virtue of this, we assume $h>0$ in what follows without loss of generality.

Modifying the set K_1 in Introduction, we put

$$K(h,k)=\left\{(l,m)\in\bar{\mathbf{N}}^2|m>\frac{h}{k}l\right\}$$

and define

$$f(h,k:u,v)=\sum_{(l,m)\in K(h,k)}u^l v^m.$$

This formal power series can be expressed by a rational function as in the following.

Lemma 3.1. *We have*

$$f(h,k:u,v)=\sum_{\alpha=0}^{k-1}\frac{u^\alpha v^{[\frac{h\alpha}{k}]+1}}{(1-u^k v^h)(1-v)}.$$

Proof. This formula is essentially the same as that for $\sigma_{K_1}(u,v)$ in Section 2 of [2], and shown by a straightforward calculation as follows:

$$\begin{aligned}
f(h,k:u,v)&=\sum_{l=0}^{\infty}\sum_{m=[\frac{hl}{k}]+1}^{\infty}u^l v^m=\sum_{\alpha=0}^{k-1}\sum_{r=0}^{\infty}u^{\alpha+kr}\sum_{m_1=0}^{\infty}v^{[\frac{h}{k}(\alpha+kr)]+1+m_1}\\
&=\sum_{\alpha=0}^{k-1}u^\alpha v^{[\frac{h}{k}\alpha]+1}\sum_{r=0}^{\infty}(u^k v^h)^r\sum_{m_1=0}^{\infty}v^{m_1}\\
&=\sum_{\alpha=0}^{k-1}\frac{u^\alpha v^{[\frac{h}{k}\alpha]+1}}{(1-u^k v^h)(1-v)}.
\end{aligned}$$

■

Now put

$$f_r(h,k:u,v)=\sum_{\alpha=0}^{k-1}\frac{u^{\alpha}v^{[\frac{h\alpha}{k}]+1}}{(1-u^kv^h)(1-v)}.$$

Since we have $[h\alpha/k]=(h\alpha/k)-\{h\alpha/k\}$, this can also be expressed as

$$f_r(h,k:u,v)=\sum_{\alpha=0}^{k-1}\frac{(u^kv^h)^{\frac{\alpha}{k}}v^{-\{\frac{\alpha}{k}\}+1}}{(1-u^kv^h)(1-v)}.$$

Put $u=e^{(s+ht)/k}$ and $v=e^{-t}$. Then $u^kv^h=e^s$ and $v^{-1}=e^t$, so that we have

$$f_r(h,k:e^{(s+ht)/k},e^{-t})=-F(h,k:s,t). \tag{5}$$

In order to proceed further, we introduce the following additive subgroup of $\mathbf{Z}^2$ for $a,d\in\mathbf{N}$ and $b\in\mathbf{Z}$:

$$A(a,d:b)=(a,-b)\mathbf{Z}+(0,d)\mathbf{Z}.$$

The following lemma plays an essential role in proving (3).

Lemma 3.2. *Let $N=ad$ with $a,d\in\mathbf{N}$ and $b\in\mathbf{Z}$ and let $(l,m)\in\mathbf{Z}^2$. Put $d_1=\gcd\{l,N\},d_2=\gcd\{l,m,N\},l'=l/d_1$ and $N'=N/d_1$. Then, we have $(l,m)\in A(a,d:b)$, if and only if the following three conditions hold:*

(i) $a|d_1$
(ii) $\frac{d_1}{a}|d_2$
(iii) $bl'\equiv-\frac{am}{d_1}\pmod{N'}$.

Proof. Suppose that $(l,m)\in A(a,d:b)$ and write

$$(l,m)=(a,-b)\mu+(0,d)\nu=(a\mu,-b\mu+d\nu) \tag{6}$$

with $\mu,\nu\in\mathbf{Z}$. Then a divides l as well as N, so that a divides d_1. We have further

$$\frac{m}{d_1/a}=\frac{a(-b\mu+d\nu)}{d_1}=-bl'+N'\nu, \tag{7}$$

which implies that d_1/a divides m as well as l and N. Hence, d_1/a divides d_2. In addition, (7) means the congruence (iii). Conversely, under the conditions (i) , (ii) and (iii), we can easily deduce equation (6). This completes the proof. ■

Corollary 3.3. *We have*

$$\sum_{\substack{ad=N\\d>0}}\sum_{b=0}^{d-1}\sum_{(l,m)\in A(a,d:b)\cap K(h,k)}u^lv^m=\sum_{\delta|N}\delta\sum_{(l,m)\in(\delta\mathbf{Z})^2\cap K(h,k)}u^lv^m. \tag{8}$$

Proof. We use the same notations as in Lemma 3.2. Note that $\gcd\{l', N'\} = 1$. Hence, if $a \in \mathbf{N}$ satisfies the conditions (i) and (ii) in Lemma 3.2, the condition (iii) shows that

$$\sharp\{b \in \mathbf{Z} | 0 \leqslant b \leqslant d-1, (l,m) \in A(a,d:b)\} = \frac{d}{N'} = \frac{dd_1}{N} = \frac{d_1}{a}.$$

This shows that the coefficient of the term $u^l v^m$ appearing in the left-hand side of (8) is $\sum_{a|d_1,(d_1/a)|d_2}(d_1/a)$. By putting $\delta = d_1/a$, this coefficient is equal to $\sum_{\delta|d_2}\delta$, which is just the coefficient of the term $u^l v^m$ appearing in the right-hand side of (8). This completes the proof. ■

Lemma 3.4. *Let $a, d \in \mathbf{N}$ and $b \in \bar{\mathbf{N}}$. Let $(l,m) \in A(a,d:b)$ and write $(l,m) = (a,-b)\mu + (0,d)\nu$ with $\mu, \nu \in \mathbf{Z}$. Then, $(l,m) \in K(h,k)$ holds if and only if $(\mu,\nu) \in K(ah+kb, dk)$.*

Proof. As in the statement, let $(l,m) = (a\mu, -b\mu + d\nu)$. Then, $(l,m) \in K(h,k)$ holds if and only if $-b\mu + d\nu > ha\mu/k \geqslant 0$, which is equivalent to $\nu > (ah+kb)\mu/(dk) \geqslant 0$, namely $(\mu,\nu) \in K(ah+kb, dk)$. ■

Now (3) is deduced as follows: Lemma 3.4 shows that the left-hand side of (8) equals

$$\sum_{\substack{ad=N\\ d>0}} \sum_{b=0}^{d-1} \sum_{(\mu,\nu)\in K(ah+kb,dk)} u^{a\mu} v^{-b\mu+d\nu} = \sum_{\substack{ad=N\\ d>0}} \sum_{b=0}^{d-1} f(ah+kb, dk : u^a b^{-b}, v^d).$$

On the other hand, note that for each $\delta | N$, we have

$$(\delta\mathbf{Z})^2 \cap K(h,k) = \{(\delta l, \delta m) | (l,m) \in K(h,k)\},$$

so that the right-hand side of (8) equals

$$\sum_{\delta|N} \delta \cdot f(h,k : u^\delta, v^\delta).$$

Then, by Lemma 3.1, equation (8) is tranformed into

$$\sum_{\substack{ad=N\\ d>0}} \sum_{b=0}^{d-1} f_r(ah+kb, dk : u^a b^{-b}, v^d) = \sum_{\delta|N} \delta \cdot f_r(h,k : u^\delta, v^\delta). \tag{9}$$

Put $u = e^{(s+ht)/k}$ and $v = e^{-t}$ as before. Then, we have $u^a v^{-b} = e^{(a(s+ht)/k)+bt} = e^{(ads+adht+bdkt)/dk} = e^{(Ns+(ah+kb)dt)/(dk)}$ and $v^d = e^{-dt}$. Note that equation (5) yields

$$f_r(ah+kb, dk : e^{(Ns+(ah+kb)dt)/(dk)}, e^{-dt}) = -F(ah+kb, dk : Ns, dt)$$

and

$$f_r(h,k:e^{-\delta s},e^{-\delta t})=-F(h,k:\delta s,\delta t).$$

Hence, equation (9) is transformed into

$$\sum_{\substack{ad=N\\ d>0}}\sum_{b=0}^{d-1}F(ah+kb,dk:Ns,dt)=\sum_{\delta|N}\delta\cdot F(h,k:\delta s,\delta t).$$

Expanding both sides at $(s,t)=(0,0)$, we see from (4) that

$$\sum_{p,q\in\bar{\mathbf{N}}}\sum_{\substack{ad=N\\ d>0}}\sum_{b=0}^{d-1}s_{p,q}(ah+kb,dk)\frac{N^{p-1}d^{q-1}s^{p-1}t^{q-1}}{p!q!}$$
$$=\sum_{p,q\in\bar{\mathbf{N}}}\sum_{\delta|N}\delta\cdot s_{p,q}(h,k)\frac{\delta^{p+q-2}s^{p-1}t^{q-1}}{p!q!}.$$

Comparing the coefficients, we obtain (3).

4. Proof of Theorem in the general case

In this section, we extend the method of the previous section to the general case and prove the Theorem.

Let $H=(h_1,\cdots,h_n)\in\mathbf{Z}^n$ and $k\in\mathbf{N}$ as before. For $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbf{Z}^n$, we put $H\cdot\alpha=h_1\alpha_1+\cdots+h_n\alpha_n$(the inner product of H and α). Let $\mathcal{A}_k=\{(\alpha_1,\cdots,\alpha_n)\in\bar{\mathbf{N}}^n\,|\,0\leqslant\alpha_i\leqslant k-1\ \text{ for }\ 1\leqslant i\leqslant n\}$ and set

$$F(H,k,\Psi:s_1,\cdots,s_n,t)=\sum_{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathcal{A}_k}\left(\prod_{i=1}^{n}\sum_{j_i=0}^{f_{\chi_i}-1}\frac{\chi_i\left(\frac{\alpha_i}{k}+j_i\right)e^{\left(\frac{\alpha_i}{k}+j_i\right)s_i}}{e^{f_{\chi_i}s_i}-1}\right)$$
$$\times\sum_{j=0}^{f_\psi-1}\frac{\psi\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)e^{\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)t}}{e^{f_\psi t}-1},$$

which is expanded at $(s_1,\cdots,s_n,t)=(0,\cdots,0,0)$ as

$$F(H,k,\Psi:s_1,\cdots,s_n,t)=\sum_{P=(p_1,\cdots,p_n,q)\in\bar{\mathbf{N}}^{n+1}}S(P,H,k,\Psi)\frac{s_1^{p_1-1}\cdots s_n^{p_n-1}t^{q-1}}{p_1!\cdots p_n!q!}.\tag{10}$$

By the periodicity of $\tilde{B}_{q,\psi}(x)$, we assume that $h_i>0$ for $1\leqslant i\leqslant n$ without loss of generality.

We put

$$K(H,k)=\left\{(l_1,\cdots,l_n,m)\in\bar{\mathbf{N}}^{n+1}|m>\frac{h_1l_1+\cdots+h_nl_n}{k}\right\}$$

and define

$$f(H,k,\Psi:u_1,\cdots,u_n,v) = \sum_{(l_1,\cdots,l_n,m)\in K(H,k)} \chi_1(l_1)\cdots\chi_n(l_n)\psi(h_1l_1+\cdots+h_nl_n-km)u_1^{l_1}\cdots u_n^{l_n}v^m.$$

Lemma 4.1. *We have*

$$f(H,k,\Psi:u_1,\cdots,u_n,v) = (\chi_1\cdots\chi_n\psi)(k)\sum_{\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathcal{A}_k}\left(\prod_{i=1}^{n}\sum_{j_i=0}^{f_{\chi_i}-1}\frac{\chi_i\left(\frac{\alpha_i}{k}+j_i\right)(u_i^kv^{h_i})^{\frac{\alpha_i}{k}+j_i}}{1-(u_i^kv^{h_i})^{f_{\chi_i}}}\right) \times\sum_{j=0}^{f_\psi-1}\frac{\psi\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)v^{-\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)+f_\psi}}{1-v^{f_\psi}}. \qquad (11)$$

Proof. For each $(l_1,\cdots,l_n,m)\in K(H,k)$, we have the following unique expressions of $l_1,\cdots,l_n$ and m:

$$l_i=\alpha_i+kj_i+kf_{\chi_i}r_i \quad \text{with} \quad 0\leqslant\alpha_i\leqslant k-1, 0\leqslant j_i\leqslant f_{\chi_i}-1 \quad \text{and} \quad r_i\in\bar{\mathbf{N}}$$

for $1\leqslant i\leqslant n$ and

$$m=\left[\frac{h_1l_1+\cdots+h_nl_n}{k}\right]+(f_\psi-j)+f_\psi m_1 \text{ with } 0\leqslant j\leqslant f_\psi-1 \text{ and } m_1\in\bar{\mathbf{N}}.$$

Then, we have $l_i=k\left(\frac{\alpha_i}{k}+j_i+f_{\chi_i}r_i\right)$ for $1\leqslant i\leqslant n$ and

$$m=\frac{H\cdot\alpha}{k}+\sum_{i=1}^{n}h_i(j_i+f_{\chi_i}r_i)-\left\{\frac{H\cdot\alpha}{k}\right\}-j+f_\psi(1+m_1),$$

where we put $\alpha=(\alpha_1,\cdots,\alpha_n)$. Hence,

$$h_1l_1+\cdots+h_nl_n-km=k\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)-kf_\psi(1+m_1)$$

and

$$u_1^{l_1}\cdots u_n^{l_n}v^m=\left(\prod_{i=1}^{n}(u_i^kv^{h_i})^{\frac{\alpha_i}{k}+j_i+f_{\chi_i}r_i}\right)v^{-\left(\left\{\frac{H\cdot\alpha}{k}\right\}+j\right)+f_\psi(1+m_1)}.$$

Consequently we derive the required formula by a straightforward calculation. ■

Let $f_r(H,k,\Psi:u_1,\cdots,u_n,v)$ denote the rational function expressed by the right-hand side of (11). Put $u_i=e^{(s_i+h_it)/k}$ for $1\leqslant i\leqslant n$ and $v=e^{-t}$. Then, $u_i^kv^{h_i}=e^{s_i}$ and $v^{-1}=e^t$, so that we have

$$f_r(H,k,\Psi:e^{(s_1+h_1t)/k},\cdots,e^{(s_n+h_nt)/k},e^{-t}) = (-1)^n(\chi_1\cdots\chi_n\psi)(k)F(H,k,\Psi:s_1,\cdots,s_n,t). \qquad (12)$$

For $a, d \in \mathbf{N}$ and $B = (b_1, \ldots, b_n) \in \mathbf{Z}^n$, let $A(a, d : B)$ denote the additive subgroup of $\mathbf{Z}^{n+1}$ generated by $(a, 0, \cdots, 0, -b_1)$, $(0, a, \cdots, 0, -b_2), \cdots$, $(0, 0, \cdots, a, -b_n)$ and $(0, \cdots, 0, d)$. Then Lemma 3.2 can be generalized in the following way:

Lemma 4.2. *Let $N = ad$ with $a, d \in \mathbf{N}, B = (b_1, \ldots, b_n) \in \mathbf{Z}^n$ and let $(l_1, \cdots, l_n, m) \in \mathbf{Z}^{n+1}$. Put $d_1 = \gcd\{l_1, \cdots, l_n, N\}$, $d_2 = \gcd\{l_1, \cdots, l_n, m, N\}$, $l'_i = l_i/d_1$ for $1 \leqslant i \leqslant n$ and $N' = N/d_1$. Then, we have $(l_1, \cdots, l_n, m) \in A(a, d : B)$, if and only if the following three conditions hold:*

(i) $a | d_1$
(ii) $\frac{d_1}{a} | d_2$
(iii) $l'_1 b_1 + \cdots + l'_n b_n \equiv -\frac{am}{d_1} \pmod{N'}$.

Proof. Suppose that $(l_1, \cdots, l_n, m) \in A(a, d : B)$ and write

$$\begin{aligned}(l_1, \cdots, l_n, m) \\ = (a, 0, \cdots, 0, -b_1)\mu_1 + \cdots + (0, 0, \cdots, a, -b_n)\mu_n + (0, \cdots, 0, d)\nu. \quad (13)\end{aligned}$$

with $\mu_1, \cdots, \mu_n, \nu \in \mathbf{Z}$. Then the conditions (i) , (ii) and (iii) follow immediately in a similar way as in the the proof of Lemma 3.2. Conversely, under the conditions (i) , (ii) and (iii), we can easily deduce equation (13). This completes the proof. ∎

Recall that $I_d = \{(b_1, \cdots, b_n) \in \bar{\mathbf{N}}^n | 0 \leqslant b_i \leqslant d - 1 \text{ for } 1 \leqslant i \leqslant n\}$ for $d \in \mathbf{N}$.

Corollary 4.3. *Let $g(l_1, \cdots, l_n, m)$ be any function on $\mathbf{Z}^{n+1}$ with values in any ring extension of $\mathbf{Q}$. Then, we have*

$$\begin{aligned}&\sum_{\substack{ad=N \\ d>0}} a^{n-1} \sum_{B \in I_d} \sum_{(l_1, \cdots, l_n, m) \in A(a,d:B) \cap K(H,k)} g(l_1, \cdots, l_n, m) u_1^{l_1} \cdots u_n^{l_n} v^m \\ &= N^{n-1} \sum_{\delta | N} \delta \sum_{(l_1, \cdots, l_n, m) \in (\delta \mathbf{Z})^{n+1} \cap K(H,k)} g(l_1, \cdots, l_n, m) u_1^{l_1} \cdots u_n^{l_n} v^m. \quad (14)\end{aligned}$$

Proof. Let $(l_1, \cdots, l_n, m) \in K(H, k)$ and suppose that a satisfies the conditions (i) and (ii). Then the condition (iii) shows that the set of n-tuples $B = (b_1, \ldots, b_n) \in \mathbf{Z}^n$ satisfying $(l_1, \cdots, l_n, m) \in A(a, d : B)$ consists of the solutions $(x_1, \cdots, x_n)$ of the congruence

$$l'_1 x_1 + \cdots + l'_n x_n \equiv -\frac{am}{d_1} \pmod{N'}.$$

Note that the map from $\mathbf{Z}^n$ to $\mathbf{Z}$ defined by mapping $(x_1, \cdots, x_n)$ onto $l'_1 x_1 + \cdots + l'_n x_n$ induces a map from $(\mathbf{Z}/N'\mathbf{Z})^n$ to $\mathbf{Z}/N'\mathbf{Z}$, which is surjective because $\gcd\{l'_1, \cdots, l'_n, N'\} = 1$. Hence, for each $y \bmod N' \in \mathbf{Z}/N'\mathbf{Z}$, the number of n-tuples $(x_1, \cdots, x_n) \bmod N' \in (\mathbf{Z}/N'\mathbf{Z})^n$ satisfying $l'_1 x_1 + \cdots + l'_n x_n \equiv y \pmod{N'}$ is $\sharp(\mathbf{Z}/N'\mathbf{Z})^n / \sharp(\mathbf{Z}/N'\mathbf{Z}) = {N'}^{n-1}$. Taking $y = -am/d_1$, we see further that

$$\sharp\{B \in I_d | (l_1, \cdots, l_n, m) \in A(a, d : B)\} = {N'}^{n-1} \left(\frac{d}{N'}\right)^n = \frac{d^n}{N'} = \frac{d^{n-1} d_1}{a}.$$

Hence, the coefficient of the term $g(l_1,\cdots,l_n,m)u_1^{l_1}\cdots u_n^{l_n}v^m$ appearing in the left-hand side of (14) is

$$\sum_{\substack{a|d_1\\(d_1/a)|d_2}} a^{n-1}\frac{d^{n-1}d_1}{a} = N^{n-1}\sum_{\substack{a|d_1\\(d_1/a)|d_2}}\frac{d_1}{a}.$$

By putting $\delta = d_1/a$, this coefficient is equal to $N^{n-1}\sum_{\delta|d_2}\delta$, which is just the coefficientof the term $g(l_1,\cdots,l_n,m)u_1^{l_1}\cdots u_n^{l_n}v^m$ appearing in the right-hand side of (14). This completes the proof. ■

Lemma 4.4. *Let $a,d\in\mathbf{N}$ and $B=(b_1,\cdots,b_n)\in\bar{\mathbf{N}}^n$. Let $(l_1,\cdots,l_n,m)\in A(a,d:B)$ be written as (13), namely*

$$(l_1,\cdots,l_n,m) = (a\mu_1,\cdots,a\mu_n,-b_1\mu_1-\cdots-b_n\mu_n+d\nu)$$

with $\mu_1,\cdots,\mu_n,\nu\in\mathbf{Z}$. Then, $(l_1,\cdots,l_n,m)\in K(H,k)$ holds if and only if $(\mu_1,\cdots,\mu_n,\nu)\in K(aH+kB,dk)$.

Proof. By (13), $(l_1,\cdots,l_n,m)\in K(H,k)$ holds if and only if

$$-(b_1\mu_1+\cdots+b_n\mu_n)+d\nu > \frac{a(h_1\mu_1+\cdots+h_n\mu_n)}{k} \quad \text{with} \quad \mu_1,\cdots,\mu_n\in\bar{N},$$

which is equivalent to

$$\nu > \sum_{i=1}^{n}(ah_i+kb_i)\mu/(dk) \quad \text{with} \quad \mu_1,\cdots,\mu_n\in\bar{N},$$

namely $(\mu_1,\cdots,\mu_n,\nu)\in K(aH+kB,dk)$. ■

Now we are going to prove the Theorem. Put

$$g(l_1,\cdots,l_n,m) = \chi_1(l_1)\cdots\chi_n(l_n)\psi(h_1l_1+\cdots h_nl_n-km).$$

If (13) holds, we have

$$\begin{aligned} h_1l_1+\cdots h_nl_n-km &= a(h_1\mu_1+\cdots+h_n\nu_n)+k(b_1\mu_1+\cdots+b_n\mu_n-d\nu)\\ &= \sum_{i=1}^{n}(ah_i+kb_i)\mu_i-dk\nu \end{aligned}$$

and

$$u_1^{l_1}\cdots u_n^{l_n}v^m = (u_1^av^{-b_1})^{\mu_1}\cdots(u_n^av^{-b_n})^{\mu_n}v^{d\nu}.$$

Hence, by Lemmas 4.1 and 4.4, equation (14) becomes

$$\begin{aligned} &\sum_{\substack{ad=N\\d>0}} a^{n-1}(\chi_1\cdots\chi_n)(a)\sum_{B\in I_d} f_r(aH+kB,dk,\Psi:u_1^av^{-b_1},\cdots,u_n^av^{-b_n},v^d)\\ &\qquad = N^{n-1}\sum_{\delta|N}\delta(\chi_1\cdots\chi_n\psi)(\delta)f_r(H,k,\Psi:u_1^\delta,\cdots,u_n^\delta,v^\delta). \quad (15) \end{aligned}$$

Put $u_i = e^{(s_i+h_it)/k}$ for $1 \leqslant i \leqslant n$ and $v = e^{-t}$ as before. Then, we have

$$u_i^a v^{-b_i} = e^{a(s_i+h_it)/k+b_it} = e^{(Ns_i+(ah_i+kb_i)dt)/dk}$$

for $1 \leqslant i \leqslant n$ and

$$v^d = e^{-dt}.$$

Then, by (10) and (12), equation (15) is transformed into

$$\begin{aligned}&\sum_{P=(p_1,\cdots,p_n,q)\in\bar{\mathbf{N}}^{n+1}} \sum_{\substack{ad=N\\ d>0}} a^{n-1}(\chi_1\cdots\chi_n)(a)\sum_{B\in I_d}(\chi_1\cdots\chi_n\psi)(dk)\\ &\qquad\times S(P,aH+kB,dk,\Psi)\frac{N^{p_1+\cdots+p_n-n}d^{q-1}s_1^{p_1-1}\cdots s_n^{p_n-1}t^{q-1}}{p_1!\cdots p_n!q!}\\ &= N^{n-1}\sum_{P=(p_1,\cdots,p_n,q)\in\bar{\mathbf{N}}^{n+1}}\sum_{\delta|N}\delta(\chi_1\cdots\chi_n\psi)(\delta)(\chi_1\cdots\chi_n\psi)(k)\\ &\qquad\times S(P,H,k,\Psi)\frac{\delta^{p_1+\cdots+p_n+q-(n+1)}s_1^{p_1-1}\cdots s_n^{p_n-1}t^{q-1}}{p_1!\cdots p_n!q!}.\end{aligned}$$

Comparing the coefficients, we complete the proof of Theorem.

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Address: Kazuhito Kozuka: Dep. Math., Miyakonojo National College of Technology, Miyakonojo, Miyazaki 885-8567, Japan.

E-mail: k31k@cc.miyakonojo-nct.ac.jp

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